REU 2013: THE COHOMOLOGY OF GROUPS

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1. Introduction

This will be an informal set of notes to be used as a crib sheet in my talks. We are taking up where Inna left off. She took as her motivation the classification of extensions

$$0 \longrightarrow K \xrightarrow{\subset} E \xrightarrow{q} Q \longrightarrow 1.$$

Here K stands for kernel and Q stands for quotient. She assumed that K is abelian (= commutative), so she wrote its product as $+$ with identity element 0. She did not assume that Q is abelian, so she wrote its product as $\cdot$ with identity element 1. That explains the joke notation with 0 at the left and 1 at the right.

The sequence is exact, which means that K is a normal subgroup of E with quotient group Q. She chose a set theoretic section t of q and found that

$$g \cdot k = t(g)kt(g)^{-1}$$

defines a well-defined action of Q on K. It is convenient to require $t(1) = 1$. That doesn't change anything seriously, but it leads to a slightly more standard and convenient description of the classification, which she described by saying that isomorphism classes of extensions of Q by K are in bijective correspondence to elements of $H^2(Q;K)$.

She did not tell you about $H^i(Q;K)$ for any other integers i. This looks kind of bizarre. I'm going to tell you about H^0 and H^1 and then I'll remind you how Inna described H^2. Then I'll begin to try to tell you the whole story and use it to desribe how to partially solve a topological problem: What finite groups can act freely on a sphere S^n?

We are going to change notation. We do not want to think of prescribed roles for Q and K, so those letters should be changed. We replace Q by G. Thus G is a fixed given group, G standing for group. We replace K by M, M standing for module. It is assumed to have an action by G. This means that M is an abelian group and for each element $g \in G$ there is a given isomorphism $g \colon M \longrightarrow M$. We write gm for $g(m)$. We require that $1m = m$ and $g(hm) = (gh)m$ for all $m \in M$ and all $g, h \in G$. A fancy way of saying this is that we are given a group homomorphism $G \longrightarrow \mathrm{Aut}(M)$, where $\mathrm{Aut}(M)$ denotes the group under composition of automorphisms of the abelian group M.

2. Rings, modules, and group actions

I shall try not to introduce too much terminology, but I cannot help but explain how such group actions fit into the general context of rings and modules. A ring R is an abelian group with an associative bilinear function $R \times R \longrightarrow R$, called multiplication. We require it to have a unit element $1 \in R$ such that $1r = r = r1$

for all $r \in R$. A left R-module M is an abelian group M with a bilinear map $R \times M \longrightarrow M$. It must be unital and associative in the sense that $1m$ and $r(sm) = (rs)m$ for all $m \in M$. Note that $r0 = 0$ for all $r \in R$. Right modules are defined similarly. For example, if R is a field, then an R-module is called a vector space over R. For another example, we have the ring $\mathbb{Z}$, and an abelian group is exactly the same thing as a $\mathbb{Z}$-module.

For a set S, the free abelian group generated by S is the set of finite linear combinations $\sum n_i s_i$, where the n_i are integers and the s_i are the elements of S. The word "finite" here means that we require $n_i = 0$ for all but finitely many of the indices i. We add elements in the obvious way:

$$(\sum m_i s_i) + (\sum n_i s_i) = \sum (m_i + n_i) s_i.$$

We write $\mathbb{Z}[S]$ for this abelian group. We can generalize this construction from the ring $\mathbb{Z}$ to an arbitrary ring R to define the free R-module $R[S]$ generated by the set S. Its elements are the finite linear combinations $\sum r_i s_i$, where now $r_i \in R$. We require a bilinear map $R \times R[S] \longrightarrow R[S]$, and that is given by

$$r(\sum_i r_i s_i) = \sum_i (r r_i) s_i.$$

If we have two R-modules M and N, we have the notion of a map $\phi\colon M \longrightarrow N$ of R-modules. It is just a homomorphism $\phi\colon M \longrightarrow N$ of abelian groups such that $\phi(rm) = r\phi(m)$ for all $r \in R$ and $m \in M$. When R is a field, ϕ is a linear map (linear transformation in the old days). We let $\mathrm{Hom}_R(M, N)$ denote the abelian group of maps of R-modules $M \longrightarrow N$, with pointwise addition:

$$(\phi + \psi)(m) = \phi(m) + \psi(m).$$

The zero homomorphism is the R-map that sends all m to 0. If R is commutative, so that $rs = sr$ for all $r, s \in R$, then $\mathrm{Hom}_R(M, N)$ is an R-module with the action of R given by $(r\phi)(m) = \phi(rm) = r\phi(m)$. This fails to give an R-module structure if R is not commutative, since $r\phi$ need not be a map of R-modules: $(r\phi)(sm) = r\phi(sm) = rs\phi(m)$, which is not $s(r\phi)(m) = sr\phi(m)$.

Observe that M can be identified with $\mathrm{Hom}_R(R, M)$ for any R-module M. The element $m \in M$ can be thought of as the unique map $\phi_m\colon R \longrightarrow M$ of R-modules that sends 1 to m. This is a special case of the following important property of the construction $R[S]$, as we see by noting that R can be identified with $R[*]$, where $*$ is the one-point set. Let $Fun(S, M)$ denote the of functions $S \longrightarrow M$. It is an abelian group under pointwise addition, and in fact it is an R-module with pointwise action by R; R need not be commutative for that to be true.

Proposition 2.1. *There is a canonical isomorphism of abelian groups*

$$\mathrm{Hom}_R(R[S], M) \cong Fun(S, M).$$

If R is commutative, it is an isomorphism of R-modules.

Proof. We have the inclusion $\eta\colon S \longrightarrow R[S]$ of sets that sends s to $1 \cdot s$. A homomorphism $\phi\colon R[S] \longrightarrow M$ of R-modules restricts to the function $f = \phi \circ \eta$, and $r\phi$ restricts to rf for $r \in R$. A function $f\colon S \longrightarrow M$ extends uniquely by R-linearity to a homomorphism ϕ such that $f = \phi \circ \eta$. $\square$

The *group ring* of G is the free abelian group $\mathbb{Z}[G]$, where we regard G as a set. Its multiplication is induced by the multiplication of G:

$$(\sum_i m_i g_i)(\sum_j n_j h_j) = \sum_{i,j} (m_i n_j) g_i h_j.$$

This makes sense since each $g_i h_j$ is an element of G. Formally, given $g \in G$, its coefficient on the right is the sum of the products $m_i n_j$ indexed on those pairs (i,j) such that $g_i h_j = g$. We are extending the multiplication $G \times G \longrightarrow G$ to $\mathbb{Z}[G]$ by bilinearity. Again, we can generalize to define the group ring $R[G]$ for a ring R, but in practice that is only used when R is commutative.

We conclude that G-modules are exactly the same thing as $\mathbb{Z}[G]$-modules. If G acts on an abelian group M, then $\mathbb{Z}[G]$ acts by

$$(\sum_i n_i g_i) m = \sum_i n_i (g_i m).$$

More generally, we can think of $R[G]$-modules as R-modules with actions by G.

If we have two G-modules M and N, we may abbreviate notation by defining

$$\mathrm{Hom}_G(M,N) = \mathrm{Hom}_{\mathbb{Z}[G]}(M,N).$$

Its elements are the homomorphisms $\phi\colon M \longrightarrow N$ of abelian groups such that that $\phi(gm) = g\phi(m)$ for all $m \in M$ and $g \in G$.

3. Invariant submodules and $H^0(G;M)$

We say that a G-module M is *trivial* if $gm = m$ for all $g \in G$ and $m \in M$. For a general G-module M, we define the *invariant submodule* M^G to be the set of all $m \in M$ such that $gm = m$ for all $g \in G$. It is the largest trivial G-module contained in M. We agree to regard $\mathbb{Z}$ as a trivial G-module. We claim that

$$M^G = \mathrm{Hom}_G(\mathbb{Z}, M).$$

This is sloppy, since we really mean that the two are so canonically isomorphic that we may as well identify them. We start out with the identification $M = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, M)$. Then $\phi_m(gn) = g\phi_m(n)$ for all $n \in \mathbb{Z}$ if and only if $\phi_m(g1) = g\phi_m(1)$. Since $g1 = 1$, this means that

$$m = \phi_m(1) = \phi_m(g1) = g\phi_m(1) = gm.$$

We use the identification of M with $\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], M)$ to reexpress this. Define the augmentation $\varepsilon\colon \mathbb{Z}[G] \longrightarrow \mathbb{Z}$ by $\varepsilon(g) = 1$ for all $g \in G$. It is a homomorphism of rings. A left ideal in a ring R is a sub R-module of R. Right ideals and two-sided ideals are defined similarly. For example, the only ideals in $\mathbb{Z}$ are (n), the integer multiples of a fixed integer n. The kernel of a ring homomorphism $R \longrightarrow S$ is a two-sided ideal in R. The kernel IG of ε is called the augmentation ideal of G. It consists of all linear combinations of the elements $g - 1$ for $g \in G$ and we have the exact sequence of $\mathbb{Z}[G]$-modules

$$0 \longrightarrow IG \longrightarrow \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0. \tag{3.1}$$

We may identify M^G with the subgroup of those $\phi \in \mathrm{Hom}_G(\mathbb{Z}[G], M)$ such that $\phi(IG) = 0$, that is, $\phi(g-1) = 0$ for all $g \in G$.

Now let us return to cohomology. Just to get it out of the way, we define $H^i(G;M) = 0$ for negative integers i. We then define

$$H^0(G;M) = M^G = \mathrm{Hom}_G(\mathbb{Z}, M).$$

4. Crossed homomorphisms and $H^1(G;M)$

This seems satisfactory, but it bears no obvious relationship with Inna's $H^2(G;M)$. We approach that by first defining $H^1(G;M)$.

Definition 4.1. Define a crossed homomorphism $f\colon G \longrightarrow M$ to be a function f such that

$$f(gh) = gf(h) + f(g) \quad \text{for all} \quad g, h \in G. \tag{4.2}$$

Observe that $f(1) = 0$ since $f(1) = f(1) + f(1)$. If G acts trivially on M, then f is just a homomorphism. Add crossed homomorphisms pointwise,

$$(f + f')(g) = f(g) + f'(g).$$

Define $Z^1(G;M)$ to be the abelian group of crossed homomorphisms. For $m \in M$, define a crossed homomorphism $f_m\colon G \longrightarrow M$ by $f_m(g) = gm - m$. It is called a *principal* crossed homomorphism. Define $B^1(G;M)$ to be the subgroup of $Z^1(G;M)$ whose elements are the principal crossed homomorphisms. Define the first cohomology group of G with coefficients in M to be

$$H^1(G;M) = Z^1(G;M)/B^1(G;M).$$

We leave it as an exercise to verify that the definitions are well defined: $f + f'$ and f_m are crossed homomorphisms and the sets asserted to be abelian groups under pointwise addition are in fact abelian groups.

We have a pleasant interpretation of $H^1(G;M)$. To see this, recall that we have the canonical *split extension*

$$0 \longrightarrow M \xrightarrow{i} M \rtimes G \xrightarrow{q} G \longrightarrow 1.$$

As a set, it is $M \times G$. The multiplication is given by

$$(m, g)(n, h) = (m + gn, gh) \quad \text{for} \quad m, n \in M \quad \text{and} \quad g, h \in G.$$

The homomorphisms i and q are given by $i(m) = (m, 1)$ and $q(m, g) = g$. Define $t\colon G \longrightarrow M \rtimes G$ by $t(g) = (0, g)$. Then t is a homomorphism such that $q \circ t = \mathrm{id}$.

An automorphism of a group G is an isomorphism $G \longrightarrow G$. An automorphism is said to be inner if it sends h to ghg^{-1} for some $g \in G$; we denote this conjugation automorphism by c_g. For any automorphism α, $\alpha c_g \alpha^{-1} = c_g$. The set $Aut(G)$ of automorphisms of G is a group under composition, the set $Inn(G)$ of inner automorphisms is a normal subgroup, and we define $Out(G)$ to be the quotient group $Aut(G)/Inn(G)$.

Define an automorphism of the extension $M \rtimes G$ to be an automorphism of the group $M \rtimes G$ that induces the identity homomorphism on both the subgroup M and the quotient group G. We agree to write $Aut(G;M)$ for the group under composition of such automorphisms. For example, every inner homomorphism of the group $M \rtimes G$ that is induced by an element $m \in M$ is an automorphism α_m of the extension $M \rtimes G$. We write $Inn(G;M)$ for the subgroup of such inner automorphisms in $Aut(G;M)$. We define $Out(G;M)$ to be the quotient group $Aut(G;M)/Inn(G;M)$.

Proposition 4.3. *The groups $Z^1(G;M)$ and $Aut(G;M)$ are isomorphic, and the isomorphism restricts to an isomorphism $B^1(G;M) \cong Inn(G;M)$. Therefore*

$$H^1(G;M) \cong Out(G;M).$$

Proof. Any $\alpha \in Aut(G;M)$ can be expressed in the form

$$\alpha(m,g) = (m + f_\alpha(g), g)$$

for some function $f_\alpha \colon G \longrightarrow M$. We leave the rest of the proof as an exercise.

Exercise 4.4. The condition that α is an automorphism is equivalent to the condition that f_α satisfies (4.2). Composition of automorphisms corresponds to addition of associated functions. The inner automorphism α_m corresponds to the principal crossed homomorphism f_m. □

We can give alternative descriptions of $H^1(G;M)$ using $\varepsilon \colon \mathbb{Z}[G] \longrightarrow \mathbb{Z}$. The first is another example of extending functions to homomorphisms by linearity. The second reinterprets crossed homomorphisms in terms of ordinary homomorphisms, giving something that is perhaps more conceptual.

Proposition 4.5. *A crossed homomorphism $f \colon G \longrightarrow M$ determines and is determined by a homomorphism of abelian groups $\phi \colon \mathbb{Z}[G] \longrightarrow M$ such that*

$$\phi(xy) = x\phi(y) + \phi(x)\varepsilon(y) \quad \textit{for all} \quad x, y \in \mathbb{Z}[G].$$

The principal crossed homomorphisms are the ϕ_m, $m \in M$ specified by

$$\phi_m(x) = xm - m\varepsilon(x) \quad \textit{for all} \quad x \in \mathbb{Z}[G].$$

Proof. We have the inclusion $\eta \colon G \longrightarrow \mathbb{Z}[G]$. The restriction $\phi \circ \eta = f$ of such a homomorphism ϕ is a crossed homomorphism f. A crossed homomorphism f extends by linearity to such a homomorphism ϕ. □

The exact sequence (3.1) is split exact as a sequence of abelian groups. There is a splitting $\mathbb{Z} \longrightarrow \mathbb{Z}[G]$ that sends n to $n \cdot 1$, and there is a projection $\pi \colon \mathbb{Z}[G] \longrightarrow IG$ that sends x to $x - \varepsilon(x)$. Its restriction to G is a crossed homomorphism $p \colon G \longrightarrow IG$ since

$$\pi(gh) = gh - 1 = g(h-1) + (g-1) = g\pi(h) + \pi(g) \quad \text{for all} \quad g, h \in G.$$

Proposition 4.6. *Restriction of crossed homomorphisms $\phi \colon \mathbb{Z}[G] \longrightarrow M$ to IG defines an isomorphism*

$$Z^1(G;M) \cong \mathrm{Hom}_{\mathbb{Z}[G]}(IG, M).$$

Restriction along the inclusion $i \colon IG \longrightarrow \mathbb{Z}[G]$ defines a homomorphism of abelian groups

$$i^* \colon M = \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], M) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G]}(IG, M),$$

and

$$B^1(G;M) \cong Im(i^*).$$

Proof. The restriction of ϕ to IG is a G-map $\psi \colon IG \longrightarrow M$. Conversely, given a G-map $\psi \colon IG \longrightarrow M$, the composite $\phi = \psi \circ \pi$ is a crossed homomorphism that restricts to ψ on IG. The restriction ψ_m of ϕ_m for $m \in M$ sends x to xm for $x \in IG$, which is the elementwise version of our description of $B^1(G;M)$. □

5. Group extensions and $H^2(G;M)$

Inna followed the historical progression of wanting to classify extensions and finding the definition of $H^2(G;M)$ as the solution to that problem. In modern expositions, one generally first defines $H^2(G;M)$ and then sees the classification of extensions as an application. We follow that route since the change of focus may help motivate one definition of $H^n(G;M)$ for $n \geq 3$. However, the motivation for the definition may be less clear this way.

Definition 5.1. Define $Z^2(G;M)$ to be the set of functions $f\colon G\times G \longrightarrow M$ such that

$$f(g,1) = 0 = f(1,g) \quad \text{for all} \quad g \in G \tag{5.2}$$

and

$$gf(h,k) - f(gh,k) + f(g,hk) - f(g,h) = 0 \quad \text{for all} \quad g,h,k \in G. \tag{5.3}$$

Here the normalization condition (5.2) is optional but the cocycle condition (5.3) is essential. The elements of $Z^2(G;M)$ are called 2-cocycles. With addition defined pointwise, $(f+f')(g,h) = f(g,h) + f'(g,h)$, $Z^2(G;M)$ is an abelian group. For a function $e\colon G \longrightarrow M$ such that $e(1) = 0$, define a 2-cocycle $\delta(e)$ by

$$\delta(e)(g,h) = ge(h) - e(gh) + e(g) \quad \text{for all} \quad g,h \in G.$$

Define $B^2(G;M)$ to be the subgroup of $Z^2(G;M)$ whose elements are these $\delta(e)$. The elements of $B^2(G;M)$ are called 2-coboundaries. Define the second cohomology group of G with coefficients in M by

$$H^2(G;M) = Z^2(G;M)/B^2(G;M).$$

We again leave the verification that everything is well-defined as an exercise. Now we review Inna's work, but taking this definition as the starting point. Define $\mathrm{Ext}(G;M)$ to be the set of equivalence classes of extensions

$$0 \longrightarrow M \xrightarrow{\subset} E \xrightarrow{q} G \longrightarrow 1$$

of G by M. Remember that we can define an action of G on M by choosing a section (not a homomorphism in general) $t\colon G \longrightarrow E$; we take sections to be functions such that $t(1) = 1$ and $q \circ t = \mathrm{id}$. We then set $gm = t(g)mt(g)^{-1}$. This makes sense since $q(gm) = 1$, so that gm is in M. We require the action of G defined this way to coincide with the given action of G on M. We say that two such extensions E and E' are equivalent if there is an isomorphism ξ making the following diagram commute.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\subset"] \arrow[d, equal] & E \arrow[r, "q"] \arrow[d, "\xi"] & G \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & M \arrow[r, "\subset"'] & E' \arrow[r, "q"'] & G \arrow[r] & 1
\end{tikzcd}$$

Theorem 5.4. *The sets* $\mathrm{Ext}(G;M)$ *and* $H^2(G;M)$ *are in canonical bijective correspondence.*

Proof. Suppose there is a multiplication on the set $E = M \times G$ that agrees with addition on $M = M \times \{1\} \subset M \times G$ and that agrees with the multiplication on

G under the projection $M \times G \longrightarrow G$ onto the second coordinate. We require the identity element to be $(0,1)$. Then the product must be given by the formula

$$(5.5)\qquad (m,g)(n,h) = (m+n+f(g,h), gh) \quad \text{for} \quad m,n \in M \quad \text{and} \quad g,h \in G,$$

where $f\colon G \times G \longrightarrow M$ is some function. It might be clearer to take $m = 0 = n$ when defining f in (5.5) and to observe that the general formula follows. Since $(0,1)$ is the identity element, we see that (5.2) holds by taking (m,g) or (n,h) to be $(0,1)$. A calculation, left as an exercise, shows that the associativity of the product together with the requirement that the action of G on M induced by the product must agree with the given action on M forces the cocycle condition (5.3) to hold. Conversely, if we start with a function f satisfying (5.2) and (5.3), then (5.5) gives a product on $M \times G$ that makes it an extension of G by M. That observation is what first motivated the cocycle condition! Thus we have constructed an extension, let us call it $M \times_f G$, from a 2-cocycle f. Implicitly, we have a fixed chosen canonical section t, namely $t(g) = (0,g)$.

Observe that the semi-direct product $M \rtimes G$ is $M \times_0 G$, where $0\colon G \times G \longrightarrow M$ is the zero 2-cocycle.

In general, given an extension E and a section t, $t(g)t(h)$ and $t(gh)$ both map to gh in G, hence we can define a function $f\colon G \times G \longrightarrow M$ by

$$f(g,h) = t(g)t(h)t(gh)^{-1}.$$

Historically, this function is called a *factor set* determined by E. The section t fixes the identification of the set $M \times G$ with E that sends (m,g) to $mt(g)$. This gives the set $M \times G$ an induced product. It clearly satisfies

$$(m,1)(n,1) = (m+n,1), \quad (m,1)(0,g) = (m,g), \quad \text{and} \quad (0,g)(m,1) = (gm,g),$$

the last because $t(g)m = (gm)t(g)$. Therefore the product is entirely determined by the value of $(0,g)(0,h)$, which by definition is $(f(g,h), gh)$. Therefore f must be a 2-cocycle and E is equivalent to the canonical extension $M \times_f G$.

If an extension E with section t and an extension E' with section t' have the same factor set f, then f is a 2-cocycle and E and E' are both equivalent to $M \times_f G$ and therefore to each other, by the argument just given.

Finally, suppose that a given extension E has sections t and t' and corresponding factor sets f and f'. Since $qt = \mathrm{id} = qt'$, we have $t'(g) = e(g)t(g)$ for some function $e\colon G \longrightarrow M$. We have $e(1) = 0$ since $t(1) = 1 = t'(1)$ and 0 maps to 1 under the inclusion $M \subset E$ (since the inclusion is a group homomorphism). We can calculate the factor set f' determined by t' in terms of e and the factor set determined by t'. Leaving the details as an exercise, we find that

$$f'(g,h) = f(g,h) + \delta(e)(g,h).$$

Moreover, f and any function e with $e(1) = 0$ determine a section t' from t by $t'(g) = e(g)t(g)$. Therefore the factor sets of a given extension E run through all representatives of a single cohomology class in $H^2(G;M)$. Since every extension is isomorphic to one of the canonical form $M \times_f G$, we conclude that equivalence classes of extensions are in bijective correspondence with elements of $H^2(G;M)$. □

6. The brute force definition of $H^n(G;M)$

As usual, we start with functions and later extend by linearity to homomorphisms. We define $C^n(G;M)$ to be the set of functions f from the nth cartesian power; G^n of G to M such that $f(g_1, \cdots, g_n) = 0$ if any $g_i = 1$; this last condition means that f is normalized; one can work instead with unnormalized functions, but working with normalized functions is more convenient. We call these functions *n-cochains*. They form an abelian group under pointwise addition. We let $C^0(G;M) = M = \mathrm{Hom}(\mathbb{Z}, M)$.

Definition 6.1. For an $n-1$-cochain $f\colon G^{n-1} \longrightarrow M$, $n \geq 1$, define an n-cochain $\delta(f)\colon G^n \longrightarrow M$ by

$$\begin{aligned}\delta(f)(g_1, \ldots, g_n) &= g_1 f(g_2, \cdots, g_n) \\ &\quad + \sum_{1 \leq i \leq n-1} (-1)^i f(g_1, \cdots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \cdots, g_n) \\ &\quad + (-1)^n f(g_1, \cdots, g_{n-1}).\end{aligned}$$

Define the group $Z^n(G;M)$ of n-cocycles to be the subgroup of $C^n(G;M)$ consisting of those n-cochains f such that $\delta(f) = 0$. Define the group $B^n(G;M)$ of n-coboundaries to be the subgroup of $Z^n(G;M)$ consisting of the elements $\delta(f)$, where f is an $(n-1)$-cochain. For $n \geq 0$, define the nth cohomology group of G with coefficients in M to be

$$H^n(G;M) = Z^n(G;M)/B^n(G;M).$$

I warned you that this first definition would be brutal, and it is. Verification that it is well-defined requires a little work, the essential point being that $\delta \circ \delta = 0$, but you will trust me on that for the moment. Note that the definition makes sense even when $n = 0$, with $B^0(G;M) = 0$: if $f(1) = m$, then $\delta(f)(g) = gm - m$. A 1-cocycle is exactly the same thing as a crossed homomorphism. Now a quick comparison with our previous definitions shows that $H^n(G;M)$ as just defined agrees with $H^n(G;M)$ as previously defined for $n = 0$, 1, and 2. For the moment, that is perhaps justification enough, but this is a terribly non-conceptual definition. Historically, I believe this is close to the original version: conceptual justification of ad hoc concrete definitions often takes time to evolve.