EXERCISES: THE COHOMOLOGY OF GROUPS

J.P. MAY

1. Let \( G \) be finite. Define the norm \( N \in \mathbb{Z}[G] \) by \( N = \sum_{g \in G} g \).
   (a) Prove that \( (\mathbb{Z}[G])^G = \{ nN | n \in \mathbb{Z} \} \).
   (b) Prove that \( IG \) is the kernel of \( N: \mathbb{Z}[G] \to \mathbb{Z}[G] \), that is \( \{ x | Nx = 0 \} \).

2. Let \( G \) be any infinite group. Prove that \( (\mathbb{Z}[G])^G = 0 \).

3. Let \( M \) be a trivial \( G \)-module. Show that \( H^1(G; M) \) is the group of homomorphisms \( G \to M \), which can be identified with \( \text{Hom}_{\mathbb{Z}}(G/[G,G], M) \), where \([G,G] \) is the commutator subgroup. In particular \( H^1(G; \mathbb{Z}) = 0 \) for any finite group \( G \).

4. We prove in class that \( \text{Ext}(G; M) \) is isomorphic as sets to the abelian group \( H^2(G; M) \). The following steps show how one can describe the resulting group structure on \( \text{Ext}(G; M) \).
   (a) (Pullback) Let \( \alpha: G' \to G \) be a homorphism of groups. Given the bottom extension \( E \) of \( G \) by \( M \) in the following diagram, construct the top extension and a homomorphism \( \beta \) such that the following diagram commutes:

   \[
   \begin{array}{c}
   0 \to M \overset{\subset}{\to} E' \overset{q}{\to} G' \to 1 \\
   0 \to M \overset{\subset}{\to} E \overset{q}{\to} G \to 1.
   \end{array}
   \]

   Here \( G' \) acts through \( \alpha \) on \( M \): \( g'm = \alpha(g')m \). Show that the construction gives a function \( \alpha^*: \text{Ext}(G; M) \to \text{Ext}(G'; M) \).
   (b) (Pushout) Let \( \gamma: M \to M' \) be a map of \( G \)-modules. Given the top extension \( E \) of \( G \) by \( M \) in the following diagram, construct the bottom extension of \( G \) by \( M' \) and a homomorphism \( \beta \) such that the following diagram commutes:

   \[
   \begin{array}{c}
   0 \to M \overset{\subset}{\to} E \overset{q}{\to} G \to 1 \\
   0 \to M' \overset{\subset}{\to} E' \overset{q}{\to} G \to 1.
   \end{array}
   \]

   Show that the construction gives a function \( \gamma_*: \text{Ext}(G; M) \to \text{Ext}(G; M') \).
   (c) With the notations of (a) and (b), show that \( \gamma_*(\alpha^*E) \cong \alpha^*(\gamma_*E) \).
   (d) (Product) Show that cartesian product of groups gives a product function

   \[ \text{Ext}(G; M) \times \text{Ext}(H; N) \to \text{Ext}(G \times H; M \oplus N) \]

   for groups \( G \) and \( H \), a \( G \)-module \( M \), and an \( H \)-module \( N \); here we have identified \( M \oplus N \) with \( M \times N \).
(e) Let $\Delta: G \to G \times G$ be the diagonal map and $\nabla: M \oplus M \to M$ the ‘fold’ map, which is the identity on each copy of $M$. Define the Baer sum

$$+ : \Ext(G; M) \times \Ext(G; M) \to \Ext(G, M)$$

to be the composite of $\times$ and $\nabla^* \Delta^* = \Delta^* \nabla^*$. Prove that this sum coincides under our bijection with the sum on $H^2(G; M)$. (Hint: do this in three steps, defining $\alpha^*, \gamma^*$ and $\times$ on $H^2$ groups and using factor sets to compare constructions).