EXERCISES: THE COHOMOLOGY OF GROUPS

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1. Let G be finite. Define the *norm* $N \in \mathbb{Z}[G]$ by $N = \sum_{g \in G} g$.

(a) Prove that $(\mathbb{Z}[G])^G$ is $\mathbb{Z}N = \{nN | n \in \mathbb{Z}\}$.
(b) Prove that IG is the kernel of $N\colon \mathbb{Z}[G] \longrightarrow \mathbb{Z}[G]$, that is $\{x | Nx = 0\}$.

2. Let G be any infinite group. Prove that $(\mathbb{Z}[G])^G = 0$.

3. Let M be a trivial G-module. Show that $H^1(G; M)$ is the group of homomorphisms $G \longrightarrow M$, which can be identified with $\mathrm{Hom}_{\mathbb{Z}}(G/[G,G], M)$, where $[G,G]$ is the commutator subgroup. In particular $H^1(G; \mathbb{Z}) = 0$ for any finite group G.

4. We prove in class that $\mathrm{Ext}(G; M)$ is isomorphic as sets to the abelian group $H^2(G; M)$. The following steps show how one one can describe the resulting group structure on $\mathrm{Ext}(G; M)$.

(a) (Pullback) Let $\alpha\colon G' \longrightarrow G$ be a homorphism of groups. Given the bottom extension E of G by M in the following diagram, construct the top extension and a homomorphism β such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\subset} & E' & \xrightarrow{q} & G' & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} & & \\ 0 & \longrightarrow & M & \xrightarrow[\subset]{} & E & \xrightarrow[q]{} & G & \longrightarrow & 1. \end{array}$$

Here G' acts through α on M: $g'm = \alpha(g')m$. Show that the construction gives a function $\alpha^*\colon \mathrm{Ext}(G; M) \longrightarrow \mathrm{Ext}(G'; M)$.

(b) (Pushout) Let $\gamma\colon M \longrightarrow M'$ be a map of G-modules. Given the top extension E of G by M in the following diagram, construct the bottom extension of G by M' and a homomorphism β such that the following digram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\subset} & E & \xrightarrow{q} & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} & & \| & & \\ 0 & \longrightarrow & M' & \xrightarrow[\subset]{} & E' & \xrightarrow[q]{} & G & \longrightarrow & 1. \end{array}$$

Show that the construction gives a function $\gamma_*\colon \mathrm{Ext}(G; M) \longrightarrow \mathrm{Ext}(G; M')$.

(c) With the notations of (a) and (b), show that $\gamma_*(\alpha^* E) \cong \alpha^*(\gamma_* E)$.

(d) (Product) Show that cartesian product of groups gives a product function

$$\mathrm{Ext}(G; M) \times \mathrm{Ext}(H; N) \longrightarrow \mathrm{Ext}(G \times H; M \oplus N).$$

for groups G and H, a G-module M, and an H-module N; here we have identified $M \oplus N$ with $M \times N$.

(e) Let $\Delta\colon G \longrightarrow G \times G$ be the diagonal map and $\nabla\colon M \oplus M \longrightarrow M$ the 'fold' map, which is the identity on each copy of M. Define the *Baer sum*

$$+\colon \operatorname{Ext}(G;M) \times \operatorname{Ext}(G;M) \longrightarrow \operatorname{Ext}(G,M)$$

to be the composite of $\times$ and $\nabla_*\Delta^* = \Delta^*\nabla_*$. Prove that this sum coincides under our bijection with the sum on $H^2(G;M)$. (Hint: do this in three steps, defining α^*, γ_* and $\times$ on H^2 groups and using factor sets to compare constructions).