GEODESICS OF HYPERBOLIC SPACE

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Abstract. Hyperbolic geometry is a non-Euclidean geometry in which the traditional Euclidean parallel postulate is false. Instead an alternate version holds; namely that given a point and a line, there exist at least two lines parallel to the first passing through the point. The purpose of this paper is to define the geodesics of the hyperbolic plane. This is accomplished by showing that the set of all isometries of the hyperbolic plane is in fact equivalent to the set of Möbius transformations that are bijective on the space. For the upper half-plane model of hyperbolic geometry, this set is PSL(2,$\mathbb{R}$). We show that the line through the y-axis of the upper half plane is a geodesic, and examine the potential results when this line is transformed by Möbius transformations to determine the geodesics of the space. We can then verify that the parallel postulate is in fact false in the upper half-plane and show that this alternate version holds.

Contents

1. Introduction to Hyperbolic Geometry 1
2. Möbius transformations 2
3. SL(2,R) and PSL(2,R) 3
4. Isometries of Hyperbolic Space 6
5. Geodesics in Hyperbolic Space 9
6. Parallel Lines in Hyperbolic Space 13
Acknowledgments 14
References 14

1. Introduction to Hyperbolic Geometry

Hyperbolic geometry cannot be isometrically embedded in Euclidean space, so several Euclidean models have been constructed to examine specific aspects of hyperbolic geometry. In this paper, we will deal with two models of hyperbolic geometry, the Poincaré disc model and the upper half-plane model.

Definition 1.1. The disc model of hyperbolic space, D, consists of the unit disc in the complex plane, that is, the set $U = \{z = x + iy | \sqrt{x^2 + y^2} < 1\}$. The metric of D is $ds^2 = \frac{4(dx^2+dy^2)}{(1-x^2+y^2)^2} = \frac{dzd\bar{z}}{(1-|z|^2)^2}$.

Definition 1.2. The upper half-plane model of hyperbolic space, H, consists of the upper half of the complex plane, not including the real line; that is, the set $H = \{z = x + iy | y > 0\}$. The metric of H is $ds^2 = \frac{dx^2+dy^2}{y^2}$

The disk and half-plane models of hyperbolic space are isomorphic, mapped conformally by the transformation $w = e^{i\theta}\frac{z-z_0}{z-\overline{z_0}}$, where θ is a constant value. Note that the real line on the edge of H maps to the edge of the disc in D.

It can be helpful to extend the upper half plane model to the upper half of the Riemann sphere. The Riemann sphere is the complex plane joined with a point at infinity. All vertical lines on the complex plane are considered to intersect at this point. This set $\mathbb{C} \cup \{\infty\}$ can be seen as a sphere, with 0 and ∞ at opposite poles. This allows for simple definitions of otherwise indeterminant functions, such as division by zero.

Definition 1.3. For every $z \neq 0$ We define $\frac{z}{0} = \infty$ and $\frac{z}{\infty} = 0$ on the Riemann sphere.

2. Möbius transformations

A class of especially well-behaved functions in hyperbolic space is the set of Möbius transformations.

Definition 2.1. Möbius transformations on the upper half plane are functions of the form $f(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. Because the quantity $ad - bc$ is suspiciously similar to the determinant of a 2×2 matrix, it is referred to as the 'determinant' of the Möbius transformation.

Möbius transformations are relatively simple transformations, and can be seen as the composition of even simpler functions.

Lemma 2.2. *Let* $m(z) = \frac{az+b}{cz+d}$. *If* $c = 0$, *then* $m(z) = \frac{a}{d}z + \frac{b}{d}$. *If* $c \neq 0$ *then* $m(z) = f(i(g(z)))$, *where* $g(z) = c^2 z + cd$, $i(z) = \frac{1}{z}$, *and* $f(z) = -z + \frac{a}{c}$.

Proof. Suppose $c = 0$. Then $m(z) = \frac{az+b}{d} = \frac{a}{d}z + \frac{b}{d}$

Suppose $c \neq 0$. Then

$$m(z) = \frac{az+b}{cz+d} = \frac{(az+b)c}{(cz+d)c} = \frac{acz+bc}{c^2z+cd}$$

Remember that $ad - bc = 1$. Thus

$$m(z) = \frac{acz+bc}{c^2z+cd} = \frac{acz+ad-(ad-bc)}{c^2z+cd} = \frac{a}{c} - \frac{1}{c^2z+cd} = f(i(g(z)))$$

□

This result can be generalized even further, to get the following description of Möbius transformations:

Theorem 2.3. *Möbius transformations consist of compositions of the following types of maps:*

- *Scalings- maps of the form* $z \mapsto kz$ *for some* $k \in \mathbb{C}$
- *Translations - maps of the form* $z \mapsto z + k$ *for some* $k \in \mathbb{C}$.
- *Inversions- maps of the form* $z \mapsto \frac{1}{z}$

Proof. As shown above, any Möbius transformation is a composition of the functions $f(z) = az + b$, $i(z) = \frac{1}{z}$, and $g(z) = c^2 z + cd$.

f is the composition of scalings and translations, namely $f_1(z) = az$ and $f_2(z) = z + b$. $i(z)$ is an inversion. $g(z)$ is also the composition of scalings and translations, namely $g_1(z) = c_2 z$ and $g_2(z) = z + cd$. □

We will now examine some general properties of Möbius transformations.

Theorem 2.4. *The set of Möbius transformations is closed under composition.*

Proof. Let f_1 and f_2 be Möbius transformations. Then $f_1(z) = \frac{a_1 z+b_1}{c_1 z+d_1}$ and $f_2(z) = \frac{a_2 z+b_2}{c_2 z+d_2}$.

$$(f_1 \circ f_2)(z) = f_1(f_2(z)) = \frac{a_1(\frac{a_2 z+b_2}{c_2 z+d_2}) + b_1}{c_1(\frac{a_2 z+b_2}{c_2 z+d_2}) + d_2}$$

$$= \frac{\frac{a_1 a_2 z+b_2+b_1 c_2 z+b_1 d_2}{c_2 z+d_2}}{\frac{c_1 a_2 z+b_2+d_1 c_2 z+d_1 d_2}{c_2 z+d_2}} = \frac{(a_1 a_2 + b_1 c_2)z + (b_1 d_2 + b_2)}{(c_1 a_2 + d_1 c_2)z + (d_1 d_2 + b_2)}$$

This is clearly another Möbius transformation. □

Theorem 2.5. *The inverse of a Möbius transformation is another Möbius transformation.*

Proof. Let f be a Möbius transformation. Thus $f = \frac{az+b}{cz+d}$. Let $g = \frac{dz-b}{-cz+a}$. Clearly, g is a Möbius transformation.

$$g(f(z)) = \frac{d\left(\frac{az+b}{cz+d}\right) - b}{-c\left(\frac{az-b}{cz-d}\right) + a} = \frac{\frac{daz+bd-bcz-bd}{cz+d}}{\frac{-caz-bc+acz+ad}{cz+d}} = \frac{daz - bcz}{da - bc} = z$$

Thus $g = f^{-1}$, so f^{-1} is a Möbius transformation.

□

3. SL(2,R) and PSL(2,R)

Not every Möbius transformation is defined from H to H. We will now examine the subset of Möbius transformations that are. To classify these transformations it is helpful to look at the Möbius transformations acting on the upper half plane as the action of a group of matrices on the upper half plane. The properties of these matrices are illustrative of the action of the transformations, and make them easier to classify.

Definition 3.1. $SL(2\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ such that } a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1 \right\}$

Definition 3.2. $PSL(2\mathbb{R}) = SL(2\mathbb{R})/\{\pm 1\}$

Theorem 3.3. *The projective action of $PSL(2, \mathbb{R})$ on the set of lines through the origin in C^2 is equivalent to the action of Möbius transformations on the complex plane.*

Proof. Let $A \in PSL(2\mathbb{R})$. Thus $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let c be a line through the origin in C^2, expressed as a vector. Thus $c = \begin{pmatrix} v \\ w \end{pmatrix}$. Because the action of $PSL(2\mathbb{R})$ is projective, we multiply this vector by the scalar $\frac{1}{w}$ to get $\begin{pmatrix} \frac{v}{w} \\ 1 \end{pmatrix}$. Defining $z = \frac{v'}{w}$ gives the vector $\begin{pmatrix} z \\ 1 \end{pmatrix}$. Multiplying these matrices together gives

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix}.$$

Using the same technique as before, this is equivalent to $\begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix}$. Thus elements of $PSL(2\mathbb{R})$ map $\begin{pmatrix} z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix}$, which is exactly the action of Möbius transformations with real coefficients. □

As we have shown, Möbius transformations are compositions of translations, scales, and inversions. To find the translations that act on the upper half plane, we will look at each in turn.

Translations of the form $z \mapsto z + k$ will be defined on H if and only if they do not translate downwards; that is, if k is a real number or k is a positive complex number. Thus if the translation is equal to $\frac{1z+b}{0z+1}$, then b is real or b is a positive complex number.

Scales ($z \mapsto kz$) will only be defined on H if k is a positive real number. If $k \in \mathbb{R} \leq 0$ then a point in H will be mapped to the real line (if $k = 0$) or to a point with a negative imaginary value, neither of which are part of H. If $k \in \mathbb{C} \setminus \mathbb{R}$, then $k = a' + b'i$. Thus $a + bi \mapsto aa' + (ab' + a'b)i - bb'$. Thus when $ab' + a'b \leq 0$, this is undefined. Solving this inequality reveals a set of points in H for which this operation maps outside the upper half plane.

The inversion, $z \mapsto \frac{1}{z}$, isn't defined on H. This can be seen quite easily; let $z = i$. $i \mapsto \frac{1}{i} = -i$, which is not in the upper half plane. To get around this, however, we look at the composition of inversions with a scalar: $z \mapsto \frac{k}{z}$. By the same reasoning, k cannot be positive, but when k is negative, the function is defined. $i \mapsto \frac{k}{i} = -ki$, which is in the upper half plane.

A pretty obvious candidate for the Möbius transformations on H is the set of Möbius transformations with all real coefficients. Clearly all coefficients must be real values to avoid simply translating or scaling out of the upper half plane. Because all Möbius transformations have determinant 1, clearly the set of all transformations from H to H is a subset of the set of Möbius transformations with real ceofficients and determinant 1.

The restriction $ad - bc = 1$ holds on Möbius translations in general, but for Möbius transformations with solely real coefficients it is a much stronger condition. For instance, let $f(z) = \frac{az+b}{cz+d}$ such that $ad - bc \neq 0$. Define $D = ad - bc$. We can define the following function:

$$g(z) = \frac{\frac{a}{\sqrt{D}} z \frac{b}{\sqrt{D}}}{\frac{a}{\sqrt{D}} z \frac{d}{\sqrt{D}}}$$

Clearly g is the same transformation; one can simply multiply both the numerator and denominator by $\sqrt{D}$ to return to f. But if we look at the "determinant" of this function, we get

$$\frac{\frac{ad+bc}{D}}{\frac{cz+d}{D}} = \frac{ad+bc}{ad+bc} = 1.$$

When we insist that all coefficients of g are real, however, we have a much stronger statement. Any function of the form $f(z) = \frac{az+b}{cz+d}$ is equivalent to a Möbius transformation with the aforementioned restrictions only if for f, $ad - bc > 0$. Otherwise,

$\sqrt{D}$ will be imaginary, which contradicts the requirements that all coefficients are real.

Theorem 3.4. *A Möbius transformation is defined from $H \to H$ if and only if all coefficients are real, and $ad - bc = 1$.*

Proof. Assume by contradiction that there exists some Möbius transformation f with real coefficients and determinant 1 that does not map H to H. f must be a composition of translations, scales, and inversions. We've already shown that translations, scales by a positive number, and inversions composed with a negative scale preserve the plane, so one of the functions composed into f must be one of the remaining types of transformation: that is, a scale by a negative number or an inversion composed by a positive scalar.

In fact, f must include an odd composition of these. Note that a scale by a negative number composed with another scale by a negative number is simply a scale by a positive number; two inversions scaled by a positive number yields simply $z \mapsto kz$ with k positive, which preserves H; and composing a scale by a negative with an inversion gives an inversion composed with a scale by a negative, which we have already shown preserves H.

Thus f is the composition of an odd number of the previously listed functions, along with any number of Möbius transformations that preserve H.

Recall that these Möbius transformations are identified with $PSL(2, \mathbb{R})$, and the determinant of each transformation is the determinant of the corresponding matrix. Thus the determinant of f is the product of the determinants of all of its components. The determinants of the H-preserving transformations will clearly be one, so all that remains is to find the determinants of the components that do not preserve H.

A negative scalar takes the form $z \mapsto \frac{kz+0}{0z+1}$ with $k < 0$, so it has determinant k. We can get the same transformation by dividing each term by $\sqrt{k}$, as detailed above, but this will give an imaginary value for all terms. Because f is the composition of an odd number of these, that would mean that f would have imaginary terms, which contradicts the original assumption. For the sake of simplicity, though, we can divide each term by $\sqrt{-k}$, which will give determinant -1.

An inversion has the form $\frac{0z+1}{1z+0}$, which has determinant -1. Once again, this determinant cannot be made real without contradicting the assumption that f has real coefficients.

The determinant of f, then, is the product of 1 an arbitrary number times and -1 an odd number of times. Thus the determinant of f is -1, which is a contradiction. So clearly if f is a Möbius transformation with real coefficients then f preserves H. □

Defining the exact Möbius transformations that are closed over H reveals that these transformations are biholomorphic, an important property that will be used to define the isometries of H.

Theorem 3.5. *All Möbius transformations defined over H are biholomorphic.*

Proof. Complex polynomials are complex differentiable over their entire domain. By the quotient rule, rational functions are as well, except at points that cause singularities. Recall that Möbius transformations defined over H are of the form $f(z) = \frac{az+b}{cz+d}$ with $a, b, c,$and $d \in \mathbb{R}$, so clearly Möbius transformations are rational functions, and will have singularities when $cz = -d$ and thus $z = -\frac{d}{c}$. Because c and d are both real $-\frac{d}{c}$ will also be real. If z is in the upper half plane, by necessity z is not real, because the real line is not included in H; thus $z \neq \frac{-d}{c}$. Thus Möbius transformations over H are holomorphic.

The inverse of a Möbius transformation over H is another Möbius transformation over H, and is thus holomorphic. So every real Möbius transformation over H is biholomorphic. □

Although we will not exactly describe the Möbius transformations over D, it is easy to show that this property transfers to D as well.

Theorem 3.6. *Every Möbius transformation over D is biholomorphic.*

Proof. Because the disc model and the upper half plane model are isomorphically mapped by the biholomorphic Möbius transformation given earlier, the Möbius transformations from $D \to D$ are simply the transformations from $H \to H$ with the isomorphism applied. Because the composition of biholomorphic functions is biholomorphic, all Möbius transformations over D are biholomorphic. □

4. Isometries of Hyperbolic Space

In order to define the isometries of hyperbolic space, we will examine how some important theorems of complex analysis apply to the disk model of hyperbolic space. Because all of the various models of hyperbolic space are isomorphic, we can use the isometries we find for the disc to determine the isometries of the upper half plane.

Because all isometries are necessarily biholomorphic, we will begin by looking at the properties that holomorphic functions have on the disc. The first complex analysis theorem we will examine is the Schwarz Lemma, showing the strength of the condition that a function on the disk be holomorphic.

Theorem 4.1 (Schwarz Lemma)**.** *Let D be the unit disk defined in 2.1. Let $f : D \to D$ be a holomorphic function with $f(0) = 0$. Then $|f(z)| \leq |z| \forall z \in D$. Moreover, if $|f(z)| = |z|$ for some $z \in D$ or if $|f'(0)| = 1$ then $f(z) = az$ for some a with $|a| = 1$.*

Proof. Consider the function $g : D \to D$ defined as follows:

$$g(x) = \begin{cases} \frac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}$$

g is a holomorphic function on all of D, including the origin.

Let $\overline{D}_r = \{z \in \mathbb{C} | |z| \leq r\}$ for $0 < r < 1$. Because g is holomorphic, we can apply the maximum modulus principle, which implies that there exists $z_r \in \partial\overline{D}_r$ such that

$$|g(z)| \leq |g(z_r)|$$

Note that $|g(z_r)| = \frac{|f(z_r)|}{|z_r|}$. Because f is a function that maps to the unit disc, $|f(z_r)| \leq 1$. Because $z_r \in \partial\overline{D}_r$, $|z_r| = 1$, giving the inequality

$$|g(z)| \leq |g(z_r)| = \frac{|f(z_r)|}{|z_r|} \leq \frac{1}{r}$$

Suppose $z \neq 0$. Thus

$$|\frac{f(z)}{z}| \leq \frac{1}{r}$$

$0 < r < 1$, so

$$|\frac{f(z)}{z}| \leq 1$$

Implying

$$|f(z)| \leq |z|$$

Suppose $z = 0$. Then $|g(0)| \leq \frac{1}{r} < 1$ implies that $|f'(0)| \leq 1$.

Suppose we have the equality $|f(z_0)| = |z_0|$ for some $z_0 \in D$. Then $|g(z_0)| = 1$. Thus by the maximum modulus principle, g is constant on D with absolute value 1. Thus $a = g(z) = \frac{f(z)}{z}$, so $f(z) = az$.

□

The next theorem is a consequence of the Schwarz Lemma. Noting the similarity of the terms of the inequality to the metric on the disc, this theorem can be used to show that a holomorphic function mapping the disc onto itself cannot increase distances.

Theorem 4.2 (Schwarz-Pick Theorem)**.** *Let $f : D \to D$ be a holomorphic function. Then for every $z_1, z_2 \in D$,*

$$|\frac{f(z_1) - f(z_2)}{1 - f(z_1)f(z_2)}| \leq |\frac{z_1 - z_2}{1 - z_1\overline{z_2}}|$$

and

$$|\frac{f'(z)}{1 - |f(z)|^2}| \leq \frac{1}{1 - |z|^2}$$

Proof. Let $g(z) = \frac{z - z_0}{\overline{z_0}z - 1}$. Not that g is a function from D to D. Fix z_1 and define the following Möbius transformations:

$$M(z) = \frac{z_1 - z}{1 - z_1 z}$$

and

$$\Phi(z) = \frac{f(z_1) - z}{1 - \overline{f(z_1)}z}$$

$M(z_1) = 0$ and M is invertible, so

$$\Phi(f(M^{-1}(0))) = 0$$

Using the Schwarz Lemma shows that

$$|\Phi(f(M^{-1}))| \leq |z|$$

Applying this function to $M(z)$ gives

$$|\Phi(f(z))| \leq |M(z)|$$

And so

$$|\frac{f(z_1) - f(z)}{1 - \overline{f(z_1)}f(z)}| \leq |\frac{z_1 - z}{1 - z_1 z}|$$

This satisfies the first part of the theorem.

Clearly,

$$\frac{|\frac{f(z_1)-f(z)}{z_1-z}|}{|1-\overline{f(z_1)}f(z)|} \leq \frac{1}{|1-z_1 z|}$$

Taking the limit as z approaches z_1 we get that

$$\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2}$$

□

Theorem 4.3. *If f is a holomorphic function on D and z_1 and z_2 are points in D, the hyperbolic distance from $f(z_1)$ and $f(z_2)$ is less than or equal to that from z_1 to z_2.*

Proof. We will now prove more explicitly the results of this theorem relating to distances between points.

For a holomorphic function $f : D \to D$ and $z_1, z_2 \in D$, the distance between between $f(z_1)$ and $f(z_2)$ is equal to the infimum over all paths $f(\gamma)$ from $f(z_1)$ to $f(z_2)$ of $\int_0^1 ||(f(\gamma))'(t)||dt$. This will be written

$$\inf f(z_1) \to f_(z_2) \int_0^1 ||(f(\gamma))'(t)||dt = \inf_{f(z_1)\to f(z_2)} \int_0^1 ||(f \circ \gamma)'(t)dt$$

By the Chain Rule and then the metric in D, we get

$$= \inf_{f(z_1)\to f(z_2)} \int_0^1 f'(\gamma(t))\gamma'(t)|| = \inf_{f(z_1)\to f(z_2)} \int_0^1 \frac{2}{1-|f(\gamma(t))|^2} \cdot \sqrt{dzd\overline{z}(f'(\gamma(t))\gamma'(t))}dt$$

Recall that $dzd\overline{z} = dx^2 + dy^2$. In this case, dx is the real part of the value, and dy is the imaginary part, so we can substitute to get the following:

$$= \inf_{f(z_1)\to f(z_2)} \int_0^1 \frac{2}{1-|f(\gamma(t))|^2} \cdot \sqrt{(\text{Re}(f'(\gamma(t))\gamma'(t)))^2 + (\text{Im}(f'(\gamma(t))\gamma'(t)))^2}dt$$

Because $|f(\gamma(t))|$ will always be greater or equal to $\text{Re}(f(\gamma(t)))$ and to $\text{Im}(f(\gamma(t)))$, we can remove it from both the real and imaginary parts.

$$\leq \inf_{f(z_1)\to f(z_2)} \int_0^1 \frac{2}{1-|f(\gamma(t))|^2} \cdot \sqrt{(|f'(\gamma(t))|^2(\text{Re}(\gamma'(t)))^2 + (\text{Im}(\gamma'(t)))^2)}dt$$

Because f maps to and from the unit disc, $|f(\gamma(t))|$ will be less than or equal to 1 for any t. In addition, since all points are in the unit disc, by definition for every $t \in D$, $\sqrt{(\text{Re}(\gamma'(t)))^2 + (\text{Im}(\gamma'(t)))^2} \leq 1$. Thus we get

$$\leq \inf_{f(z_1)\to f(z_2)} \int_0^1 \frac{2}{1-|f(\gamma(t))|^2} \cdot \sqrt{(\Re(\gamma'(t)))^2 + (\text{Im}(\gamma'(t)))^2}dt \leq \inf_{f(z_1)\to f(z_2)} \int_0^1 \frac{2}{1-|f(\gamma(t))|^2}$$

By the final Schwarz-Pick inequality we can see that

$$\leq \inf_{f(z_1)\to f(z_2)} \int_0^1 \frac{2}{1-|\gamma(t)|^2}$$

Thus $d(f(z_1), f(z_2)) \leq d(z_1, z_2)$. □

Theorem 4.4. *If f is biholomorphic on D, f is an isometry of D.*

Proof. Let z_1 and z_2 be points in D. Because f is holomorphic, the distance between z_1 and z_2 is less than or equal than the distance between $f(z_1)$ and $f(z_2)$; thus we have that

$$d(f(z_1), f(z_2)) \leq d(z_1, z_2)$$

Since f is biholomorphic, f^{-1} is also holomorphic. Thus

$$d(f^{-1}(f(z_1)), f^{-1}(f(z_2))) = d(z_1, z_2) \leq d(f(z_1), f(z_2))$$

This implies that $d(z_1, z_2) = d(f(z_1), f(z_2))$, so f is an isometry. □

Remark 4.5. Because as we have shown earlier Möbius transformations over the disc are biholomorphic, this shows that all Möbius transformations from D to D are isometries of D. Now all that remains is to prove the converse.

Lemma 4.6. *If f is an isometry of D, then f is biholomorphic.*

Proof. Let f be an isometry of D. Then for any $z_1, z_2 \in D$, $d(z_1, z_2) = d(f(z_1), f(z_2))$. Using the same notation as above, this means that

$$\inf_{z_1 \to z_2} \int_0^1 ||\gamma'(t)||dt = \inf_{f(z_1) \to f(z_2)} \int_0^1 ||(f \circ \gamma)'(t)||dt$$

This requires that $f'(\gamma(t))$ exists; if it does not, then f cannot be an isometry. Thus f is holomorphic.

Because f is an isometry, f is injective. Because $d(z_1, z_2) = d(f(z_1), f(z_2))$ for any $z_1, z_2 \in D$, f is surjective. The function f is thus bijective, and so f is invertible. Thus $f^{-1}(f(z_1)) = z_1$ and $f^{-1}(f(z_2)) = z_2$. Because $d(z_1, z_2) = d(f(z_1), f(z_2))$, f^{-1} is clearly an isometry as well, and is thus holomorphic. Thus f is biholomorphic. □

Theorem 4.7. *If f is an isometry, f is a Möbius transformation.*

Proof. Let $f : D \to D$ be an isometry. Thus f is a biholomorphism. Compose f with some Möbius transformation g such that $f(g(0)) = 0$. Thus $|f(g(0)) = |0|$, so by the Schwarz Lemma$(f \circ g)(z) = az$ For some a such that $|a| = 1$. Clearly $f = ag^{-1}(z)$. g is a Möbius transformation on D, so g^{-1} is as well, and thus $f(z) = ag^{-1}(z)$ is a Möbius transformation. Thus the isometries of D are the Möbius transformations from $D \to D$. □

Remark 4.8. Because H and D are isomorphic, this property will hold for H as well; that is, the isometry group of H is the set of Möbius transformations from H to H. Thus the isometries of H are $PSL(2, \mathbb{R})$.

For the sake of simplicity, I will now turn to H to examine the geodesics of hyperbolic space. To determine the geodesics of D, simply apply the mapping from H to D listed above.

5. Geodesics in Hyperbolic Space

Definition 5.1. In hyperbolic space, and in other Reimannian manifolds, a geodesic is a distance minimizing path from one point to another.

Remark 5.2. The specific geodesic through a point with a specific tangent at that point is the solution to a second order ordinary differential equation. The exact details of the geodesic equation are beyond the scope of this paper. Nonetheless, because solutions to ODEs are unique given initial conditions, we can conclude that for a given point and a given tangent at that point, the geodesic will be unique.

Theorem 5.3. *The vertical line through the origin perpendicular to the real axis is a geodesic in H.*

Proof. Let z be a point on the y-axis. We will attempt to determine the geodesic through z with a vertical tangent at z.

Assume for the sake of contradiction that there exists a geodesic that is not simply the line $l = \{z \in H = x + iy | x = 0\}$. An example is shown in Figure 1.

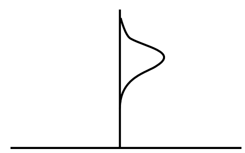

FIGURE 1

The map given by the reflection of the hyperbolic plane across the y-axis is an isomorphism, so the reflection of this geodesic across the axis would have the same length. Since geodesics in H are distance-reducing paths, this reflection would also be a geodesic, with the same initial point and tangent vector.

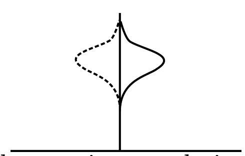

FIGURE 2. If the solid curve is a geodesic, the dotted curve must be as well.

This contradicts the uniqueness of geodesics, so the only geodesic through z with a vertical tangent vector is the a vertical line. □

To define the other geodesics in the hyperbolic plane, we simply look at the possible results when Möbius transformations are applied to this line. A function is an isometry of the hyperbolic plane if and only if it is a Möbius transformation, so by examining how every possible Möbious transformation affects this line, we can determine all of the geodesics of the hyperbolic plane.

To do this we need to describe more properties of Mobius transformations. Recall that Möbius transformations consist of compositions of the following types of maps:

a) Scalings- maps of the form $z \mapsto kz$ for some $k \in \mathbb{R}$
b) Translations - maps of the form $z \mapsto z + k$ for some $k \in \mathbb{R}$.
c) Inversions- maps of the form $z \mapsto \frac{1}{z}$

The result when l is scaled is fairly straightforward- it is simply l. The result when l is translated by k is another line, perpendicular to the real axis, $l_k = \{z \in \mathbb{C} = x + iy | x = k\}$. Inversions, however, are more complicated.

Theorem 5.4. *Inversions behave in the following manner:*

a) *The vertical line through the origin is mapped to itself.*
b) *Lines not through the origin are mapped to circles through the origin.*
c) *Circles through the origin map to lines not through the origin.*
d) *Circles not through the origin are mapped to circles not through the origin.*

Proof. Let f be the inversion transformation; thus $f(z) = \frac{1}{z}$.

The vertical line through the origin in $\mathbb{C}$ takes the form $z = ci$ with $c \in \mathbb{C}$. Thus $f(z) = \frac{1}{ci} = \frac{-i}{c}$. For every $c \in \mathbb{R}$, $\frac{-1}{c} \in \mathbb{R}$, so $f(z)$ is also in this vertical line. Thus the vertical line through the origin maps to itself.

a) A complex line through the origin is of the form $z = x + iy$ where $x, y \in \mathbb{R}$ and satisfy $ax + by = c$ for fixed $a, b, c \in \mathbb{R}$. Let $f(z) = w = u + vi$. Thus

$$z = \frac{1}{w} = \frac{1}{u + vi} = \frac{u - vi}{u^2 + v^2}$$

Recall that $z = x + iy$. So

$$x = \frac{u}{u^2 + v^2} \text{ and } y = -\frac{v}{u^2 + v^2}$$

Thus

$$\frac{au}{u^2 + v^2} + \frac{-bv}{u^2 + v^2} = c$$

This can be rewritten as

$$u^2 + v^2 - \frac{a}{c}u + \frac{b}{c}v = 0$$

which is the equation of a circle passing through the origin in $\mathbb{C}$.

b) Note that the inversion function is its own inverse- that is, $f(f(z)) = z$, because $\frac{1}{(\frac{1}{z})} = z$. Thus, because lines not through the origin were mapped to circles through the origin, circles through the origin will be mapped back to lines not through the origin.

c) A circle not passing through the origin consist of points $z = x + iy$ where $x, y \in \mathbb{R}$ and $x^2 + y^2 + ax + by = c$ for a, b, c fixed in $\mathbb{R}$ and $c \neq 0$. The image of z under f is of the form $w = u + iv$ such that $f(z) = w$; that is, $\frac{1}{z} = w$

As above,

$$x = \frac{u}{u^2 + v^2} \text{ and } y = -\frac{v}{u^2 + v^2}$$

Combining this with the expression for a circle gives us that

$$(\frac{u}{u^2 + v^2})^2 + (-\frac{v}{u^2 + v^2})^2 + \frac{au}{u^2 + v^2} - \frac{bv}{u^2 + v^2} = c$$

We simplify to get

$$\frac{u^2 + v^2}{(u^2 + v^2)^2} + \frac{au - bv}{u^2 + v^2} = c$$

So

$$\frac{1 + au + bv}{u^2 + v^2} = c$$

Clearing the denominator of the fraction gives

$$1 + au + bv = c(u^2 + v^2)$$

Dividing both sides by c gets us that

$$\frac{1}{c} + \frac{au}{c} - \frac{bv}{c} = u^2 + v^2$$

This equation can be easily modified into a more recognizable form:

$$u^2 + v^2 - \frac{av}{c} + \frac{bv}{c} = \frac{1}{c}$$

This is the equation for another circle not passing through the origin.

□

Theorem 5.5. *Translations take vertical lines to other vertical lines, and circles to other circles whose centers have the same real value.*

Proof. Translations are of the form $z \mapsto z + k$, with k a real number. Clearly this moves every point in a vertical line to the point $x + k + qi$, which is another vertical line shifted by k.

A transformation of a circle functions exactly the same way- k is added to every point, so the whole curve is shifted by k parallel to the real axis. □

Theorem 5.6. *Scales map vertical lines to other vertical lines, and circles to other circles.*

Proof. Vertical lines are of the form $x + yi$ for x fixed. Scales map $z \mapsto kz$, so the line is mapped to $kx + kyi$. This is another vertical line, meeting the real axis at kx instead of k.

Circles are of the form $z = x + iy$ where $x^2 + y^2 + ax + by = c$. This is mapped to $kx + kyi$. $x^2 + y^2 + ax + by = c$, so $k^2x^2 + k^2y^2 + k^2ax + k^2by = k^2c$. Because k^2a, k^2b, and k^2c are real, these points form another circle. □

Lemma 5.7. *In the upper half plane, Möbius transformations map the vertical line through the origin to other vertical lines and to half-circles orthogonal to the real line.*

Proof. Let l denote the vertical line through the origin. Möbius transformations are compositions of translations, scales, and inversions, so we will look at each in turn. Translations of l will produce other vertical lines intersecting the real line at any $k \in \mathbb{R}$. Scales will not change any vertical line.

Inverting l will yield l, but inverting a translation of l will produce a circle through the origin. l and all translations of l meet the real line perpendicularly, so these circles must be orthogonal to the real line. The upper half plane doesn't include the real axis or anything below it, however, so these 'circles' take the form of half-circles, centered on the real axis. All possible half-circles orthogonal to the real axis can be reached through Möbius transformations of l, because these half-circles can be shifted along the real axis by translations, and their radius changed by scales.

Inverting these half-circles will give more half-circles orthogonal to the real axis, if the circles are not through the origin, or it will produce more vertical lines perpendicular to the real axis. Clearly, the only possible results when Möbius translations are applied to l are vertical lines perpendicular to the real axis, and half-circles centered on the real axis. □

Theorem 5.8. *The geodesics of H are lines perpendicular to the real line, and half-circles orthogonal to the real line.*

Proof. We have shown that the vertical line through the origin is a geodesic of H, and that Möbius transformations are the isometries of H. All geodesics are isometric, so the geodesics of H are all constructions isometric to the vertical line through the origin. That is to say, the geodesics of the upper half plane are all of the possible results when a Möbius transformation is applied to this vertical line. As we have just shown, these results are lines perpendicular to the real line and half-circles orthogonal to the real line. □

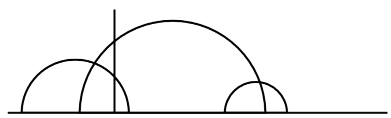

FIGURE 3. Hyperbolic Lines

When we picture the upper half plane as half of the Reimann sphere, these geodesics make much more sense. Lines intersecting the real axis at only one point can be mapped conformally to half-circles meeting the real axis at two points because the lines can be considered to also meet the real axis at the point at infinity.

Moreover, we can consider these geodesics as half-circles along the Riemann sphere. The vertical lines are simply half-circles that pass through the point at infinity, while the half-circles on the upper half plane are clearly half-circles on the Riemann sphere.

6. Parallel Lines in Hyperbolic Space

A defining characteristic of hyperbolic geometry is its modified version of the parallel postulate, which can be derived from these geodesics.

The definition of parallel hyperbolic lines is drawn from one of the definitions of parallel lines in two-dimensional Euclidean space, namely that lines that have no intersection are parallel.

Definition 6.1. Two lines in H are said to be parallel if they are disjoint.

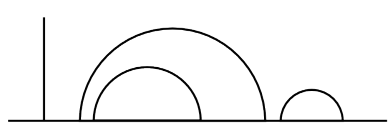

FIGURE 4. Examples of Parallel Hyperbolic Lines

Theorem 6.2. *Let l be a hyperbolic line in H and p be a point in H not contained in l. There are at least two lines through p parallel to H.*

Proof. First suppose that l is a vertical line. Because $p \notin l$, the vertical line k through p will be disjoint with l, and thus is parallel to l.

Now take some $x \in \mathbb{R}$ such that x lies between K and L on the real axis. Because $\text{Re}(x) \neq \text{Re}(p)$, there exists some circle A centered on the real axis such that A passes through both x and p, as shown in the following figure.

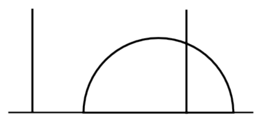

FIGURE 5. Hyperbolic lines parallel to a vertical line

A is necessarily disjoint from l. There are infinitely many values for x that will produce this result, so there are in fact infinitely many hyperbolic lines parallel to l through p.

Now suppose that l is a half-circle centered on the real axis. Let B be the half-circle concentric to l that passes through p. Concentric circles are disjoint, so B is one line through p parallel to l.

Again, let x be a point on $\mathbb{R}$ between l and B. Let C be the half-circle centered on $\mathbb{R}$ that passes through x and p. C and l are necessarily disjoint, so C is another line through p parallel to l.

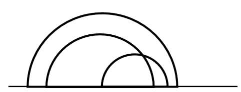

FIGURE 6. Hyperbolic lines parallel to a half-circle

Because there are infinitely many such values for x, there are again infinitely many lines through p parallel to l. □

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