SEMI-SIMPLE LIE ALGEBRAS AND THEIR REPRESENTATIONS

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Abstract. This paper presents an overview of the representations of Lie algebras, particularly semi-simple Lie algebras, with a view towards theoretical physics. We proceed from the relationship between Lie algebras and Lie groups to more abstract characterizations of Lie groups, give basic definitions of different properties that may characterize Lie groups, and then prove results about root systems, proceeding towards a proof of the Theorem of Highest Weight. After proving this theorem, we show how it may be applied to the the Lorentz group SO(1,3) through representations of $\mathfrak{su}(2)$.

Contents

1. An introduction to Lie algebras

1.1. **Lie groups and Lie algebras: some beginning motivation.** When studying physics, understanding the symmetries of a system is often key to understanding the system itself. The symmetries of a system will always form a group (with the natural group action being the composition of two symmetries); furthermore, since these symmetries often vary continuously, they will form a Lie group: a topological group which is also a manifold, such that the group action and the action of taking inverses are smooth functions.

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When studying manifolds, we often pass to studying the tangent space, which is simpler since it is linear; since Lie groups are manifolds, we can apply this technique as well. The question then arises: if the tangent space is an infinitesimal approximation to the Lie group at a point, how can we infinitesimally approximate the group action at this point? The study of Lie algebras gives us an answer to this question.

The study of matrix groups is well-understood, so a first step in understanding Lie groups is often representing the group as a group of matrices; for an example consider the following:

Example 1.1. *Recall the circle group, which may be written* $\{e^{i\theta}|\theta \in [0, 2\pi)\}$*—we know we may also represent this group as the set of matrices*

$$\left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} |\theta \in [0, 2\pi) \right\}.$$

All the Lie groups that we give as examples in this paper will be finite-dimensional matrix groups. Thus when motivating the study of Lie algebras, let us assume our group G is a matrix group. We know we may write the tangent space at the identity to this group as

$$\mathfrak{g} = \{c'(0)|c : \mathbb{R} \to G \text{ is a curve with } c(0) = \mathbb{1}_G \\ \text{that is smooth as a function into matrices}\}.$$

Note that the maps $t \mapsto c(kt)$ and $t \mapsto c(t)b(t)$ shows that $\mathfrak{g}$ is closed under addition and multiplication by scalars. Now, we wish to consider how the group action in G affects this space. Consider the map $t \mapsto gc(t)g^{-1}$ for $g \in G$; differentiating this operation shows that this space is closed under the operation $[X, Y] = XY - YX$. This bracket operation provides the structure for the Lie algebra, which leads to an understanding of the group action; the bracket form $[X, Y]$ is the differential form of the operation $x^{-1}yx$.

The above provides a way to obtain a Lie algebra from a Lie group; it is also sometimes possible to reconstruct the Lie group given the Lie algebra. It can be shown that for such matrix groups, the map $X \mapsto e^{tX}$ (where this is just the standard matrix exponential) is a map from $\mathfrak{g}$ to G, and for certain groups, this is map is a bijection. Hence understanding the Lie algebras of these matrix groups can give useful information about the group itself. In this paper, we do not discuss how this exponential map forms such a correspond; the curious reader is encouraged to consult [2].

In this paper, we first study general Lie algebras and then move towards studying the representations of semi-simple Lie algebras, which are Lie algebras with no Abelian sub-algebras (i.e. sub-algebras consisting of elements X such that $[X, Y] = 0$ for all elements Y in the Lie algebra), with the motivation being that these semi-simple Lie algebras are the most common Lie algebras in areas of theoretical physics. We conclude by demonstrating how these Lie algebras arise in physics by briefly sketching how the study of semi-simple Lie algebras relates to the Lorentz group and its corresponding Lie algebra.

1.2. **Definitions and examples.** A **Lie algebra** is $\mathfrak{g}$ is a vector space over a field $\mathbb{K}$, equipped with an operation $[\cdot,\cdot] : L \times L \to L$ satisfying the following properties: for all $X, Y, Z \in \mathfrak{g}$, and $a, b \in \mathbb{K}$.

- **linearity:** $[aX + bY, Z] = a[X, Z] + b[Y, Z]$
- **anti-symmetry**: $[X, Y] = -[Y, X]$
- **the Jacobi identity:**$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$

Example 1.2. *If $\mathfrak{g}$ is an algebra, then $\mathfrak{g}$ becomes a Lie algebra under the bracket operation defined by the commutator:*

$$[X, Y] = XY - YX.$$

In particular for a vector space K, $\mathrm{End}_{\mathbb{K}}(V)$ becomes a Lie algebra under this operation.

As we might expect, a **Lie algebra homomorphism** is a map between Lie algebras that respects the vector space and bracket structure: if $\varphi : \mathfrak{g} \to \mathfrak{g}'$ is a Lie algebra homomorphism, then $\varphi([X, aY + Z]) = [\varphi(X), a\varphi(Y) + \varphi(Z)]$ for all $X, Y, Z \in \mathfrak{g}$ and all a in the underlying field. If $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$ are subsets of a Lie algebra $\mathfrak{g}$, then we take $[\mathfrak{a}, \mathfrak{b}] = \mathrm{span}\{\{[a, b]|\{a \in \mathfrak{a}, b \in \mathfrak{b}\}$. A **subalgebra** $\mathfrak{h} \subseteq \mathfrak{g}$ is a vector subspace of $\mathfrak{g}$ such that $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$; if in addition $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$, then $\mathfrak{h}$ is an **ideal**. $\mathfrak{g}$ is said to be **Abelian** if $[\mathfrak{g}, \mathfrak{g}] = 0$; this definition makes sense since the bracket operation often arises as the commutator of operators[1]. We may also define in the usual way a direct sum of Lie algebras.

If $\mathfrak{g}$ is a vector space over a field $\mathbb{K}$, and V is a vector space over the same field, and $\pi : \mathfrak{g} \to \mathrm{End}_{\mathbb{K}}V$ is a homomorphism, then π is a **representation** of $\mathfrak{g}$ on V. If $\pi : \mathfrak{g} \to \mathrm{End}_{\mathbb{K}}V$ and $\pi' : \mathfrak{g} \to \mathrm{End}_{\mathbb{K}}V'$ are two representations, these representations are equivalent if there exists an isomorphism $E : V \to V'$ such that $E\pi(X) = \pi'(X)E$ for all $X \in \mathfrak{g}$. An invariant subspace for a representation is a subspace U such that $\pi(X)U \subseteq U$ for all $X \in \mathfrak{g}$.

We can consider the representation of $\mathfrak{g}$ on itself denoted ad and defined by $\mathrm{ad}(X)(Y) = [X, Y]$; that this is a linear map that respects the bracket structure follows from the linearity of the bracket operation and the Jacobi identity. Furthermore, for any subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, the adjoint representation restricted to $\mathfrak{h}$ is a representation of $\mathfrak{h}$ on $\mathfrak{g}$.

Example 1.3. *The **Heisenberg Lie algebra** is a Lie algebra H over $\mathbb{C}$ generated by elements $\{P_1, \dots, P_n, Q_1, \dots, Q_n, C\}$ satisfying*

$$[P_i, P_j] = [Q_i, Q_j] = [P_i, C] = [Q_j, C] = [C, C] = 0 \tag{1.4}$$

$$[P_i, Q_j] = \delta_{ij}C. \tag{1.5}$$

(We use C since this denotes the center of the Lie algebra). If we take $C = i\hbar\mathbb{1}$, we may note that this models the quantum mechanical position and momentum operators, with P_i and Q_i being the momentum and position operators of the i-th coordinate, respectively.

In the case where $n = 1$, one representation of this is given by

$$\pi(aP + bQ + cC) = \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}.$$

[1]Indeed, when studying the universal enveloping algebra of a Lie group, we will see that every bracket operation of two elements can be regarded as a commutator of the two elements

1.3. **Characterizing properties of Lie algebras.**

Definition 1.6. Define recursively:

$$\mathfrak{g}^0 = \mathfrak{g}, \qquad \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}], \qquad \mathfrak{g}^{j+1} = [\mathfrak{g}^j, \mathfrak{g}^j].$$

This is the **commutator series** for $\mathfrak{g}$. Note that $\mathfrak{g}^j$ is an ideal for all j. $\mathfrak{g}$ is **solvable** if $\mathfrak{g}^j = 0$ for some j.

Definition 1.7. Similarly, define recursively

$$\mathfrak{g}_0 = \mathfrak{g}, \qquad \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}], \qquad \mathfrak{g}_{j+1} = [\mathfrak{g}, \mathfrak{g}_j].$$

This is the **lower central series** for $\mathfrak{g}$. Again note that $\mathfrak{g}_j$ is an ideal for all j. $\mathfrak{g}$ is **nilpotent** if $\mathfrak{g}_j = 0$ for some j.

Proposition 1.8. *A Lie algebra is solvable if it is nilpotent.*

Proof. Note that $\mathfrak{g}^j \subseteq \mathfrak{g}_j$ for all j. □

Proposition 1.9. *Any sub-algebra or quotient algebra of a solvable Lie algebra is solvable. Any sub-algebra or quotient Lie algebra of a nilpotent Lie algebra is nilpotent.*

Proof. If $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, then $\mathfrak{h}^k \subseteq \mathfrak{g}^k$, so if $\mathfrak{g}$ is solvable, $\mathfrak{h}$ is solvable. If $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is the projection to the quotient algebra, then $\pi(\mathfrak{g}^k) = \mathfrak{h}^k$, so $\mathfrak{g}$ solvable imples $\mathfrak{h}^k$ solvable. We may replace 'solvable' with 'nilpotent' in the above without changing anything else to obtain the same results for nilpotent Lie algebras. □

Definition 1.10. A finite-dimensional Lie algebra $\mathfrak{g}$ is **simple** if $\mathfrak{g}$ has no nonzero proper ideals.

Definition 1.11. A finite-dimensional Lie algebra $\mathfrak{g}$ is **semi-simple** if $\mathfrak{g}$ has no nonzero proper solvable ideals.

This paper focuses on finite-dimensional semi-simple Lie algebras, as many Lie algebras encountered in physics are of this type. The above definition does not give as much insight into the structure of a semi-simple Lie algebra as other characterizations do; we list those alternate characterizations below, after first introducing a bilinear form on Lie algebras, the **Killing form**:

Let $\mathfrak{g}$ be a a finite-dimensional Lie algebra over $\mathbb{R}$. For $X, Y \in \mathfrak{g}$, note that $\operatorname{ad} X \operatorname{ad} Y \in \operatorname{End}_V \mathfrak{g}$, and we can define

$$B(X, Y) = \operatorname{Tr}(\operatorname{ad} X, \operatorname{ad} Y).$$

Then B is a symmetric bilinear form which we call the Killing form. It is also associative, in the sense that $B([X, Y], Z) = B(X, [Y, Z])$ for all $X, Y, Z \in \mathfrak{g}$, which is clear from the fact that $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$.

Having introduced the Killing form, we may now list useful alternate characterizations of semi-simplicity:

Theorem 1.12. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. The following are equivalent:*

(a) $\mathfrak{g}$ is semi-simple.
*(b) its Killing form is non-degenerate (*Cartan's criterion for semi-simplicity*).*
(c) it is the direct sum of simple Lie algebras.
(d) if it has no Abelian sub-algebras.

We omit the proof of these statements. In many cases it is obvious that a Lie algebra is semi-simple from (*b*) or (*c*), and some texts (especially physics texts) often take (c) as the definition.

1.4. **A useful example: $\mathfrak{sl}(2,\mathbb{C})$.** One of the most commonly encountered Lie algebraz is

$$\mathfrak{sl}(n,\mathbb{C}) = X \in \mathfrak{gl}(n,\mathbb{C}) |\, \mathrm{Tr}(X) = 0\}$$

That this is a Lie algebra follows from the fact that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for all matrices A, B, so $\mathrm{Tr}([A,B]) = 0$ for any $A, B \in \mathfrak{gl}(n,\mathbb{C}) |\, \mathrm{Tr}(X) = 0$.

Remark 1.13. *$\mathfrak{sl}(2,\mathbb{C})$ has a basis given by*

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The study of the representations of $(2,\mathbb{C})$ has implications in classifying the structure of semi-simple Lie algebras. While we do not prove the major theorems about these representations found in e.g. [1], we do present a result that has important implications in our study. If we let φ be a representation of a Lie algebra $\mathfrak{g}$ on a vector space V, then φ is **completely reducible** if there exist invariant subspaces $U_1, \ldots, U_n$ such that $V = U_1 \oplus \ldots \oplus U_n$ and the restriction of φ to U_i is irreducible.

We have the following (which is unproven here).

Theorem 1.14. *If φ is a representation of a semi-simple Lie algebra on a finite dimensional vector space, then φ is completely reducible; in particular, if φ is a representation of $\mathfrak{sl}(2,\mathbb{C})$, then φ is completely reducible.*

We also have the following, which is specific to $\mathfrak{sl}(2,\mathbb{C})$:

Theorem 1.15. *With h the basis vector for $\mathfrak{sl}(2,\mathbb{C})$ above, $\varphi(h)$ is diagonalizable with all eigenvalues integers and the multiplicity of each eigenvalue k equal to the multiplicity of $-k$.*

For a proof of these results, see [1].

2. Nilpotent sub-algebras and diagonalization results

Our study of semi-simple Lie algebras begins with the following results, which allows us to understand how solvable Lie algebras can be understood to act jointly on a space:

Theorem 2.1 (Lie's theorem)**.** *Let $\mathfrak{g}$ be solvable, let $V \neq 0$ be a finite-dimensional vector space over $\mathbb{C}$, and let $\pi : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}} V$ be a representation. Then there is a simultaneous eigenvector $v \neq 0$ for all members of$\pi(\mathfrak{g})$. More generally, for V a vector space over a subfield $\mathbb{C} \subset \mathbb{C}$, and if all the eigenvalues of $\pi(X)$, $X \in \mathfrak{g}$ lie in K, then there exists such an eigenvector.*

Proof. Proceed by induction on the dimension n of $\mathfrak{g}$. If $n = 1$, then $\pi(\mathfrak{g})$ is one dimensional and the result is immediate.

Now, assume the result for Lie algebras of dimension $k < n$. Consider the commutator ideal $\mathfrak{h} = [\mathfrak{g}, \mathfrak{g}]$; this must be a proper ideal since otherwise the commutator series would never terminate. As $[\mathfrak{g}, \mathfrak{g}]$ is a proper subspace of $\mathfrak{g}$, it must have dimension less than n; we may therefore find a subspace $\mathfrak{h} \subset \mathfrak{g}$ of codimension 1 such that $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{h}$. Now, note that $[\mathfrak{g}, \mathfrak{h}] \subseteq [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{h}$, so $\mathfrak{h}$ is an ideal. As any subalgebra of a solvable Lie algebra is solvable, we have that $\mathfrak{h}$ is solvable, and since it has dimension $n-1$, we may apply the induction hypothesis to obtain a vector $v \in V$ such that v is a simultaneous eigenvector for all $H \in \mathfrak{h}$. We then have for each $H \in \mathfrak{h}$, $\pi(H)v = \lambda(H)v$, where $\lambda : \mathfrak{h} \to \mathbb{K}$ is a functional.

Now, choose nonzero $X \in \mathfrak{g}$ not in $\mathfrak{h}$, and define recursively:

$$e_{-1} = 0, \qquad e_0 = v, \qquad e_p = \pi(X)e_{p-1}$$

and let $E = \operatorname{span}\{e_0, e_1, \ldots\}$. Note that $\pi(X)E \subseteq E$, so we may regard $\pi(X)$ as an operator on E; then, since all the eigenvalues of $\pi(X)$ lie in K, we may chose an eigenvector $w \in E$ of $\pi(X)$.

Denote $E_p = \operatorname{span}\{e_0, e_1, \ldots, e_p\}$. We first show that $\pi(H)e_p \equiv \lambda(H)e_p \mod E_{p-1}$. We may proceed by induction on p. We chose $e_0 = v$ such that this condition holds. Now assume the result for p and let $H \in \mathfrak{h}$. We have

$$\begin{aligned} \pi(H)e_{p+1} &= \pi(H)\pi(X)e_p \\ &= \pi([H,X])e_p + \pi(X)\pi(H)e_p. \end{aligned}$$

Now, note that $\pi([H,X]) \in \mathfrak{h}$, so by induction we have $\pi([H,X])e_p \equiv \lambda([H,X])e_p \mod E_{p-1}$. We also have that $\pi(H)e_p \equiv \lambda(H)e_p \mod E_{p-1}$, so

$$\pi(X)\pi(H)e_p \equiv \pi(X)\lambda(H)e_p \mod \operatorname{span}\{E_{p-1}, \pi(X)E_{p-1}\}.$$

But

$$\operatorname{span}\{E_{p-1}, \pi(X)E_{p-1}\} = \operatorname{span}\{e_0, \ldots, e_{p-1}, \pi(X)e_0, \ldots, \pi(X)e_{p-1}\} = E_p,$$

so we have that

$$\begin{aligned} \pi(H)e_{p+1} &\equiv \lambda(H)e_p + \lambda(H)\pi(X)e_p \mod E_p \\ &\equiv \lambda([H,X])e_p + \lambda(H)\pi(X)e_p \mod E_p \\ &\equiv 0 + \lambda\pi(H)\pi(X)e_p \mod E_p \\ &\equiv \lambda\pi(H)e_{p+1} \mod E_p \end{aligned}$$

and so the result follows by induction.

Now, we may chose n such that $\{e_1, \ldots, e_n\}$ form a basis for E. Then as $\pi(H)e_p \equiv \lambda(H)e_p \mod E_{p-1}$, relative to this basis, $\pi(H)$ when considered as an operator on E has the form

$$\begin{pmatrix} \lambda(H) & & & \star \\ & \lambda(H) & & \\ \vdots & & \ddots & \\ 0 & & & \lambda(H) \end{pmatrix}$$

(i.e. is upper triangular with all diagonal entries $\lambda(H)$.) We see then that $\operatorname{Tr}(\pi(H)) = \lambda(H)\dim E$. Applying this result to $[H,X]$, we obtain that

$$\lambda[H,X]\dim E = \operatorname{Tr}\pi([H,X]) = \operatorname{Tr}[\pi(H), \pi(X)] = 0$$

(where here we have used the fact that $\operatorname{Tr}[A,B] = 0$ for any matrices A and B). Since $\dim E \neq 0$, we have that $\lambda[H,X] = 0$. We may use this to refine our previous result. Before, we had $\pi(H)e_p \equiv \lambda(H)e_p \mod E_{p-1}$; we wish to refine this to $\pi(H)e_p = \lambda(H)e_p$. Again, we proceed by induction, with the result true by definition for $p = 0$. Then using the inductive hypothesis, we have that

$$\begin{aligned}\pi(H)e_{p+1} &= \pi(H)\pi(X)e_p \\ &= \pi([H, X])e_p + \pi(X)\pi(H)e_p \\ &= \lambda([H, X])e_p + \lambda(H)\pi(X)e_p \\ &= \lambda(H)\pi(X)e_p = \lambda(H)e_{p+1}\end{aligned}$$

and we may conclude that $\pi(H)e_p = \lambda(H)e_p$ for all p. Then $\pi(H)e = \lambda(H)e$ for all $e \in E$ and in particular $\pi(H)w = \lambda(H)w$. Since then w is an eigenvector for all $H \in \mathfrak{h}$, and we chose w to be an eigenvector of X, we conclude that w is a simultaneous eigenvector for all $G \in \mathfrak{g}$.

□

Corollary 2.2. *Under the assumptions on $\mathfrak{g}, V, \pi$, and $\mathbb{K}$ as above, there exists a sequence of subspaces*

$$V = V_0 \supseteq V_1 \supseteq \ldots \supseteq V_m = 0$$

such that each V_i is stable under $\pi(\mathfrak{g})$ and $\dim V_i/V_{i+1} = 1$. This means that there exists a basis with respect to which all the matrices of $\pi(\mathfrak{g})$ are upper triangular. **??**

Proof. Proceed by induction on the dimension of V. If $m = 1$, the result is obvious. For $m > 1$, the above theorem tells us that we may find a simultaneous eigenvector v of $\pi(X)$ for all $X \in \mathfrak{g}$. Let U be the subspace spanned by this vector. Then since $\pi(\mathfrak{g})U = U$, π acts a representation $\tilde{\pi}$ on the quotient space V/U, which has dimension strictly less than V. By the inductive hypothesis we may thus find a sequence of subspaces

$$W = W_0 \supseteq V_1 \supseteq \ldots \supseteq W_m = 0$$

such that each W_i is stable under $\tilde{\pi}(\mathfrak{g})$ and $\dim W_i/W_{i+1} = 1$. Now, consider the corresponding sequence of subspaces in V, where each V_i has image W_i under the projection to the quotient space; we obtain the sequence

$$V = V_0 \supseteq V_1 \supseteq \ldots \supseteq V_m = 0$$

which satisfies the desired properties. Hence by induction, such a sequence may always be found.

Now, chose a vector $v_i \in V_{i-1}$ for $1 \leq i \leq m$ such that $\mathbb{K}v_i + V_i = V_{i-1}$; with respect to the basis $\{v_1, \ldots, v_m\}$, all matrices of $\pi(\mathfrak{g})$ are upper triangular. □

The following result allows us to relate the property of nilpotency of a Lie algebra and nilpotentcy of its adjoint representation:

Theorem 2.3. *A Lie algebra $\mathfrak{h}$ is nilpotent if and only if the Lie algebra* $\operatorname{ad} H$ *is nilpotent.*

Proof. Let $H \in \mathfrak{h}$. We have

$$[[...[H_{k+1}, H_k], H_{k-1}, \ldots, H_1] = \operatorname{ad}[...[H_{k+1}, H_k], ..., H_2](H_1)$$

so using the fact that

$$\operatorname{ad}[...[H_{k+1}, H_k], ..., H_2] = [\ldots[\operatorname{ad} H_{k+1}, \operatorname{ad} H_k], \ldots, \operatorname{ad} H_2]$$

shows us that if $\mathfrak{h}$ is nilpotent, then the left side is 0 for high enough k, and conversely, if $\operatorname{ad}\mathfrak{h}$ is nilpotent, then the right side is 0 for high enough k and hence $\mathfrak{h}$ is nilpotent. $\square$

In the future, these theorems will allow us to put matrices in our Lie algebras into useful forms.

3. Weight Spaces and Root spaces

3.1. **Weight Spaces.** Let $\mathfrak{h}$ be a finite-dimensional Lie algebra over $\mathbb{C}$, and let π be a representation of $\mathfrak{h}$ onto a complex vector space V. For such π and V, whenever α is in the dual $\mathfrak{h}^*$, let V_α be defined as

$$\{v \in V | (\pi(H) - \alpha(H)\mathbb{1})^n v = 0 \text{ for all } H \in \mathfrak{h} \text{ and some } n = n(H, v).\}$$

These look like generalized eigenspaces. Clearly, we may assume that $n \leq \dim V$.

Definition 3.1. If $V_\alpha \neq 0$, V_α is called a **generalized weight space** and α is a **weight.** Members of V_α are **generalized weight vectors**.

Proposition 3.2. *Suppose that $\mathfrak{h}$ is a nilpotent Lie algebra over $\mathbb{C}$ and that π is a representation of $\mathfrak{h}$ on a finite-dimensional complex vector space V. Then there are only finitely many generalized weights, each generalized weight space is stable under $\mathfrak{h}$, and V is the direct sum of all the generalized weight spaces.*

Proof. First , we will show each V_α is stable under $\mathfrak{h}$. For $H \in \mathfrak{h}$, let

$$V_{\alpha,H} = \{v \in V | (\pi(H) - \alpha(H)\mathbb{1})^n v = 0 \text{ for some } n = n(H, v).\}$$

We wish to show that $V_{\alpha,H}$ is invariant under $\pi(H)$; to do this, we use the nilpotency of $\mathfrak{h}$. Since $\mathfrak{h}$ is nilpotent, $\operatorname{ad} H$ is nilpotent as a linear map by Theorem 2.3. We may define

$$\mathfrak{h}_{(m)} = \{Y \in \mathfrak{h} | (\operatorname{ad} H)^m Y = 0\},$$

and the nilpotency guarantees that $\mathfrak{h} = \cup_{m=0}^d \mathfrak{h}_{(m)}$ (where d is the dimension of $\mathfrak{h}$). We wish to show that $\pi(Y)V_{\alpha,H} \subseteq V_{\alpha,H}$ for $Y \in \mathfrak{h}_{(m)}$ by induction on m.

For $m = 0$, we have that $\mathfrak{h}_{(0)} = 0$ and the result follows trivially. Now, assume the result is true for $m-1$. Let $Y \in \mathfrak{h}_{(m)}$, and note then that $[H, Y] \in \mathfrak{h}_{(m-1)}$. We have that

$$\begin{aligned}(\pi(H) - \alpha(H)\mathbb{1})\pi(Y) &= \pi(H)\pi(Y) - \alpha(H)\pi(Y) \\ &= \pi([H, Y]) + \pi(Y)\pi(H) - \alpha(H)\pi(Y) \\ &= \pi([H, Y]) + \pi(Y)(\pi(H) - \alpha(H)\mathbb{1}).\end{aligned}$$

Some computation shows that

$$\begin{aligned}&(\pi(H) - \alpha(H)\mathbb{1})^2\pi(Y) \\ &= (\pi(H) - \alpha(H)\mathbb{1})\pi(Y)(\pi(H) - \alpha(H)\mathbb{1}) + (\pi(H) - \alpha(H)\mathbb{1})\pi([H, Y]) \\ &= \pi(Y)(\pi(H) - \alpha(H)\mathbb{1})^2 + (\pi(H) - \alpha(H)\mathbb{1})\pi([H, Y]) + \pi([H, Y])(\pi(H) - \alpha(H)\mathbb{1}),\end{aligned}$$

and we see that for a given n,

$$\begin{aligned}(\pi(H) - \alpha(H)\mathbb{1})^n\pi(Y) &= \pi(Y)(\pi(H) - \alpha(H)\mathbb{1})^n \\ &\quad + \sum_{s=0}^{n-1}(\pi(H) - \alpha(H)\mathbb{1})^{n-s-1}\pi([H, Y])(\pi(H) - \alpha(H)\mathbb{1})^s.\end{aligned}$$

Now, recall that for $m \geq \dim V$, for $v \in V_{\alpha,H}$, $(\pi(H) - \alpha(H)\mathbb{1})^m v = 0$. In the above expression, take $n = 2\dim V$.Then for $v \in V_{\alpha,H}$, we have that

$$\begin{aligned}(\pi(H) - \alpha(H)\mathbb{1})^n \pi(Y)v &= \pi(Y)(\pi(H) - \alpha(H)\mathbb{1})^n v \\ &\quad + \sum_{s=0}^{n-1} (\pi(H) - \alpha(H)\mathbb{1})^{n-s-1}\pi([H,Y])(\pi(H) - \alpha(H)\mathbb{1})^s v \\ &= \sum_{s=0}^{n/2} (\pi(H) - \alpha(H)\mathbb{1})^{n-s-1}\pi([H,Y])(\pi(H) - \alpha(H)\mathbb{1})^s v\end{aligned}$$

(all other terms vanish). Now, consider each remaining term of the form

$$(\pi(H) - \alpha(H)\mathbb{1})^{n-s-1}\pi([H,Y])(\pi(H) - \alpha(H)\mathbb{1})^s v.$$

Note that $(\pi(H) - \alpha(H)\mathbb{1})^s v \in V_{\alpha,H}$, and since $[H,Y] \in \mathfrak{h}_{(m-1)}$, we have by the inductive hypothesis that $[H,Y]$ leaves $V_{\alpha,H}$ stable, so

$$\pi([H,Y])(\pi(H) - \alpha(H)\mathbb{1})^s v \in V_{\alpha,H}.$$

But then since $n - s - 1 \geq \dim V$, $(\pi(H) - \alpha(H)\mathbb{1})^{n-s-1}$ acts as 0 on $V_{\alpha,H}$, so we obtain that

$$\sum_{s=0}^{n/2} (\pi(H) - \alpha(H)\mathbb{1})^{n-s-1}\pi([H,Y])(\pi(H) - \alpha(H)\mathbb{1})^s v = 0.$$

The sum above then gives us

$$(\pi(H) - \alpha(H)\mathbb{1})^n \pi(Y)v = 0.$$

Thus $\pi(Y)v \in V_{\alpha,H}$, and we conclude that $V_{\alpha,H}$ is stable under $\pi(Y)$; then by induction, $\pi(Y)V_{\alpha,H} \subseteq V_{\alpha,H}$ for all $Y \in \mathfrak{h}$.

Now we are in a position to prove that V may be written as the direct sum of the weight spaces. First, chose a basis $H_1, \ldots, H_r$ for $\mathfrak{h}$. Since we work over $\mathbb{C}$, we know we may write V as the direct sum of the generalized eigenspaces of $\pi(H_1)$:

$$V = \bigoplus_{\lambda} V_{\lambda,H_1}$$

Now, note that we may regard each λ as $\mu(H_1)$ for some $\mu \in \mathfrak{h}^*$.

We may then write

$$V = \bigoplus_{\text{values of } \mu(H_1)} V_{\mu(H_1),H_1}$$

(this direct sum decomposition makes since there are only finitely many $\mu(H_1)$ such that $V_{\mu(H_1),H_1} \neq 0$). Now, note that for each $\mu \in \mathfrak{h}^*$, $V_{\mu(H_1),H_1}$ is invariant under $\pi(\mathfrak{h})$, so in particular is invariant under H_2; Thus we may further decompose V in the same way:

$$V = \bigoplus_{\text{values of } \mu(H_1)} \bigoplus_{\text{values of } \mu(H_2)} V_{\mu(H_1),H_1} \cap V_{\mu(H_1),H_2},$$

and so on, iterating to obtain

$$V = \bigoplus_{\text{values of } \mu(H_1) \ldots} \left(\bigcap_{j=1}^{r} V_{\mu(H_j),H_j} \right).$$

Each of these spaces is invariant under $\pi(\mathfrak{h})$. Now, by Corollary **??** (which is applicable since $\mathfrak{h}$ is nilpotent and hence solvable), there is some basis in which all $\pi(H_i)$ are upper triangular matrices. Then in this basis, for each H_i, $\pi(H_i)$ must have diagonal entries $\mu(H_i)$. Thus $\pi(\sum c_i H_i)$ must be a triangular matrix with all diagonal entries $\sum c_i \mu(H_i)$. If we define a linear functional α by $\alpha(\sum_{i=1}^r c_i H_i) = c_i \mu(H_i)$, we conclude that $\bigcap_{j=1}^r V_{\mu(H_j),H_j} = V_\alpha$. Hence $V = \bigoplus_\alpha V_\alpha$. □

3.2. **Root spaces.** Now, let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ a subalgebra, and recall that the adjoint representation defined by $\operatorname{ad} H(X) = [H, X]$ is a representation of $\mathfrak{h}$ on $\mathfrak{g}$. The weights of $\operatorname{ad}\mathfrak{h}$ relative to this representation are called **roots**. There are many statements we can make about the roots.

Proposition 3.3. *If $\mathfrak{g}$ is any finite-dimensional Lie algebra over $\mathbb{C}$ and $\mathfrak{h}$ is a nilpotent Lie subalgebra, then the generalized weight spaces of $\mathfrak{g}$ relative to $\operatorname{ad}_{\mathfrak{g}} \mathfrak{h}$ satisfy*
(a) $\mathfrak{g} = \bigoplus \mathfrak{g}_\alpha$
(b) $\mathfrak{h} \subseteq \mathfrak{g}_0$
(c) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$. *If $\alpha+\beta$ is not a root, then this statement says that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$.*

Proof. (a) This is by the above proposition.
(b) Since $\mathfrak{h}$ is nilpotent, for all $H \in \mathfrak{h}$, there is an n such that $(\operatorname{ad} H)^n = 0$ on H, which immediately gives that $\mathfrak{h} \subseteq \mathfrak{g}_0$.
(c) Let $A \in \mathfrak{g}_\alpha$ and $B \in \mathfrak{g}_\beta$. Let $H \in \mathfrak{h}$. Note that we have

$$\begin{aligned}
&(\operatorname{ad}(H) - \alpha(H) - \beta(H))[X, Y] \\
&\quad = [H, [X, Y]] - \alpha(H)[X, Y] - \beta(H)[X, Y] \\
&\quad = [[H, X], Y] + [X, [H, Y]] - \alpha(H)[X, Y] - \beta(H)[X, Y] \\
&\quad = [(\operatorname{ad} H - \alpha(H))X, Y] + [X, (\operatorname{ad} H - \beta(H))Y].
\end{aligned}$$

Induction gives that

$$\begin{aligned}
&(\operatorname{ad}(H) - \alpha(H) - \beta(H))^n[X, Y] \\
&\quad = \sum_{k=1}^{n} \binom{k}{n} \left[(\operatorname{ad} H - \alpha(H))^k X, (\operatorname{ad} H - \beta(H))^{n-k} Y \right]
\end{aligned}$$

(this is just the Binomial Theorem and the calculations above). Then chosing $n > 2 \dim V$ guarantees that either $k > \dim V$ or $n - k > \dim V$, so $[(\operatorname{ad} H - \alpha(H))^k X, (\operatorname{ad} H - \beta(H))^{n-k} Y] = 0$. Thus $[A, B] \in \mathfrak{g}_{\alpha+\beta}$.

□

Note that (c) above implies that $\mathfrak{g}_0$ is a subalgebra. If $\mathfrak{g}_0 = \mathfrak{h}$, we call $\mathfrak{h}$ a **Cartan subalgebra**. It can be shown ([1, p. 90]) that any finite-dimensional Lie algebra over $\mathbb{C}$ has a Cartan subalgebra.

The above definition of a Cartan subalgebra is hard to work with; the following characterization is more useful:

Proposition 3.4. *If $\mathfrak{g}$ is a finite-dimensional Lie algebra and $\mathfrak{h}$ is a nilpotent subalgebra, then $\mathfrak{h}$ is a Cartan subalgebra if and only if*

$$\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h}) = \{X \in \mathfrak{g} | [X, \mathfrak{h}] \subseteq \mathfrak{h}\}.$$

Proof. First note that if $\mathfrak{h}$ is any nilpotent subalgebra, we have that $\mathfrak{h} \subseteq N_{\mathfrak{g}}(\mathfrak{h}) \subseteq \mathfrak{g}_0$: the first of these inclusions holds because $\mathfrak{h}$ is a subalgebra, and the second holds because $\operatorname{ad} H^n X = \operatorname{ad} H^{n-1}[H, X]$, and $\operatorname{ad} H^{n-1}$ is 0 on $\mathfrak{h}$ for large enough n, since $\mathfrak{h}$ is nilpotent.

Now, assume that $\mathfrak{h}$ is not a Cartan subalgebra, so $\mathfrak{g}_0 \neq \mathfrak{h}$. Then $\mathfrak{g}_0/\mathfrak{h}$ is nonzero, and as the quotient of a solvable Lie algebra, is solvable. By Lie's theorem, there exists a nonzero $\tilde{X} + \mathfrak{h} \in \mathfrak{g}_0/\mathfrak{h}$ that is a simultaneous eigenvector of $\operatorname{ad} \mathfrak{h}$, and its eigenvalue has to be zero (as otherwise X would not be in $\mathfrak{g}_0$). But then for $H \in \mathfrak{h}$, $[H, X] \in \mathfrak{h}$, so X is not in $\mathfrak{h}$ but is in $N_{\mathfrak{g}}(\mathfrak{h})$. Thus $\mathfrak{h} \neq N_{\mathfrak{g}}(\mathfrak{h})$.

Conversely, note that if $\mathfrak{h}$ is a Cartan subalgebra, $\mathfrak{g}_0 = \mathfrak{h}$, so $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$. □

In particular, semi-simple Lie algebras have easily understood Cartan sub-algebras.

Proposition 3.5. *If $\mathfrak{g}$ is a complex finite-dimensional semi-simple Lie algebra and $\mathfrak{h}$ is a Cartan subalgebra then $\mathfrak{h}$ is Abelian.*

Proof. Since $\mathfrak{h}$ is nilpotent, $\operatorname{ad} \mathfrak{h}$ is solvable and we may apply Lie's theorem to obtain a basis for $\mathfrak{g}$ in which $\operatorname{ad} \mathfrak{h}$ is simultaneously triangular. Now, note that for triangular matrices, $\operatorname{Tr}(ABC) = \operatorname{Tr}(BAC)$, so for any $H_1, H_2, H \in \mathfrak{h}$, we have

$$B(\operatorname{ad}[H_1, H_2], \operatorname{ad} H) = 0$$

Now, let $H \in \mathfrak{h}$ and $X \in \mathfrak{g}_\alpha$ where α is any nonzero weight. By the above proposition, may find a basis

$$\{G_{\alpha_1,1}, \ldots, G_{\alpha_1,n_1}, \ldots, G_{\alpha_m,1}, \ldots, G_{\alpha_m,n_m}\}$$

where each set $\{G_{\alpha_i,1}, \ldots, G_{\alpha_i,n_i}\}$ is a basis for $\mathfrak{g}_{\alpha_i}$, and the set of all such $G_{\alpha,i}$ forms a basis for $\mathfrak{g}$. Now, recall that for any $B \in \mathfrak{g}_\beta$,

$$\operatorname{ad} H(\operatorname{ad} XB) = \operatorname{ad} H([X, B]) \in g_{\alpha+\beta}.$$

In this basis, we see that $\operatorname{ad} H \operatorname{ad} X$ has all zeros on the diagonal, so $\operatorname{Tr}(\operatorname{ad} H \operatorname{ad} X) = 0$. In particular, for any $H_1, H_2 \in \mathfrak{h}$, $\operatorname{Tr}(\operatorname{ad}[H_1, H_2] \operatorname{ad} X) = 0$.

We thus have proven for any $X \in \mathfrak{g}_\alpha$ or $X \in \mathfrak{h}$, for any $H_1, H_2 \in H$, $\operatorname{Tr}(\operatorname{ad}[H_1, H_2] \operatorname{ad} X) = 0$. Since any $X \in \mathfrak{g}$ may be written as a linear combination of such elements and because trace is linear, we obtain that for any $H_1, H_2 \in \mathfrak{h}$, any $X \in \mathfrak{g}$,

$$B([[H_1, H_2], X]) = 0.$$

Since $\mathfrak{g}$ is semi-simple, the Killing form is non-degenerate; hence $[H_1, H_2] = 0$ for any $H_1, H_2 \in \mathfrak{h}$. □

We also have a partial converse:

Proposition 3.6. *If $\mathfrak{g}$ is a complex semi-simple Lie algebra and $\mathfrak{h}$ is a maximal Abelian sub-algebra such that $\operatorname{ad}_{\mathfrak{g}} \mathfrak{h}$ is diagonalizable, then $\mathfrak{h}$ is a Cartan subalgebra.*

Proof. $\mathfrak{h}$ is Abelian and hence nilpotent, so we may decompose $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha.$$

Then since $\operatorname{ad}_{\mathfrak{g}} \mathfrak{h}$ is simultaneously diagonalizable, we know we may write $\mathfrak{g}_0$ as $\mathfrak{g}_0 = \mathfrak{h} \bigoplus \tau$, where $[\mathfrak{h}, \tau] = 0$. Now, note that if τ is nonzero, then there exists $X \in \tau$, and $[\mathfrak{h} + \mathbb{C}X, \mathfrak{h} + \mathbb{C}X] = [\mathfrak{h}, \mathfrak{h}] + [\mathfrak{h}, \mathbb{C}X] + [X, X] = 0$. Hence $\mathfrak{h} + \mathbb{C}X$ is an Abelian subalgebra properly containing $\mathfrak{h}$, a contradiction. Hence $\tau = 0$ and $\mathfrak{h}$ is a Cartan subalgebra. □

We will show later that all maximal Abelian sub-algebras are diagonalizable, so we may leave out that assumption. Many texts take this as the definition of a Cartan subalgebra.

Furthermore, we have the following: (for a proof, see [1], pp. 92-93).

Proposition 3.7. *Any two Cartan sub-algebras of a finite-dimensional complex Lie algebra $\mathfrak{g}$ are conjugate by an inner automorphism of $\mathfrak{g}$.*

Corollary 3.8. *All Cartan sub-algebras $\mathfrak{h}$ of $\mathfrak{g}$ have the same dimension; this is the **rank** of $\mathfrak{g}$.*

4. Structure of Root Systems

Again, let $\mathfrak{g}$ be a complex semi-simple Lie algebra, B its Killing form, and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Recall that if we define for $\alpha \in \mathfrak{h}^*$

$$\{X \in \mathfrak{g} | (\operatorname{ad} H - \alpha(H)\mathbb{1})^n X = 0 \text{for some n } \in \mathbb{N}, \text{for all H} \in \mathfrak{h}\}$$

and denote Δ to be the set of nonzero roots (i.e. nonzero $\alpha \in \mathfrak{h}^*$ such that $\mathfrak{g}_\alpha \neq 0$), then we may write

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

We will soon show that we may take $n = 1$, which simplifies this expression significantly.

There are many statements we can make about these roots:

Proposition 4.1. *(a) If α and β are in $\Delta \cup \{0\}$ and $\alpha + \beta \neq 0$, then $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$.*
(b) If $\alpha \in \Delta \cup \{0\}$, then $B(\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}) \neq 0$.
(c) If $\alpha \in \Delta$, then $-\alpha \in \Delta$
(d) $B|_{\mathfrak{h},\mathfrak{h}}$ is nondegenerate. Hence for each $\alpha \in \Delta$, there is a corresponding $H_\alpha \in \mathfrak{h}$ with $\alpha(H) = B(H, H_\alpha)$ for all $H \in \mathfrak{h}$.
(e) Δ spans $\mathfrak{h}^*$, and $\{H_\alpha\}$ spans $\mathfrak{h}$

Proof. (a) We know that $\operatorname{ad} \mathfrak{g}_\alpha \operatorname{ad} \mathfrak{g}_\beta(\mathfrak{g}_\gamma) \subseteq \mathfrak{g}_{\alpha+\beta+\gamma}$, so we see than in a basis compatible with the root space decomposition of $\mathfrak{g}$, for any $A \in \mathfrak{g}_\alpha$ and any $B \in \mathfrak{g}_\beta$, $\operatorname{ad} A \operatorname{ad} B$ must have all zeros on the diagonal. Hence $\operatorname{Tr}(\operatorname{ad} A \operatorname{ad} B) = B(A, B) = 0$, and we conclude that $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$.

(b) Recall that since $\mathfrak{g}$ is semi-simple, B is nondegenerate. Since we may write $\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$ and we know $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ for all β with $\beta \neq -\alpha$, but $B(\mathfrak{g}_\alpha, \mathfrak{g}) \neq 0$, we must have $B(\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}) \neq 0$.

(c) This is immediate from the above, since if there were no root space $\mathfrak{g}_{-\alpha}$, we would have $B(\mathfrak{g}_\alpha, \mathfrak{g}) = 0$, which is impossible since B is nondegenerate.

(d) By the above, we see that $B|_{\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate: if $H \in \mathfrak{h}$ is such that $B(H, X) = 0$ for all $X \in H$, then the fact that $B(H, \mathfrak{g}_\beta) = 0$ for all $\beta \neq 0$ gives that H is such that $B(H, Y) = 0$ for all $Y \in \mathfrak{g}$. Thus $H \mapsto B(\cdot, H)$ is an isomorphism from $\mathfrak{h}$ to $\mathfrak{h}^*$, and thus there exists H_α such that $\alpha(H) = B(H, H_\alpha)$ for some $H_\alpha \in \mathfrak{g}$.

(e) Note that if Δ spans $\mathfrak{h}^*$, then for all nonzero $H \in \mathfrak{h}$, there exists $\alpha \in \mathfrak{h}^*$ such that $\alpha(H) \neq 0$. Thus suppose $H \in \mathfrak{h}$ has $\alpha(H) = 0$ for all $\alpha \in \Delta$. Then since we may decompose $\mathfrak{g}$ into root space and on each root space H is nilpotent, $\operatorname{ad} H$ is nilpotent. Then since $\operatorname{ad}\mathfrak{h}$ is Abelian, $(\operatorname{ad} H' \operatorname{ad} H)^n = \operatorname{ad} H'^n \operatorname{ad} H^n$ for any $H' \in \mathfrak{h}$, so $\operatorname{ad} H' \operatorname{ad} H$ is also nilpotent. Hence $\operatorname{Tr}(\mathfrak{h}, H) = B(\mathfrak{h}, H) = 0$; but since B is non-degenerate, this implies $H = 0$. Hence Δ spans $\mathfrak{h}^*$. Since $\mathfrak{h}$ is finite-dimensional it is isomorphic to $\mathfrak{h}^*$, and we conclude that $\{H_\alpha\}$ spans $\mathfrak{h}$.

□

Now, for each $\alpha \in \Delta \cup \{0\}$ and consider the action of $\mathfrak{h}$ on $\mathfrak{g}_\alpha$. Since $\mathfrak{h}$ fixes $\mathfrak{g}_\alpha$ and $\operatorname{ad}\mathfrak{h}$ is nilpotent and hence solvable, by Lie's theorem that we may find a simultaneous eigenvector for $\operatorname{ad}\mathfrak{h}$ on $\mathfrak{g}_\alpha$. Let E_α be such an eigenvector; then for all $H \in \mathfrak{h}$, $[H, E_\alpha] = \alpha(H)E_\alpha$.

We have the following further information about the roots of $\mathfrak{h}$:

Proposition 4.2. *(a) If α is a root, X is in $\mathfrak{g}_{-\alpha}$, then $[E_\alpha, X] = B(E_\alpha, X)H_\alpha$.*
(b) If $\alpha, \beta \in \Delta$, then $\beta(H_\alpha) = q\alpha(H_\alpha)$ for some $q \in \mathbb{Q}$.
(c) If $\alpha \in \Delta$, then $\alpha(H_\alpha) \neq 0$.

Proof. (a) We have that

$$\begin{aligned} B([E_\alpha, X], H) &= B(X, [H, E_\alpha]) \\ &= \alpha(H)B(X, E_\alpha) \\ &= B(H_\alpha, H)B(E_\alpha, X) \\ &= B(B(E_\alpha, X)H_\alpha, H) \end{aligned}$$

and since B is nondegenerate, we conclude that $B(E_\alpha, X)H_\alpha = [E_\alpha, X]$.

(b) By Proposition 4.1b we can chose $X'_{-\alpha} \in \mathfrak{g}$ such that $B(E_\alpha, X_{-\alpha}) \neq 0$; furthermore, any such $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$. After normalizing, we have that $B(E_\alpha, X_{-\alpha}) = 1$. Then by a, we have that $[E_\alpha, X_{-\alpha}] = H_\alpha$.

Now, consider the subspace $\mathfrak{g}' = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_{\beta+n\alpha}$, which by Proposition 4.1b is invariant under $\operatorname{ad} H_\alpha$. Now, consider the trace of $\operatorname{ad} H_\alpha$ considered as an operator on this subspace in two ways. First, note that for any n, $\operatorname{ad} H_\alpha$ acts on $\mathfrak{g}_{\beta+n\alpha}$ with generalized eigenvalue $\beta(H_\alpha) + n\alpha(H_\alpha)$, so we have

$$\operatorname{Tr}(\operatorname{ad} H_\alpha) = \sum_{n\in\mathbb{Z}} (\beta(H_\alpha) + n\alpha(H_\alpha)) \dim \mathfrak{g}_{\beta+n\alpha}.$$

We also know that $\mathfrak{g}'$ is invariant under E_α and $X_{-\alpha}$, so we regard these as operators on this subspace. Since we have that $[E_\alpha, X_{-\alpha}] = H_\alpha$, we then have $\mathrm{Tr}(\mathrm{ad}\, H_\alpha) = \mathrm{Tr}([E_\alpha, X_{-\alpha}]) = 0\,\mathrm{Tr}(\mathrm{ad}[E_\alpha, X_{-\alpha}]) = 0$. Thus the equation above shows that $\beta(H_\alpha)$ is a rational multiple of $\alpha(H_\alpha)$.

(c) If $\alpha(H_\alpha) = 0$, then the above shows that $\beta(H_\alpha) = 0$ for all $\beta \in \mathfrak{g}$. But since we know that Δ forms a basis for $\mathfrak{h}^*$, this implies that $H_\alpha = 0$, which is impossible by how we chose H_α. Hence $\alpha(H_\alpha) \neq 0$.

□

Proposition 4.3. *For all $\alpha \in \Delta$, $\dim \mathfrak{g}_\alpha = 1$, and $n\alpha \notin \Delta$ for any $n \geq 2$.*

Proof. Again choose $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ with $B(E_\alpha, X_{-\alpha}) = 1$ to obtain $H_\alpha = [E_\alpha, X_{-\alpha}]$. Let $\mathfrak{g}' = \mathbb{C}E_\alpha \oplus \mathbb{C}H_\alpha \oplus \bigoplus_{n<0} \mathfrak{g}_{n\alpha}$. Then this subspace is invariant under $\mathrm{ad}\, H_\alpha, \mathrm{ad}\, E_\alpha$, and $\mathrm{ad}\, X_{-\alpha}$. We know that $H_\alpha = [E_\alpha, X_{-\alpha}]$ and $E_\alpha, X_\alpha \in \mathfrak{g}'$, so when considered as an operator on $\mathfrak{g}'$, $\mathrm{Tr}(H_\alpha) = \mathrm{Tr}([E_\alpha, X_\alpha]) = 0$. However, we also know that $\mathrm{ad}\, H_\alpha(E_\alpha) = \alpha(H_\alpha)E_\alpha$, that $\mathrm{ad}\, H_\alpha(\mathbb{C}H_\alpha) = 0$, and that $\mathrm{ad}\, H_\alpha$ acts on each summand with generalized eigenvalue $n\alpha(H_\alpha)$. We thus see that

$$\mathrm{Tr}(\mathrm{ad}\, H_\alpha) = \alpha(H_\alpha) + \sum_{n<0} n\alpha(H_\alpha) \dim \mathfrak{g}_{n\alpha}.$$

Setting this equal to zero, we see that

$$\alpha(H_\alpha) = -\sum_{n<0} n\alpha(H_\alpha) \dim \mathfrak{g}_{n\alpha}$$

and since the above tells us that $\alpha(H_\alpha) \neq 0$, we have that

$$\sum_{n<0} n \dim \mathfrak{g}_{n\alpha} = -1$$

or reversing the sign of the indices,

$$\sum_{n>0} n \dim \mathfrak{g}_{-n\alpha} = 1.$$

Thus we see that $\dim_{-n\alpha} = 1$ for $n = 1$, and 0 otherwise. □

This result also allows for some very useful results: We had

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | (\mathrm{ad}\, H - \alpha(H)\mathbb{1})^n X = 0 \text{ for some n } \in \mathbb{N}, \text{for all}\, H \in \mathfrak{h}\}.$$

We can now simplify this:

Corollary 4.4. *For any $\alpha \in \Delta$, $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | \, \mathrm{ad}\, HX = \alpha(H)X\}$; in other words we can always take $n = 1$ above.*

Proof. By the above, we know that $\dim_\alpha = 1$ for all $\alpha \in \Delta$. □

Corollary 4.5. *The action of* $\mathrm{ad}\, \mathfrak{h}$ *on* $\mathfrak{g}$ *is simultaneously diagonalizable.*

Proof. This follows from the fact that $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ and that each $\mathfrak{g}_\alpha$ is one dimensional. □

This gives the promised generalization of Proposition 3.6.

Corollary 4.6. *For a semi-simple complex Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra if and only if $\mathfrak{h}$ is a maximal Abelian subalgebra.*

Corollary 4.7. *The pair of vectors $E_\alpha, E_{-\alpha}$ may be normalized so $B(E_\alpha, E_{-\alpha}) = 1$: that is, $B(E_\alpha, E_{-\alpha}) \neq 0$.*

Proof. $\mathfrak{g}_\alpha$ is one dimensional for all α, and $B(\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}) \neq 0$. □

With this normalization and proposition 4.2 above, we have the relations

$$\begin{aligned}
[H_\alpha, E_\alpha] &= \alpha(H_\alpha)E_\alpha, \\
[H_\alpha, E_{-\alpha}] &= -\alpha(H_\alpha)E_{-\alpha}, \\
[E_\alpha, E_{-\alpha}] &= H_\alpha
\end{aligned}$$

After re-normalization, we obtain that

$$\begin{aligned}
[H'_\alpha, E'_\alpha] &= 2E_\alpha, \\
[H'_\alpha, E_{-\alpha'}] &= -2E_{-\alpha}, \\
[E'_\alpha, E'_{-\alpha}] &= H'_\alpha.
\end{aligned}$$

Then some checking shows that

$$H'_\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E'_\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_{-\alpha'} \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

defines an isomorphism between the vector space spanned by H'_α, E'_α, and $E'_{-\alpha}$ and the vector space spanned by these three matrices. This the vector space

$$\left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} |a, b, c \in \mathbb{C} \right\} = \mathfrak{sl}(2, \mathbb{C}).$$

So, we conclude that every semi-simple Lie algebra contains within it embedded copies of $\mathfrak{sl}(2, \mathbb{C})$.

5. The Universal Enveloping Algebra

With all this in mind, we are almost in a position to state our final results, but we need to define one more concept: that of the universal enveloping algebra.

Definition 5.1. For a complex Lie algebra $\mathfrak{g}$, the **universal enveloping algebra** $U(\mathfrak{g})$ is a complex associative algebra with identity having the following property: there exists a Lie algebra homomorphism $i : \mathfrak{g} \to U(\mathfrak{g})$ such for any complex associative algebra A with identity and Lie algebra homomorphism $\varphi : \mathfrak{g} \to A$ (so that for all $X, Y \in \mathfrak{g}$, $\varphi(A)\varphi(B) - \varphi(B)\varphi(A) = \varphi([X, Y])$), there exists a unique extension $\tilde{\varphi} : U(\mathfrak{g}) \to A$ such that the diagram

$$\begin{tikzcd}
\mathfrak{g} \arrow[rr, "\varphi"] \arrow[dr, "i"'] & & A \\
& U(\mathfrak{g}) \arrow[ur, dotted, "\tilde{\varphi}"'] &
\end{tikzcd}$$

commutes.

A universal enveloping algebra $\mathfrak{g}$ can be explicitly constructed for any complex Lie algebra; for the details of this construction, refer to [1, pp. 164–180]. Furthermore, any two universal enveloping algebras are the same up to unique isomorphism.

Proposition 5.2. *Representations of $\mathfrak{g}$ on complex vector spaces are in one-to-one correspondence with left $U(\mathfrak{g})$ modules, with the correspondence being $\pi \mapsto \tilde{\pi}$.*

Proof. If $\pi : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(V)$ is a representation of $\mathfrak{g}$ on V, then by the definition of $U(\mathfrak{g})$, there exists an extension $\tilde{\pi} : U(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}}(V)$, and V becomes a left $U(\mathfrak{g})$ module with the left action being defined as $uv = \tilde{\pi}(u)v$ for all $u \in U(\mathfrak{g})$.

Conversely, if V is a left V module, then we may define for all $X \in \mathfrak{g}$ $\pi(X)v = i(X)v$, and since i is a Lie algebra homomorphism, this defines a representation. Since $\tilde{\pi} \circ i = \pi$ these two operations are inverses, and thus this correspondence is one-to-one. □

To proceed, we need an important theorem about the universal enveloping algebra which describes a basis for $U(\mathfrak{g})$. We want to describe this basis in terms of the basis for $\mathfrak{g}$, and we may do this as follows:

Theorem 5.3. *(Poincaré-Birkhoff-Witt). Let $\mathfrak{g}$ be a complex Lie algebra, and let $\{X_i\}_{i\in A}$ be a totally ordered basis for $\mathfrak{g}$ (note that here this basis can be uncountable, although we will always work with A finite). Then the set of all finite monomials*

$$i(X_{i_1})^{j_1} \cdots i(X_{i_k})^{j_k}$$

with all $j_k \geq 0$ forms a basis for $U(\mathfrak{g})$.

Corollary 5.4. *The canonical map $i : \mathfrak{g} \to U(\mathfrak{g})$ is one-to-one.*

The proof of the Poincaré-Birkhoff-Witt theorem, while not difficult, is relatively long, and is omitted here; for reference, see [[1] pp. 168-171]. This theorem gives us many useful results, among of which is the following, which follows almost immediately:

Corollary 5.5. *If $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, then the associative subalgebra of $U(\mathfrak{g})$ generated by 1 and $\mathfrak{h}$ is canonically isomorphic to $U(\mathfrak{h})$.*

6. Relating roots and weights

Finally, we are in a position to tie together all of this information. Before stating the final theorem, we must define an ordering on the roots α; one way to do this is pick a basis and order the roots lexographically relative to this basis, so $\alpha_1 > \alpha_2$ if $\alpha_1(H_i) = \alpha_2(H_i)$ for all $i < j$, and $\alpha_1(H_j) > \alpha_2(H_j)$.

We also make two more definitions that describe linear functionals on the Cartan subalgebra $\mathfrak{h}$:

Definition 6.1. Let $\lambda \in \mathfrak{h}^*$; we call λ **algebraically integral** if $2\langle \lambda, \alpha\rangle/\|\alpha\|^2$ is an integer for all roots. We call λ **dominant** if $\langle \lambda, \alpha\rangle \geq 0$ for all positive roots.

Proposition 6.2. *Let $\mathfrak{g}$ be a complex semi-simple Lie algebra, let $\mathfrak{h}$ be a Cartan subalgebra, let Δ be the nonzero roots of the adjoint representation of $\mathfrak{h}$ over $\mathfrak{g}$, and let $\mathfrak{h}_0 = \sum_{\alpha\in\Delta} \mathbb{R}H_\alpha$. If φ is a representation of $\mathfrak{g}$ on the finite-dimensional complex vector space V, then:*

(a) $\varphi(\mathfrak{h})$ acts diagonally on V, so that every generalized weight vector is a weight vector and V is the direct sum of all the weight spaces.
(b) Every weight is real-valued on $\mathfrak{h}_0$.
(c) Every weight is algebraically integral.
(d) Roots and weights are related by $\varphi(\mathfrak{g}_\alpha)V_\lambda \subseteq V_{\lambda+\alpha}$.

Proof. *(a, b)* We saw above that for any nonzero root α and corresponding root vectors E_α, $E_{-\alpha}$ spans a subalgebra $\mathfrak{sl}_\alpha$ isomorphic to $\mathfrak{sl}(2,\mathbb{C})$ with $2|\alpha|^{-2}H_\alpha$ corresponding to $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ (where the $2|\alpha|^{-2}$ is just a normalization constant.)Thus $\pi(\mathfrak{sl}_\alpha)$ is a finite-dimensional representation of $\mathfrak{sl}_\alpha$, and thus Example 2 shows that $\varphi(2|\alpha|^{-2}H_\alpha)$ is diagonalizable with integer eigenvalues. Since the collection of H_α span $\mathfrak{h}$ and since diagonalizable commuting operators are simultaneously diagonalizable, this proves that $\varphi(\mathfrak{h})$ acts diagonally on V and is real valued, which proves (a) and (b).

(c) Let λ be a weight and choose a nonzero $V \in V_\lambda$. Then the above shows that

$$\varphi(2|\alpha|^{-2}H_\alpha -) = 2|\alpha|^{-2}\varphi(H_\alpha)v = 2|\alpha|^{-2}\lambda(H_\alpha)v = 2|\alpha|^{-2}\langle\lambda,\alpha\rangle$$

is an integer multiple of v, so $2|\alpha|^{-2}\langle\lambda,\alpha\rangle$ is an integer and λ is algebraically integral.

(d) Let $E_\alpha \in \mathfrak{g}_\alpha$, let $v \in V_\lambda$, and let $H \in \mathfrak{h}$. Then

$$\begin{aligned}\varphi(H)\varphi(E_\alpha)v &= \varphi(E_\alpha)\varphi(H)v + \varphi([H,E_\alpha])v\\ &= \lambda(H)\varphi(E_\alpha)v + \alpha(H)\varphi(E_\alpha)v\\ &= (\lambda+\alpha)HE_\alpha v.\end{aligned}$$

so we conclude that $\varphi(E_\alpha)v \in V_{\lambda+\alpha}$.

□

Definition 6.3. A representation φ of a Lie algebra $\mathfrak{g}$ on a vector space V is **irreducible** if $\varphi(\mathfrak{g})$ has no irreducible subspaces besides for 0 and V.

With all of this built up, we can finally prove the following:

Theorem 6.4. *(Highest Weight Theorem) Let $\mathfrak{g}$ be a complex semi-simple Lie algebra, let $\mathfrak{h}$ be a Cartan subalgebra, let Δ be the set of roots of the adjoint representation, and introduce an ordering on Δ. Let $\mathfrak{h}_0$ be as above. Let Δ^+ denote the positive roots.*

If two irreducible representations φ_1 and φ_2 of $\mathfrak{g}$ have the same highest weight λ, then φ_1 and φ_2 are equivalent and may be labeled φ_λ. In addition:

(a) λ is independent of the ordering on $\mathfrak{h}_0$.
(b) V_α, the weight space of λ, is one dimensional.
(c) For all $\alpha \in \Delta^+$, the root vector E_α annihilates V_λ, and the members of V_λ are the only vectors with this property.
(d) every weight of φ_λ is of the form $\lambda - \sum_{i=1}^{\ell} n_i\alpha_i$ with each $n_i \geq 0$ and the $\alpha_i \in \Delta^+$.

We also have the following, which we will not prove, but is useful in the following section where we apply these results:

Theorem 6.5. *For any dominant algebraically integral linear functional $\lambda \in \mathfrak{h}^*$, there exists such an irreducible representation φ_λ.*

Proof. (of Theorem 6.4). First, we prove that such a correspondence exists. Let φ be an irreducible representation of $\mathfrak{g}$ on a vector space V. Let λ be the highest weight of this representation. By Proposition 6.2, we have that λ is algebraically integral.

If $\alpha \in \Delta^+$, then $\lambda+\alpha$ cannot be a weight, since we chose λ to be the highest weight. Then by Proposition X, $\varphi(E_\alpha)V_\lambda \subseteq V_{\lambda+\alpha} = 0$, and the first conclusion in c follows.

Now, extend φ to be defined on all of $U(\mathfrak{g})$ with $\varphi(1) = 1$, as discussed above in the definition of $U(\mathfrak{g})$. First, we may show that $\varphi(U(\mathfrak{g}))v = V$ for all $v \in V$.

Proceed as follows: Let $w \in V$. Define recursively $W_0 = \mathbb{C}v$, $W_n = \varphi(U(\mathfrak{g}))W_{n-1}$. Then since φ is irreducible, we have that $W_0 \subset W_1 \subset \dots W_n = V$, with each inclusion strict since φ is irreducible; this process must thus terminate at $n \leq \dim V$. But then we have that there exists $X_1, \dots, X_n \in U(\mathfrak{g})$ such that $\varphi(w_1)\dots\varphi(w_n)v = w$, and this gives that $\varphi(X_1)\dots\varphi(X_n)v = \varphi(X_1 \dots X_n)v = w$, so we see that we must have $U(\mathfrak{g})v = V$.

Now, if $\beta_1, \dots, \beta_k$ is an ordering of Δ^+, $H_1, \dots, H_l$ be a basis for H, and note that the set $\{E_{-\beta_1}, \dots, E_{-\beta_k}, H_1, \dots, H_k, E_{\beta_1}, \dots, E_{\beta_k}\}$ forms a basis for $\mathfrak{g}$. By the Poincaré-Birkhoff-Witt theorem, the monomials of the form

$$E_{-\beta_1}^{p_1} \dots E_{-\beta_k}^{p_k} H_1^{m_1} \dots H_l^{m_l} E_{\beta_1}^{q_1} \dots E_{\beta_k}^{q_k}$$

form a basis for $U(\mathfrak{g})$.

Let $v \in V_\lambda$, and consider

$$\begin{aligned} &\varphi\left(E_{-\beta_1}^{p_1} \dots E_{-\beta_k}^{p_k} H_1^{m_1} \dots H_l^{m_l} E_{\beta_1}^{q_1} \dots E_{\beta_k}^{q_k}\right) v \\ &= \varphi\left(E_{-\beta_1}^{p_1}\right) \dots \varphi\left(E_{\beta_k}^{p_k}\right) v \end{aligned}$$

We know from the above that $E_{\beta_k}v = 0$, and that $E_{-\beta_k}v \in V_{\lambda-\beta}$, and $H_\ell v = \lambda v$ from Proposition X. We conclude that if the mutliplication by the representation of the monomial gives a result in V_λ then $q_1, \dots q_k, p_1, \dots, p_k = 0$, and hence we obtain a multiple of v. Since we know that $U(\mathfrak{g})v = V$, we conclude that V_λ consists precisely of the span of v, proving (b).

Furthermore, we know that if the monomial acts nontrivially on v, then the result lands in the weight space with weight

$$\lambda - \sum_{i=1}^{k} \partial_i \beta_i.$$

Again, since $U(\mathfrak{g})v = V$, we obtain that these are all the weight spaces there are; in other words, every weight is of the form $\lambda - \sum_{i=1}^k \partial_i \beta_i$, and d is proved. Since λ is the only weight in Δ that can have this property, and this is independent of the ordering chosen, a is proven.

We can now prove that the second half of c: that if $E_\alpha v = 0$ for all $\alpha \in \Delta^+$, then $v \in V_\lambda$. Assume not, so there exists some such $v \notin V_\lambda$. Without loss of generality, we may assume that v has no component in V_λ. Let λ_0 be the largest weight such that v has a nonzero component in V_{λ_0} and let v' be this component. Then since $E_\alpha v = 0$ for all α, and in particular the component of $E_\alpha v$ in $V_{\alpha+\lambda_0}$ is 0, we must have $E_\alpha v' = 0$ for all α (as $E_\alpha(v - v')$ will always have 0 component in $V_{\alpha+\lambda_0}$). We also have that $\varphi(\mathfrak{h})v' \subseteq \mathrm{span} v'$. Then applying a general monomial to v and using the result that $U(\mathfrak{g})v = V$, we conclude that

$$V = \sum \left(E_{-\beta_1}^{p_1}\right) \dots \varphi\left(E_{\beta_k}^{p_k}\right) \mathbb{C}v = V$$

but this is impossible since $\left(E_{-\beta_1}^{p_1}\right) \dots \varphi\left(E_{\beta_k}^{p_k}\right) \mathbb{C}v \in V_{\lambda_0} - \sum_i \beta_i p_i$, so we would obtain that V is contained within only weight spaces with weight less than or equal to λ_0, a contradiction. Hence if $E_\alpha v = 0$ for all $\alpha \in \Delta^+$, then $v \in V_\lambda$, and this property characterizes V_λ.

Now, we may prove that λ is dominant, i.e. that $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Delta^+$. For $\alpha \in \Delta^+$, form H_α, E_α, and $E_{-\alpha}$, normalized to obey equations X. As we have seen, these vectors span an isomorphic copy of $\mathfrak{sl}(2,\mathbb{C})$ that we label $\mathfrak{sl}_\alpha$. For $v \neq 0$ in V_λ, the subspace of V spanned by all monomials

$$\varphi(E_{-\alpha})^p \varphi(H_\alpha)^q \varphi(E_\alpha)^r v$$

is stable under $\mathfrak{sl}_\alpha$, and the fact that E_α annihilates the members of V_λ shows that this is equal to the span of all $\varphi(E_{-\alpha})^p$. On these vectors $\varphi(H'_\alpha)$ has eigenvalue

$$(\lambda - p\alpha)(H'_\alpha) = \frac{2\langle \lambda, \alpha \rangle}{|\alpha|^2} - 2p$$

and thus the largest eigenvalue is thus $\frac{2\langle \lambda, \alpha \rangle}{|\alpha|^2}$. Since we know from Theorem 1.15 that for any representation of $\mathfrak{sl}(2,\mathbb{C})$ on a finite-dimensional vector space, $\pi(h)$ has positive eigenvalues, this shows that $\langle \lambda, \alpha \rangle \geq 0$, so λ is therefore dominant.

Now, we can prove that this correspondence is one-to-one. Let φ and φ' be irreducible finite-dimensional representations on V and V' with the same highest weight, λ. We know we may regard φ and φ' as module representations of $U(\mathfrak{g})$. Let $v_0 \in V_\lambda$ and $v'_0 \in V'_\lambda$ be nonzero, and consider the representation $\varphi \oplus \varphi'$ on $V \oplus V'$. Consider the subspace

$$S = \varphi \oplus \varphi'(U(\mathfrak{g}))(v_0 \bigoplus v'_0)$$

of $V \oplus V'$. Certainly S is invariant under $\varphi \oplus \varphi'(U(\mathfrak{g}))$; we claim that it is irreducible.

Let $T \subseteq S$ be an irreducible invariant subspace and let $v \oplus v'$ be a nonzero highest weight vector of T. Then for $\alpha \in \Delta^+$, we have that

$$0 = (\varphi \oplus \varphi')E_\alpha v \oplus v' = \varphi(E_\alpha)v \oplus \varphi'(E'_\alpha)v'$$

so $\varphi(E_\alpha)v = \varphi'(E'_\alpha)v' = 0$. Since the c shows that the only vectors that are annihilated by all weight vectors are the vectors in V_λ, we must have that $v = cv_0$ and $v' = cv'_0$. By assumption $cv_0 \oplus c'v'_0 \in \varphi \oplus \varphi'(U(\mathfrak{g}))v_0 \oplus v'_0$, since v_0 and v'_0 are weight vectors.

Now, note that when we apply one of the monomials in equation X, we obtain that the E_β annihilate $v \oplus v'$ and the $E_{-\beta}$ lower the weights, and the $H \in \mathfrak{h}$ act by

$$\varphi \bigoplus \varphi'(H)(v_0 \oplus v'_0) = \varphi(H)v_0 \oplus \varphi'(H)v'_0 = \lambda(H)v_0 \oplus v'_0.$$

Since these monomials act as a basis, we conclude that $c = c'$ and thus

$$S = (\varphi \oplus \varphi')E_\alpha v_0 \oplus v'_0 \subseteq T$$

so $T = S$, and S is irreducible.

Now, note that the projection $\pi : S \to V$ commutes with the representations and is not identically 0, and we may apply the following Lemma:

Lemma 6.6. *(Schurr's Lemma) Let φ and φ' be irreducible representations of a Lie algebra $\mathfrak{g}$ on finite-dimensional vector spaces V and V'. If $L : V \to V'$ is a nontrivial linear map such that $\varphi'(X)L = L\varphi(X)$ for all $X \in \mathfrak{g}$, then L is a bijection.*

Proof. Note that $\ker L$ is invariant under φ, so if $\ker L \neq L$, then $\ker L = 0$; similarly, $\operatorname{im} L$ is invariant under φ', so since $\operatorname{im} L \neq 0$, $\operatorname{im} L = L$. □

This gives that π is a bijection between S and V. Similarly, $\pi' : S \to V'$ is a bijection from S to V'. Hence φ and φ' are equivalent. □

7. Applying these results: the Lorentz group and its Lie algebra.

In this section we apply the results proven above to sketch out how the study of the Lie algebra of the Lorentz group gives information in theoretical physics. For the sake of brevity, we assume the reader has some familiarity with theoretical physics and in particular the Lorentz group and Einstein summation convention; we also leave out many computational details that are easy to check but lengthy to write out. For much more detail on the applications of Lie algebras to particle physics, the reader is referred to [2].

Briefly, the Lorentz group is the group of symmetries that acts on events in Minkowski space that leaves the relativistic interval between two events invariant. The the characterizing condition states that for two events x and y, for all Λ in the Lorentz group

$$g_{\alpha\nu}(\Lambda x)^{\alpha}(\Lambda y)^{\nu} = g_{\alpha\beta}x^{\alpha}y^{\beta}.$$

(where $g_{\alpha\nu}$ is the Minkowski metric). If we use the matrix representation where

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and we consider only transformations that preserve orientation and the direction of time, we obtain the group $SO(1,3)$, which can be represented as matrices X satisfying the equation

$$X^T G + GX = 0.$$

This is a quadratic equation in the entries of X that defines the manifold $SO(1,3)$. In identifying the Lie algebra with the tangent space $\mathfrak{so}(1,3)$ at the identity by differentiating, we obtain that for $X \in \mathfrak{so}(1,3)$, we must have $X + X^T = 0$, so $\mathfrak{so}(1,3)$ is the space of skew symmetric matrices (which is clearly a Lie algebra, as the commutator of two skew symmetric matrices is skew symmetric). Note that $\mathfrak{so}(1,3)$ has dimension 6, since any element is characterized by its entries above the diagonal.

Our study of this Lie algebra takes us through the study of a related Lie algebra, $\mathfrak{su}(2)$ (denoted as such since it is the Lie algebra of the group $SU(2)$). Consider the *real* Lie algebra of dimension three with the following basis:

$$\tau_1 = 1/2\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \qquad \tau_2 = 1/2\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \tau_3 = 1/3\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

These matrices satisfy $v_j v_k + v_s v_r = \delta_{jk}\mathbb{1}$.

Now, consider the adjoint representation of $\mathfrak{su}(2)$ on itself. For example, computations will show that

$$\begin{aligned} \mathrm{ad}(\tau_1)\tau_1 &= 0\tau_1 + 0\tau_2 + 0\tau_3 \\ \mathrm{ad}(\tau_1)\tau_2 &= 0\tau_1 + 0\tau_2 + 1\tau_3 \\ \mathrm{ad}(\tau_1)\tau_3 &= 0\tau_1 - 1\tau_2 + 0\tau_3 \end{aligned}$$

and so on; we obtain that with respect to the basis $\{\tau_1, \tau_2, \tau_3\}$, the adjoint representation has matrices

$$\mathrm{ad}(\tau_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \mathrm{ad}(\tau_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \mathrm{ad}(\tau_2) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Consider now the **complexification** of the aforementioned Lie algebras–i.e., the set of matrices when considered as a *complex* vector space. Ticciati identifies the complexification of $\mathfrak{so}(1,3)$ with $\mathbb{C} \otimes \mathfrak{so}(1,3)$ and the complexification of $\mathfrak{su}(\mathfrak{2})$ with $\mathbb{C} \otimes \mathfrak{su}(2)$; we follow Knapp in denoting these instead as $\mathfrak{so}(1,3)_{\mathbb{C}}$ and $\mathfrak{su}(2)_{\mathbb{C}}$. This is useful because it allows us to apply our above results with roots and weights, which all depended on having a complex Lie algebra. Note that $\mathfrak{so}(1,3)_{\mathbb{C}}$ is the Lie algebra of anti-Hermitian matrices, and $\mathfrak{su}(2)_{\mathbb{C}}$ is exactly $\mathfrak{sl}(2,\mathbb{C})$.

Consider the matrices

$$X_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

and the basis of $\mathfrak{so}(1,3)_{\mathbb{C}}$ given by $T_i = 1/2(X_i + iB_1)$, $\overline{T_i} = 1/2(X_i - iB_1)$. Computations will show that the maps $T_i \mapsto \tau_i$, $\overline{T_i} \mapsto \tau_i$ are isomorphisms, so we conclude that $\mathfrak{so}(1,3)_{\mathbb{C}} \cong \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}}$. Thus studying the representations of $\mathfrak{su}(2)_{\mathbb{C}}$ will give us information on the representations of $\mathfrak{su}(2)_{\mathbb{C}}$.

To study the representations of $\mathfrak{su}(2)_{\mathbb{C}}$, we first choose a Cartan subalgebra. Since $\mathfrak{su}(2)_{\mathbb{C}}$ is semi-simple and we cannot chose two linearly independent elements of $\mathfrak{su}(2)_{\mathbb{C}}$ that commute with each-other, the subspace generated by any element is a maximal Abelian subalgebra and hence a Cartan subalgebra. For notational convenience's sake, denote $\rho_j = i\tau_j$ for $j = 1, 2, 3$; then, following the usual convention in theoretical physics (where our z axis is always the axis of choice), we chose $\rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as our basis for this algebra. We wish to find then operators X such that

$$[\rho_3, X] = \lambda X$$

where such X will be in our root spaces. Computation will show that for operators of the form $R = i\rho_1 + i\rho_2$, $\lambda = 1$, and for operators of the form $L = \rho_1 - i\rho_2$, $\lambda = -1$. Dimension considerations and Proposition 3.2 then show that we may write $X = \mathbb{C}\rho_3 \oplus \mathbb{C}L \oplus$ Ŗ; this is one root space decomposition of $\mathfrak{su}(2)_{\mathbb{C}}$.

We may now apply the Theorem of Highest Weight to note that for any dominant linear functional λ on $\mathbb{C}\rho_3$, it makes sense to talk about the irreducible representation D_λ where λ is the highest weight. The linear functionals of $\mathbb{C}\rho_3$ are just determined by the value they send ρ_3 to; the condition that these are dominant amounts to the condition that this value must be greater than zero, and the fact that they are algebraically integral just amounts to the fact that this value must be $n/2$ for some integer $n \geq 0$. Furthermore, given that our roots here are just α and $-\alpha$, we see that the weights of this representation are of the form $\lambda - n$ for

$0 \leq n \leq 2\lambda$, and the operator L acts as a lowering operator. The reader familiar with quantum mechanics will recognize this as precisely the conditions describing the spin of a particular particle.

We can therefore see how the choice of a particular representation of this Lie algebra can be useful in describing the actual physical system that the Lie algebra acts upon; furthermore, we may chose the highest weight as appropriate to the physical system at hand. The above shows briefly how the representations of $\mathfrak{su}(2)_{\mathbb{C}}$ can be useful in describing physical systems; in more complicated systems described by $SO(1,3)$, similar methods apply.

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