EXPLORATION OF VARIOUS ITEMS IN LINEAR ALGEBRA

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Abstract. This paper will attempt to explore some important theorems of linear algebra and the application of these theorems. The "Clubtown" combinatorics problem will receive special attention with respect to principles of linear algebra.

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1. Introduction

Linear maps and matrices account for a large part linear algebra as a whole. Understanding the properties of nonsingular matrices helps in proofs where it may not be obvious if a matrix has full rank. The Cayley-Hamilton theorem shows us that a square matrix satisfies its own characteristic polynomial. The Clubtown analogy deals with the problem of producing a "maximal" solution based on specified parameters. The Clubtown problem also entertains the difference between "maximal" and "maximum" solutions. We will then prove that a maximum solution does exist for the Clubtown problem.

2. Singular and Nonsingular Matrices

The study of linear algebra is concerned with linear maps, and therefore, matrices. A linear map, a type of function, maps one vector space to another ($V \to W$). Linear maps sometimes map a vector space to the same vector space ($V \to V$). When a linear map maps a vector space to itself, we can then take the inverse of such linear map. When a linear map maps a vector space bijectively to itself, we can then take the inverse of such linear map. Such matrices are called nonsingular or invertible.

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Definition 1. *A square matrix A is invertible if there exists a matrix B such that $AB = BA = I$, where I is the identity matrix.*

Unlike many matrices, invertible matrices represent linear maps where we can "retrace our steps" and see the input of the function. Because invertible matrices are special compared to most other matrices, it is important that we understand some basics about invertible matrices.

Theorem 1. *The following are equivalent for a square matrix $M_{n\times n}$.*

(1) *M is invertible*
(2) *$det(M) \neq 0$*
(3) *For $X_{n\times 1}$, if $MX = 0$, then $X = 0$*

Proof. To prove if M is invertible then $det(M) \neq 0$, all we need is to use Cramer's rule. We know that

$$adj(M)M = det(M)I$$

Where $adj(M)$ is the adjugate matrix of M. We can manipulate this equation, therefore

$$M^{-1} = \frac{adj(M)}{det(M)}$$

from this equation, we can see that if M^{-1} exists, then $det(M) \neq 0$. But this statement also proves the reverse direction, in that if $det(M) \neq 0$ then M^{-1} exists.

Now we want to prove if $det(M) \neq 0$, then for $MX = 0$, $X = 0$. Because $det(M) \neq 0$, we know that the columns of M are linearly independent, i.e. for all columns v_i in M

$$\alpha_1 v_1 + \ldots + \alpha_n v_n = 0 \tag{1}$$

only when $\alpha_1 = \ldots = \alpha_n = 0$. The columns, $v_1 \ldots v_n$, can be rewritten in matrix form

$$\begin{pmatrix} v_1 & \ldots & v_n \end{pmatrix}$$

and equation (1) can be rewritten as

$$\begin{pmatrix} v_1 & \ldots & v_n \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2}$$

Because we know that the columns of M are linearly independent, there is only one way that we can obtain equation (2), and that is when $\alpha_1 = \ldots = \alpha_m = 0$. Therefore, if $det(M) \neq 0$, then $MX = 0$ when $X = 0$.

Now all we have to prove is if $MX = 0$ when $X = 0$, then $det(M) \neq 0$. Suppose $det(M) = 0$. Because $MX = 0$ when $X = 0$, we know that M is not a zero matrix. From Cramer's Rule, we know that

$$\frac{adj(M)M}{det(M)} = I$$

but this is contradicts our assumption that $det(M) = 0$. Therefore, if $MX = 0$ when $X = 0$, then $det(M) \neq 0$.

□

3. Cayley-Hamiltion Theorem

The Cayley-Hamiltion Theorem tells us for any square matrix $M_{n\times n}$, $p(M) = 0$, where p is the characteristic polynomial of M, defined as $p(\lambda) = det(\lambda I - M)$, and I is an $n \times n$ identity matrix. We will prove the Cayley-Hamilton Theorem for the special case when a matrix M is diagonalizable. The proof for the general case involves the use of real analysis and therefore will not be included here.

Theorem 2. *Let M be an $n \times n$ matrix, and let $p(\lambda) = det(\lambda I - M)$ be the characteristic polynomial of M. Then $p(M) = 0$.*

Proof. First, let us assume that M is a diagonalizable matrix, therefore there exists an invertible matrix S and a diagonal matrix T where $M = STS^{-1}$ such that

$$T = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}. \tag{3}$$

Based on how matrix multiplication is defined, we know that T^k will be a diagonal matrix where all the eigenvalues are to the k power, as below

$$T^k = \begin{pmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \lambda_n^k \end{pmatrix}. \tag{4}$$

We can infer from the above equation that

$$p(T) = \begin{pmatrix} p(\lambda_1) & 0 & \dots & 0 \\ 0 & p(\lambda_2) & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & p(\lambda_n) \end{pmatrix}. \tag{5}$$

Because $\lambda_1, \dots, \lambda_n$ are eigenvalues, we know that $p(\lambda_1) = \dots = p(\lambda_n) = 0$, and therefore $p(T) = 0$. We know the change of basis identity $M = STS^{-1}$, therefore we know that $M^k = ST^kS^{-1}$ because

$$M^k = STS^{-1} \dots STS^{-1} = ST \dots TS^{-1} = ST^kS^{-1} \tag{6}$$

which implies $p(M) = Sp(T)S^{-1}$. We already know though that $p(T) = 0$, therefore we know that $p(M) = 0$, which proves the Cayley-Hamiltion Theorem for the special case. □

4. Background of Clubtown

The Clubtown problem is a way to visualize a combinatorics problem based on specific parameters to regulate the establishment of clubs in a town. There are two different types of Clubtowns, Eventown and Oddtown. The Eventown and Oddtown problems are separate problems. In both Clubtowns, there are n number of people, and there are three laws regarding the establishment of clubs; (1) in either Eventown or Oddtown, no two clubs can have the same set of members; (2) if on is in Eventown, there must be an even number of people in each club, and if one is in Oddtown, there must be an odd number of people in each club; (3) in either Eventown or Oddtown, the intersection of any two clubs must have an even number of members. Let us now translate these parameters into in more mathematical language.

Clubtown Rules.

(1) if $i \neq j$, then $C_i \neq C_j$
(2) (Eventown) $\forall k$, $|C_k| =$ even or (Oddtown) $\forall k$, $|C_k| =$ odd
(3) $\forall\, i \neq j$, we have $|C_i \bigcap C_j| =$ even

Finding any particular solution to these set of rules is not hard. The problem becomes more difficult when one wants to find a maximal or maximum solution. Before we proceed any further, it is important to note the difference between maximal and maximum. A maximal solution is one where you cannot add any more elements to a set and still have the set satisfy the original rules, i.e. you would not be able to add anymore clubs to either Eventown or Oddtown and still have the set of clubs satisfy the original rules. A maximum set is the largest maximal set of solutions, i.e. the solution that produces the most number of clubs which still satisfies the rules. A maximum set must be a maximal set, but a maximal set is not necessarily a maximum set.

4.1. **Maximal Eventown Solution.** Now let us try and find a maximal solution to the Eventown problem, which happens to be quite simple. We know that there are n number of people in the town. If there is an even number of people, pair everyone with someone else, e.g. $\{1,2\},\{3,4\},\dots,\{n-1,n\}$. If there is an odd number of people, pair the people as before, but leave the nth person in a club of their own. Because we have paired everyone beforehand, we have guaranteed that the 2nd and 3rd rules will follow when n is even; no matter how we group these paired clubs, the number of people in the union of paired clubs will be even, and the intersection of any two clubs will be either 0 or a multiple of 2. For Eventown, because we now have $\frac{n}{2}$ paired clubs, we know that a maximal solution to the Eventown problem is $2^{\frac{n}{2}}$, because each club has the option of being in or out (2 options), and there are $\frac{n}{2}$ clubs.

A maximal Eventown solution when n is odd happens to be $2^{\frac{n-1}{2}}$ because there is only one place for the nth person to be without breaking any of the rules, which is not in any club (if we add the nth person to any club, that club will have an odd number of people in it). A maximal solution for Eventown when n is either even or odd is $2^{\lfloor \frac{n}{2} \rfloor}$

4.2. **Maximal Oddtown Solution.** Now that we have found a maximal solution for Eventown, lets find a maximal solution for Oddtown. There are n number of people in Oddtown. A maximal solution happens to be n number of clubs. n

number of clubs is maximal because if you try to add another club, which would have an odd number of members, then the intersection of such club and the correct single person club would produce an intersection equal to 1, which breaks the third rule of Oddtown.

Now that we have shown maximal solutions for Eventown and Oddtown, we want to prove the maximum solution to the Eventown and Oddtown problems. The maximum solutions to each problem happens to be the maximal solutions that we have obtained here.

5. Eventown

Now we will prove that $2^{\lfloor \frac{n}{2} \rfloor}$ is the maximal solution to the Eventown problem.

Proof. Let us first restate the rules of eventown.

Eventown Rules.

(1) if $i \neq j$, then $C_i \neq C_j$
(2) (Eventown) $\forall k$, $|C_k| =$ even or (Oddtown) $\forall k$, $|C_k| =$ odd
(3) $\forall\, i \neq j$, we have $|C_i \bigcap C_j| =$ even

We know that there are n number of people in Eventown. Let's define a set $S = \{1, 2, \ldots, n\}$, which is a set that represents all of the n people in Eventown. Now lets define the power set $P(S) = \{\emptyset, S, \ldots\}$, which is a set that contains all of the possible subsets of S. The power set $P(S)$ is very similar to the field F_2 over n-tuples. In a slight digression, F_2 is a field of two elements, 1 and 0, where we define the addition operator as follows; $0 + 0 = 0$, $0 + 1 = 1$, and $1 + 1 = 0$. We define multiplication over F_2 as follows; $0 \times 0 = 0$, $0 \times 1 = 0$, $1 \times 1 = 1$. F_2^n, therefore, is a vector with n positions where each position takes on the value 0 or 1. As before, F_2^n is very similar to $P(S)$, therefore we will use F_2^n to prove the Eventown maximum. To prove Eventown, though, we need to modify our rules slightly so as they apply to vectors in F_2^n. First, we will define the dot product.

Definition 2. *Let v_1, v_2 be vectors where $v_1 = (\alpha_1, \dot{;} \alpha_n)$ and $v_2 = (\beta_1, \dot{;} \beta_n)$. We define the the dot product of v_1 and v_2 to be $v_1 \times v_2 = \alpha_1\beta_1 + \cdot + \alpha_n\beta_n$.*

Now we will state the modified Eventown rules.

Modified Eventown Rules.

(1) if $i \neq j$ and $v_i, v_j \in M$, then $v_i \neq v_j$.
(2) $\forall\, v \in M$, there are an even number of 1's in v.
(3) $\forall\, v_i, v_j \in M$, we have $v_i \cdot v_j = 0$.

M is a set of vectors in F_2^n, which represents a set of clubs that satisfy the Eventown Rules. We will prove the maximal solution in two parts. First what we want to do for M maximal is prove that M is a vector subspace of F_2^n. We want to show that the zero vector is contained in M, M is closed under scalar multiplication, and M is closed under vector addition. We already know that the zero vector is contained in M because the zero vector has an even number of 1's, and it's dot product with any other vector is zero.

Next we will show that M is closed under scalar multiplication. $\forall\, v \in M$, we have $0 \times v = 0$ and $1 \times v = v$. Therefore, M is closed under scalar multiplication.

Now we want to prove that for $v_1, v_2 \in M, v_1 + v_2 \in M$. Assume that $v_1, v_2 \neq M$. We know that $v_1 + v_2$ has an even number of 1's because v_1, v_2 have an even number of 1's and their intersection has an even number of 1's by the equality $|A \cap B| = |A| + |B| - |A \cup B|$ (if $|A \cap B|$, $|A|$, and $|B|$ are even, then $|A \cup B|$ must also be even). Therefore $v_1 + v_2$ satisfies the third Eventown condition. Now we want to show that $v_1 + v_2$ satisfies the second Eventown condition. $\forall\, v \in M$,

$$(v_1 + v_2) \cdot v = v_1 \cdot v + v_2 \cdot v = 0 + 0 = 0. \tag{7}$$

Therefore, $M \cup \{(v_{1)} + v_2)\} = M'$. M' is a strict superset of M, contradicting our assumption that M is a maximal set satisfying the Eventown conditions, therefore $v_1 + v_2 \in M$. We have now proved that M is a vector subspace of F_2^n.

We will begin the second part of our proof with the Degeneracy Condition.

Theorem 3. *For any vector $v \in F_2^n$ and $v \neq 0$, there exists a $v' \in F_2^n$ such that $v \cdot v' = 1$.*

For M maximal of $2^{\lfloor \frac{n}{2} \rfloor}$ clubs, let $m = \frac{n}{2}$, therefore $2m = n$. Now that we know that M maximal is a vector space of F_2^n, we know that M admits a basis of r vectors such that $v_i \cdot v_i' = 1$, where $v_1' \in F_2^n$. Now we want to construct v_i' such that $v_j \cdot v_i' = 0\,\forall\, j \neq i$. If v_i does not satisfy the condition, replace v_i with $v_i - v_1$, therefore $(v_i - v_1) \cdot v_1' = 1 - 1 = 0$. Repeat this process for all v_i in the basis of M where $v_i \cdot v_1' = 1$. Continue this process for v_2', v_3', through v_r' in order. If $v_i \cdot v_k' = 1$, then replace v_i with $v_i - v_k$. We know that this will not mess up our new v_i's relation with any of the v_k''s as we can verify with the dot product.

$$(v_i - v_k) \cdot v_i' = 1 - 0 = 1 \tag{8}$$

where $i < k$. We can choose $v_i' \in F_2^{2m}$ such that $v_i \cdot v_i' = 1$ and $v_j \cdot v_i' = 0\,\forall\, j \neq i$.

Let, if possible, $\exists\, c_i, c_i' \in F_2$ such that

$$c_1 v_1 + \ldots + c_r v_r + c_1' v_1' + \ldots + c_r' v_r' = 0.$$

Now we will multiply both sides with the dot product of v_1 to obtain

$$v_1(c_1 v_1 + \ldots + c_r v_r + c_1' v_1' + \ldots + c_r' v_r') = c_1' v_1' = 0$$

therefore

$$c_1' = 0$$

because $v_1' \neq 0$. If we continue this process for all v_i's, then we find that $\forall\ i$, $c_i' = 0$. We also know that

$$c_1 v_1 + \ldots + c_r v_r = 0 \tag{9}$$

because the v_i's are linearly independent, therefore there are $2r$ linearly independent elements in F_2^{2m}, therefore $2r \leq 2m$, which implies $r \leq m$. Because there are only two elements in F_2, the maximum number of clubs in Eventown equals $2^m = 2^{\lfloor \frac{n}{2} \rfloor}$.

$\square$

6. Oddtown

Now we will prove that n is the maximum number of clubs in Oddtown. First, let us restate the Oddtown rules slightly so as they apply to vectors in F_2^n:

(1) if $i \neq j$ and $v_i, v_j \in M$, then $v_i \neq v_j$.
(2) $\forall\, v \in M$, there are an odd number of 1's in v.
(3) $\forall\, v_i, v_j \in M$, we have $v_i \cdot v_j = 0$.

Now we will state a general linear algebra fact

Theorem 4. *If there exists $v_1, \dots, v_k$ such that $v_i \cdot v_i = 1$ for all $1 \leq i \leq k$ and $v_i \cdot v_j = 0$ for all $i \neq j$, then $v_1, \dots, v_k$ are linearly independent.*

Since we can produce vectors $v_1, \dots, v_k$ that satisfy the above condition, we know that $k \leq n$ because the dimension of any linearly independent list of vectors for a vector space is less than or equal to the dimension of any spanning list of vectors. Therefore, the maximum number of clubs in Oddtown is n clubs. You cannot add any more clubs as one is bound to create a club that has an intersection with odd cardinality with another club.

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