2-SYLOW THEORY FOR GROUPS OF FINITE MORLEY RANK

SALMAN SIDDIQI

Abstract. We explore 2-Sylow theory in groups of finite Morley rank, developing the language necessary to begin to understand the structure of such groups along the way, and culminating in a proof of a theorem of Borovik and Poizat that all Sylow 2-subgroups are conjugate in a group of finite Morley rank.

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0. Introduction

The analysis of groups of finite Morley rank begins with the following conjecture (often called the algebraicity conjecture) of Cherlin and Zil'ber.

Conjecture 0.1 (Cherlin-Zil'ber)**.** *A simple ω-stable group is algebraic over an algebraically closed field.*

Though we will not elaborate much further, it is known that all algebraic groups over algebraically closed fields are simple groups of finite Morley rank, with rank equal to their dimension. In turn, ω-stable groups are also of finite Morley rank. If this conjecture were true, all three of these would be equivalent: simple ω-stable groups, simple groups of finite Morley rank and simple algebraic groups over algebraically closed fields. This characterisation would single-handedly resolve almost all open problems in the field.

There is also a weaker conjecture that simple groups of finite Morley rank are algebraic over algebraically closed fields, however, and this already poses significant challenges. Borovik has provided a possible path to understanding groups of finite Morley rank, transferring tools and techniques from the classification of finite simple groups. We will explore some of these tools, and attempt to gain some understanding of the structure of Sylow 2-subgroups of groups of finite rank. Our aim will be to provide a proof of the following theorem:

Theorem 0.2 (Borovik-Poizat)**.** *Sylow 2-subgroups of a group of finite Morley rank are conjugate.*

We will assume some familiarity with the basics of model theory (in particular, the concept of definability and the notion of a model), and the fundamentals of group theory. There are a few theorems – mostly related to the internal structure of nilpotent-by-finite 2-groups – which we will take for granted, but otherwise we aim for this paper to be self-contained.

1. Morley Rank on Groups

For an introduction to the basics of model theory, the first chapter of Chang & Keisler's *Model Theory* is an excellent reference. Alternatively, the third and fourth chapters of Borovik & Nesin's *Groups of Finite Morley Rank*, whose presentations of many of the theorems in this paper we will follow more-or-less exactly, provides exactly the basic requisite definitions from model theory.

The central notion, that of a *Morley rank*, is perhaps one that requires some time to appreciate.

Definition 1.1. The *Morley rank* of a definable subset is at least 0 if the set is nonempty. If α is not a limit ordinal, the rank is at least α if there are countably many disjoint definable subsets of rank at least $\alpha - 1$. If α is a limit ordinal, the rank is at least α if it is at least β for all $\beta < \alpha$.

The Morley rank, in a sense, measures the extent to which we can infinitely subdivide a set into definable subsets. That this serves as a good measure of complexity should not at all be immediately obvious; however, having a finite Morley rank on a structure will immediately give us enormous control over the definable subsets, as we will soon see.

In our particular case, we will be concerned with the Morley rank of a group G, by which we will mean the rank of the structure $\langle G, \cdot, e, g\rangle_{g\in G}$ in the pure theory of groups, with constants for each element of the group.

An extremely useful tool that goes hand-in-hand with the notion of rank is that of the degree of a definable set of ordinal-valued rank.

Definition 1.2. The *degree* of a definable set X is the maximum number of disjoint definable subsets of X, where each is required to have rank equal to the rank of X.

The Morley rank need not be finite, and may well be larger than every ordinal. However, if the rank on a set is ordinal-valued, the degree is necessarily finite. To see why, first we observe that the degree is additive:

Proposition 1.3. *If X is a set with an ordinal-valued rank and we can write $X = A \amalg B$ where $\mathrm{rk}(A) = \mathrm{rk}(B) = \mathrm{rk}(X)$, then $\deg(X) = \deg(A) + \deg(B)$.*

Proof. Clearly, $\deg(X) \geq \deg(A) + \deg(B)$. For the converse, suppose X contained d-many disjoint subsets of the same rank, and consider their intersections with A and B. Either $D_i \cap A$, $D_i \cap B$ or both must have rank $rk(X)$ (as we will prove shortly, the rank of a disjoint union is the maximum of the ranks). So A contains n and B contains $n - d$ disjoint definable subsets of rank $rk(X)$, giving us equality. □

Now, if X has rank, say, α, and infinite degree, we can write it as the disjoint union of at least 2 sets with rank equal to $rk(X)$. By the above, one of these necessarily has infinite degree, and so we can repeat this process with that set. We continue indefinitely and see that this gives us a division of X into infinitely many sets of rank $rk(X)$, which cannot be the case. Hence, X must have had finite degree.

Often, we will have proofs by contradiction in which we pick a counterexample of 'minimal rank and degree' – by this we mean a counterexample whose rank is minimal among other counterexamples, and whose degree is minimal among counterexamples of the same rank. In this case, we will also often consider subgroups of 'smaller rank and degree', in which case we mean that either the rank is smaller or the rank is the same and the degree is smaller.

While we will not attempt to compute the Morley ranks of particular groups, there are a few traditional examples that should help provide some intuition.

Definition 1.4. The *Prüfer p-group* is the group $Z(p^\infty) = \{x \in \mathbb{C} \mid x^{p^n} = 1 \text{ for some } n \in \mathbb{N}\}$.

The Prüfer p-group is an infinite, divisible abelian group of Morley rank 1.

Definition 1.5. An *algebraic group* is an algebraic variety together with a group operation so that the inverse and multiplication maps are regular.

An algebraic group over an algebraically closed field will have finite Morley rank equal to its dimension as an algebraic variety, and degree equal to the maximal number of disjoint irreducible subvarieties of the same dimension.

Moreover, a finite direct product of groups of finite Morley rank is a group of finite Morley rank, and an arbitrary direct sum of copies of $\mathbb{Q}$ is also a group of finite Morley rank. However, as we will see by the end of the section, neither free nor free abelian groups can have finite rank. Moreover, while we will not prove this, no group of finite rank can contain as a subgroup either an infinite free product of nonabelian groups or an infinite symmetric group.

Philosophically, one of the great advantages on imposing an ordinal-valued rank on a structure is that we now have control on descending chains. Here, we get a descending chain condition on definable subgroups; to see this, we will first need a lemma.

Lemma 1.6. *If G can be written as the union of definable subsets $H \cup K$, then if G is of finite Morley rank, so are H and K and we have* $\mathrm{rk}(G) = \max(\mathrm{rk}(H), \mathrm{rk}(K))$.

Proof. We proceed by induction. If $rk(G) = 0$, this is clear. Now suppose this holds for groups of rank n, and let G be of rank $n+1$. Clearly, $rk(H), rk(K) \leq rk(G)$ - suppose, then, that these inequalities were strict, so H and K must then be of rank n.

Since G is of rank $n+1$, we can write G as the disjoint union of countably many sets of rank n, say $\{G_i\}_{i<\omega}$. Now consider the intersections $H_i = H \cap G_i$ and $K_i = K \cap G_i$. G_i is the union of these two and has rank n, so by our inductive hypothesis, one of these must have rank n. Then, either infinitely many H_i must have rank n, in which case H has rank $n+1$, or infinitely many K_i have rank n, in which case K has rank $n+1$ (or both), as required. □

And now, we can really begin to understand how the Morley rank and group structure interact.

Lemma 1.7. *If $H \subset G$ is a definable subgroup of a group of finite Morley rank G, then:*

(i) *If $[G : H] < \infty$, then* $\mathrm{rk}(H) = \mathrm{rk}(G)$.

(ii) *If* $[G : H] = \infty$*, then* $\mathrm{rk}(H) < \mathrm{rk}(G)$.
(iii) *If* $H \subsetneq G$*, either* $\mathrm{rk}(H) < \mathrm{rk}(G)$ *or* $\deg(H) < \deg(G)$*. In the latter case, we have* $\deg(G) = [G : H] \deg(H)$.

Proof. If $[G : H]$ is finite, we consider the cosets of H in G. Clearly, the rank of all of these cosets must be the same (since gH is definable and there is a definable bijection between H and gH) and, moreover, being definable subgroups of a group of finite Morley rank, they clearly have finite rank, say n. From induction on the above lemma, it is clear then that $rk(G) = n$ as well.

On the other hand, if $[G : H]$ is infinite, since H is definable and the cosets gH are therefore also definable, G contains the infinitely many disjoint cosets of H, all of rank equal to $rk(H)$. Hence, $rk(G) > rk(H)$.

Finally, suppose H is a proper definable subgroup of G of the same rank. Then G is the disjoint union of $[G : H]$ many cosets of H, each of degree $\deg(H)$ (following the same reasoning that H and gH are in definable bijection). But then, since degrees are additive, we see that $\deg(G) = [G : H] \deg(H)$. □

Note that, as an easy consequence of this lemma, we see that a subgroup H of the same Morley rank and degree as G must be of index 1, and hence must be equal to G. With what we have, we can show that the descending chain condition holds on definable subgroups of groups of finite Morley rank, which will be one of the most useful techniques in our arsenal.

Theorem 1.8. *Let* $\{D_i\}_{i<\omega}$ *be an infinite descending chain of definable subgroups. Then* $\{D_i\}_{i<\omega}$ *stabilizes at some* $n \in \mathbb{N}$.

Proof. The ordinals are well-ordered, so the ranks cannot decrease indefinitely, and so the ranks must stabilize. Then, since the degree is also an ordinal, it too cannot decrease indefinitely. So, for some $n \in \mathbb{N}$, all D_i with $i > n$ must have the same rank and degree. By the preceding lemma, this implies that the chain must stabilize. □

From this, we can also easily conclude that any infinite intersection must be finitely generated, and that the centralizer of any set (which is the intersection of the necessarily definable centralizers of its elements) is definable.

At this point, we can begin to appreciate the additional structure and control that a finite Morley rank imposes on the group. We start with a fact from group theory. As a word of warning which should hopefully help with the clarity of the proof, n, n', p, q and m will be integers, with p prime.

Proposition 1.9. *If* $D \subset A$ *is a divisible subgroup of an abelian group* A*, then* A *can be written as the direct sum* $D \oplus H$ *for some subgroup* H.

Proof. Consider the set of subgroups of A intersecting D trivially, partially ordered by inclusion. This set is nonempty, since it contains the trivial group. Moreover, given any chain of subgroups of A intersecting D trivially, the union is a subgroup intersecting D trivially. Hence, we have a maximal subgroup H by Zorn's lemma.

Now, suppose that $D \oplus H$ were a proper subgroup of A, and say $g \in A$ could not be written as a sum $d + h$. The group generated by H and g has nontrivial intersection with D by the maximality of H, and so we can pick n' minimal so that $n'g + h = d$ for some $d \in D, h \in H$ (equivalently, $n'g \in D \oplus H$). Now, write $n' = pn$ for p prime. We will show that $ng \in D \oplus H$, from which we can immediately conclude that $g \in D \oplus H$ (from a simple induction on the prime factors of n').

Suppose that $ng \notin D \oplus H$. We saw earlier that $png + h = d$, and since D is divisible, we can write $png + h = pd'$. Hence, we see that $p(ng - d') = -h \in H$. But now, $ng \notin D \oplus H$ so, clearly, $ng - d' \notin D \oplus H$. The group generated by H and $ng - d'$ intersects D nontrivially, so $m(ng - d') = d'' + h'$ for some nontrivial d''. From here, we see that p cannot divide m, since otherwise we would have $m = pq$ and hence $m(ng - d') = -qh = d'' + h'$, contradicting our choice of H (namely, the intersection of H and D must be trivial).

Hence, p and m must be coprime. But then, we can find $a, b \in \mathbb{Z}$ so that $ap + bm = 1$ and hence $a(p(ng - d')) + b(m(ng - d')) = (ap + bm)(ng - d') = ng - d'$. Since $p(ng - d') \in H$ and $m(ng - d') \in D$, we then have that $ng - d' \in D \oplus H$, and hence $ng \in D \oplus H$, as required. □

With this in hand, we can prove the following theorem of Macintyre, which provides some useful insight on the structure of abelian groups of finite Morley rank.

Theorem 1.10 (Macintyre)**.** *An abelian group of finite Morley rank A can be expressed as the direct sum $A = D \oplus B$ of a definable, divisible subgroup D and a subgroup B of bounded exponent.*

Proof. Write $A(n) = \{a^n \mid a \in A\}$, and note that these are all definable. Moreover, we clearly have $A((n + 1)!) \subset A(n!)$ for all $n \in \mathbb{N}$. The intersection D of all the $A(n!)$ is clearly divisible and, by the descending chain condition, is generated by finitely many of these subgroups and hence is definable – since the subgroups form a chain, this intersection is in fact just $A(n!)$ for some n. By the previous proposition, we can then write $A = D \oplus B$ for some subgroup B.

It remains to be shown that B has bounded exponent. Suppose that some element b of B had order greater than $n!$. Then we would have $b^{n!} \in B$ nontrivial even though $b^{n!} \in D$ by construction, which is a contradiction (since $B \cap D$ is trivial), as required. □

It is important to note that B is not necessarily definable; however, if we let $m = \exp(B)$ and consider the group $C = \{a \in A \mid a^m = 1\}$, it is easy to show that we do in fact have a decomposition of $A = D * C = DC$ as a central product, with C a definable characteristic subgroup of A containing B. In fact, this latter decomposition generalizes to nilpotent groups: if G is nilpotent, we can write G as the central product $B * D$ for a definable characteristic subgroup B of bounded exponent and a definable characteristic divisible subgroup D.

Using this decomposition, we can obtain an interesting criterion for n-divisibility, which we will use later.

Proposition 1.11. *Let G be a group of finite Morley rank. If G has no element g whose order divides n, then G is n-divisible.*

Proof. Suppose G were abelian, so we have the decomposition $G = D \oplus B$ by the previous theorem. Now, let $g \in G$ and write $g = d + b$. Since D is n-divisible, we can find a $d' \in D$ so that $d = nd'$. Since B is of bounded exponent, b has finite order and thus generates a finite cyclic subgroup, say of order m.

If p is a prime dividing m, by Cauchy's theorem, there is an element of G of order p. Hence, since no element has order dividing n, m and n must be coprime. Thus, b must have an n^{th} root in B, and so $g = nd' + nb'$, as required.

Now, if G is not abelian, let $g \in G$. The centralizer $C_G(g)$ of g in G is a definable subgroup, and so is its center $Z(C_G(g))$. Observe that the center is a definable abelian subgroup containing g of a group of finite Morley rank, and is hence a group of finite Morley rank containing g. Since G has no elements of order n, neither does $Z(C_G(g))$, and so g has n^{th} root in $Z(C_G(g))$, and hence in G. $\square$

Note that, as an immediate consequence, we see that free and free abelian groups are not groups of finite Morley rank.

2. Connectedness, Nilpotency and Definable Closures

With the basic properties of the Morley rank that we have just developed, we can further investigate the structure of these groups. One of the fundamental notions in our analysis will be that of a connected group, which is precisely analogous to the notion of connectedness in the context of algebraic groups.

Definition 2.1. Let G be a group of finite Morley rank. The *connected component* G° of G is the intersection of all definable subgroups of finite index in G. We say that G is connected if $G = G^\circ$.

Note that, by the descending chain condition, G° is necessarily given by a finite intersection, and is therefore itself a definable subgroup of finite index. Clearly, if $\deg(G) = 1$, there can be no definable subgroups of finite index, and so G must be connected. The converse is also true, but the proof is slightly more involved and we will have no need of it, so we omit it.

Moreover, the connected component of a group is characteristic in the following sense.

Proposition 2.2. *If G is a group of finite rank, its connected component is invariant (as a subgroup of G) under any definable automorphism.*

Proof. If φ is a definable automorphism, then $\varphi(G^\circ)$ is a definable subgroup of finite index in G, and so must necessarily contain G°. Applying the same argument to φ^{-1} completes the proof. $\square$

In particular, conjugation by any element is a definable automorphism, so this shows that for any group G of finite rank, G° is normal.

It will be useful later on to note that a quotient of a connected group is itself connected.

Theorem 2.3. *The quotient of a connected group of finite Morley rank G by a normal definable subgroup H is connected.*

Proof. Consider the connected component $\overline{K}$ of G/H, corresponding to the subgroup K of G containing H. The cosets of $\overline{K}$ in G/H correspond exactly to the cosets of K in G, and so there cannot be more than $n = [G/H : \overline{K}]$ many cosets of K in G. However, the only subgroup of finite index in G is itself, so we must have $K = G$, showing that $n = 1$ and hence $\overline{K} = G/H$, as required. $\square$

One of our main techniques in approaching Theorem 0.2 will be in considering groups in which there is a nilpotent subgroup of finite index.

Definition 2.4. A group G is said to be nilpotent-by-finite if there is a nilpotent normal subgroup H so that G/H is finite.

We will show later that 2-subgroups of a group of finite Morley rank are nilpotent-by-finite, and the main step in proving this will rely on the normalizer condition, stated below. Recall from basic group theory that nilpotent groups have nontrivial centers and satisfy the normalizer condition. This is also true of nilpotent-by-finite p-groups, though we omit the proof.

Theorem 2.5. *Nilpotent-by-finite p-groups have nontrivial center and satisfy the normalizer condition: if $X \subset G$ is a proper subset, then $X \subsetneq N_G(X)$.*

If G is nilpotent (not necessarily a p-group) and X is a proper definable subgroup, we can actually get something stronger: if X is of infinite index, then $rk(X) < rk(N_G(X))$.

In nilpotent-by-finite groups of finite rank, one may hope that the connected component (being a normal subgroup of maximal finite index) is itself nilpotent. This turns out to be true, and will be easy to see with definable closures.

Definition 2.6. The definable closure $d(X)$ of a set X in a group of finite rank G is the intersection of all definable subgroups of G containing X.

By the descending chain condition, this intersection is given by finitely many such subgroups and is therefore itself a definable subgroup of G containing X. It should be apparent that conjugation commutes with taking a definable closure, from which it follows that:

Proposition 2.7. *If $A, X \subset G$ are subsets of a group of finite rank and A normalizes X, then $d(A)$ normalizes $d(X)$.*

Proof. Conjugation commutes with taking a definable closure, from which it is immediate that, for any $a \in A$, $d(X) = a^{-1}d(X)a$. Then, A normalizes $d(X)$, and so $A \subset N_G(d(X))$. Since $N_G(d(X))$ is definable, it follows that $d(A)$ also normalizes $d(X)$. □

We can use this to extract information on definable closure of terms in the lower central series. In general, we have:

Lemma 2.8. *If $X, Y \subset G$ are subgroups of a group of finite rank, then $[d(X), d(Y)] \subset d([X, Y])$.*

The proof of this lemma is rather technical and unenlightening, and so we leave it to the reader to follow the details in Borovik & Nesin's chapter on definable closures.

From this, we can show that solvability and nilpotency are transferred to definable closures.

Theorem 2.9. *If $H \subset G$ is solvable or nilpotent of class k, then so is $d(H)$.*

Proof. Note that, by induction on the preceding lemma, we have the containment of characteristic subgroups $d(H)_n = [d(H)_{n-1}, d(H)] \subset d([H_{n-1}, H]) = H_n$ and similarly $d(H)^{(n)} \subset H^{(n)}$. So $d(H)$ is solvable or nilpotent of class at most k, and since $H \subset d(H)$ is solvable or nilpotent of class exactly k, so is $d(H)$. □

It is certainly desirable, though perhaps not obvious without taking definable closures, that nilpotent-by-finite groups of finite rank have a *definable* nilpotent subgroup of finite index. However, we now have all the tools necessary to show that this is, in fact, true.

Theorem 2.10. *If a group G of finite Morley rank is nilpotent-by-finite, then G has a definable, normal nilpotent subgroup of finite index.*

Proof. Since G is nilpotent-by-finite, there is a normal nilpotent subgroup of finite index, say H. Since G noramlizes H, by an earlier proposition, G also normalizes $d(H)$. By the preceding theorem, $d(H)$ is nilpotent since H is. Hence, $d(H)$ is a normal nilpotent subgroup, and necessarily of finite index since it contains H. $\square$

In particular, this shows that the connected component G° is a normal nilpotent subgroup of finite index, since $G^\circ \subset H$ by definition.

The last major notion we will need is that of uniform definability.

Definition 2.11. Let G be a group of finite rank and $\mathscr{F} = \{F_i \mid i \in I\}$ be a family of subgroups. We say that $\mathscr{F}$ is uniformly definable if we can find an indexing set of n-tuples of G, $I \subset G^n$ for some $n \in \mathbb{N}$, and there is a formula $\varphi(x, \overline{y})$ in $n+1$ free variables so that F_i is defined by $\varphi(x, i)$ for $i \in I$.

The central theorem here is due to Baldwin and Saxl, which gives us enormous control on uniformly definable families.

Theorem 2.12 (Baldwin-Saxl)**.** *Let $\mathscr{F}$ be a uniformly definable family of subgroups of a group G of finite rank. Then there is a uniform bound n on the length of any chain of subgroups in $\mathscr{F}$.*

The proof of this theorem is largely model-theoretic, so we omit the proof and instead refer the reader to the section on chain conditions in Altinel & Borovik's book. However, note that we have the following immediate corollary:

Corollary 2.13. *There are no infinite ascending or descending chains of centralizers.*

3. 2-Sylow Theory

We are finally in a position to begin the proof of Theorem 0.2:

Theorem 0.2 (Borovik, Poizat)**.** *Sylow 2-subgroups of a group of finite Morley rank are conjugate.*

It is an open conjecture that, for any prime p, the p-Sylow subgroups of a group of finite Morley rank are conjugate. That this holds in the particular case that $p = 2$, however, has inherent interest, and plays a great role in Borovik's program towards providing an answer to the classification conjecture for groups of finite Morley rank.

It turns out that, in the case $p = 2$, the fact of critical importance is that the possible groups generated by two involutions (that is, elements of order 2) are very well understood. In fact, we have the following characterization:

Proposition 3.1. *A group generated by two involutions is a (possibly infinite) dihedral group.*

Proof. Let i and j be two involutions. The group generated by these involutions is just the set of words $\{1, i, j, ij, ji, iji, \ldots\}$, from which it is easy to see that we have the presentation $\langle i, j \mid i^2 = j^2 = 1, (ij)^n = 1\rangle$ if the order of the product ij is finite, say n, giving us the dihedral group of order $2n$. If the order of ij is infinite, we have the presentation $\langle i, j \mid i^2 = j^2 = 1\rangle$, which is just the infinite dihedral group. $\square$

Moreover, the ways in which two involutions in a group of finite Morley rank can interact are well-understood. First, we note that the action of either involution on their product is simply inversion:

Proposition 3.2. *If i and j are two involutions with product $x = ij$, then we have $i^{-1}xi = j^{-1}xj = x^{-1}$.*

Proof. It is easy to see that $i^{-1}xi = i^{-1}iji = ji = x^{-1}$. For the second assertion, we use the fact that j is an element of order 2, and so $j = j^{-1}$. Hence, we have $j^{-1}xj = jxj^{-1} = jijj^{-1} = ji = x^{-1}$, as required. □

With this, we can obtain the following characterization of the interaction of two involutions:

Proposition 3.3. *In a group of finite Morley rank with $x = ij$ the product of two involutions, i and j are either $d(x)$-conjugate or commute with a third involution k.*

Proof. Consider the action of i by conjugation on the subgroup $d(x)$. The set of elements of $d(x)$ inverted by this action, that is those elements z for which $i^{-1}zi = z^{-1}$, is a definable subgroup of $d(x)$, which contains x by Proposition 3.2. So the action of i must invert every element of $d(x)$ (because $d(x)$ is the minimal definable subgroup of G containing x), and a similar argument shows that the action of j must invert element of $d(x)$. Now, if $d(x)$ contains an involution y, we see that $i^{-1}yi = y^{-1} = y$ and $j^{-1}yj = y^{-1} = y$. Hence, i and j commute with y.

Otherwise, if $d(x)$ contains no involutions, by Proposition 1.11, $d(x)$ is 2-divisible. So we can find a $y \in d(x)$ so that $y^2 = x$. Since the action of i inverts every element of $d(x)$, we have that $i^{-1}y^{-1}i = y$. But then, we have that $y^{-1}iy = ii^{-1}y^{-1}iy = iy^2 = ix = iij = j$, as required. □

We can now begin to understand the structure of 2-subgroups of groups of finite Morley rank. We will first need a lemma: recall that $Z_2(G)$ is the second center of G.

Lemma 3.4. *If G is connected and $Z(G)$ is finite, then $G/Z(G)$ is centerless.*

Proof. Let $z \in Z_2(G)$, and consider the homomorphism $h : G \to G$ given by $h(g) = [z, g]$. Since $z \in Z_2(G)$, we have by definition that $[z, g] \in Z(G)$ for all $g \in G$. Hence, the image of h is finite since $Z(G)$ is, and therefore its kernel $\ker(h)$ has finite index in G. Clearly, $[z, g] = 1$ if and only if $g \in C_G(z)$, so $\ker(h) = C_G(z)$. Since G is connected, however, it has no proper subgroups of finite index, and so $C_G(z) = G$. Hence, $z \in Z(G)$ and so $Z_2(G) \subset Z(G)$. We already have the reverse inclusion by definition, so $Z_2(G) = Z(G)$. Hence, the quotient is trivial, and so the center of $G/Z(G)$, which is precisely the quotient $Z_2(G)/Z(G)$, is also trivial. □

Recall from finite group theory that all finite p-groups are nilpotent. While this may not be true of all p-subgroups of groups of finite rank, under the additional condition that every finitely generated subgroup is finite, all p-subgroups have a normal nilpotent subgroup of finite index. However, when $p = 2$, this assumption is redundant, and we have the following theorem:

Theorem 3.5. *A 2-subgroup of a group of finite Morley rank is nilpotent-by-finite.*

Proof. Suppose we had a group G of minimal rank with a 2-subgroup P that was not nilpotent-by-finite. Without loss of generality, we can assume that G is the definable closure of P. Moreover, we can assume that P, and hence $G = d(P)$, is connected: if P° were nilpotent-by-finite, P would also be nilpotent-by-finite since P° has finite index in P.

If $Z(P)$ were infinite, then $d(P)/Z(P) = G/Z(P)$ would have a strictly lower Morley rank. Then, $P/Z(P)$ is a 2-subgroup of $G/Z(P)$, and is hence nilpotent-by-finite because we chose G to be of minimal rank. But $P/Z(P)$ must be connected since $Z(P)$ is definable and P is connected. Hence, if $P/Z(P)$ were nilpotent-by-finite, $P/Z(P)$ would be nilpotent, and so P would be nilpotent as well, a contradiction (we would have a chain of normal subgroups containing $Z(P)$ that end with $Z(P)$, which is nilpotent of class 1, where each quotient is abelian).

So we may take $Z(P)$ to be finite, and it will suffice to show that this will lead to a contradiction. By Lemma 3.4, $P/Z(P)$ is centerless. Now take $G' = G/Z(P)$ and $P' = P/Z(P)$. Observe that $C_{P'}(x) \subset C_{G'}(x)$ is a 2-subgroup for any $x \in P'$. By the minimality of our choice of G (and hence, of G'), $C_{P'}(x)$ must be nilpotent-by-finite unless x is the identity.

By Corollary 2.13, centralizers satisfy the ascending chain condition, so P' has maximal centralizers. Moreover, pairwise intersections of distinct maximal centralizers also satisfy the ascending chain condition, so we can pick a maximal intersection, say $D = A \cap B$ where $A = C_{P'}(a)$ and $B = C_{P'}(b)$ are maximal. By the previous paragraph, these two are nilpotent-by-finite.

Since A and B are nilpotent-by-finite and D was chosen to be a proper subgroup of each, we have the normalizer condition $D \subsetneq N_A(D)$ and $D \subsetneq N_B(D)$. Since D is normal in each, and recalling that these are all 2-subgroups, we can pick nontrivial elements $\overline{a} \in N_A(D)/D$ and $\overline{b} \in N_B(D)/D$ of order 2. Now, $\overline{a}\overline{b}$ has finite order in $N_P(D)/D$, so the subgroup $\langle \overline{a}, \overline{b} \rangle$ is finite. Let H be the full preimage of this subgroup under the quotient map, which clearly contains D.

Since $\langle \overline{a}, \overline{b} \rangle$ is a finite 2-group, D has finite index in H. Since D is nilpotent-by-finite, H must also be nilpotent-by-finite. Then, by Theorem 2.5, H has nontrivial center, and so we can pick a nontrivial central element $c \in Z(H)$. There is some maximal centralizer $C \supset C_{P'}(c)$ in P'. Now, since c is central in H, we have $D \subsetneq H \subset C$. Considering intersections with A and B, we see that $D \subsetneq H \cap A$ since $H \cap A$ contains an element in the preimage of $\overline{a}$, and likewise $D \subsetneq H \cap B$. Hence, $D \subsetneq C \cap A$ and $D \subsetneq C \cap B$, which contradicts the maximality of D, completing the proof. □

Note that proof above need only be modified slightly to show that locally finite p-subgroups are nilpotent-by-finite, since we only needed to know that the subgroup $\langle \overline{a}, \overline{b} \rangle$ was finite.

In the case of 2-groups, a useful nilpotent subgroup turns out to be the connected component. Moreover, we have a structure theorem for the connected component, though we will need a definition first.

Definition 3.6. A p-torus is a divisible abelian p-group of finite Morley rank.

For instance, any finite direct sum of Prüfer p-groups is a p-torus. An interesting fact that will be useful later is that, in a group G of finite Morley rank, we have a uniform constant n so that $[N_G(T) : C_G(T)] < n$ for any p-torus T in G. As an immediate corollary, we see that the normalizer of a p-torus is always definable.

Understanding 2-tori will prove instrumental to us, not least because of their role in the decomposition of 2-groups:

Theorem 3.7. *If P is a 2-subgroup of a group of finite Morley rank, then P° is nilpotent and can be written as the central product $P^\circ = B * T$ where B is a nilpotent group of bounded exponent and T is a 2-torus.*

When P is maximal – that is a Sylow 2-subgroup – B is definable and B and T are unique. In this case, we will denote B by $B(P)$ and T by $T(P)$, and call them the *bounded part* and the *maximal torus* of P respectively.

Clearly, if B is definable and $T(P)$ is trivial, then P has a definable subgroup of finite index and is therefore definable.

Now, we can begin to set up the machinery required for the proof. Throughout the rest of the paper, G will be a group of finite Morley rank chosen to be of minimal rank and degree among those with a nonconjugate pair of Sylow 2-subgroups. We will make a series of simplifying assumptions, ruling out the possibility of G having a nontrivial normal 2-subgroup or having any finite Sylow 2-subgroups, and in light of these, set up a rather powerful inductive argument to show that no such group can exist.

We will have need of the following theorem, whose proof we will omit, to make the first simplification; while not necessary, it greatly simplifies the argument.

Theorem 3.8 (Altinel-Cherlin-Corredor-Nesin)**.** *Let π be a set of primes. Any two Hall π-subgroups (that is, subgroups in which the order of each element is a product of primes in π) of a solvable group of finite Morley rank are conjugate.*

First, we will make the simplifying assumption that the maximal normal 2-subgroup of G is trivial.

Proposition 3.9. *If G is of minimal Morley rank and degree with nonconjugate Sylow 2-subgroups, then its maximal normal 2-subgroup $O_2(G)$ is trivial.*

Proof. If $O_2(G)$ is nontrivial but finite, then clearly $G/O_2(G)$ is of smaller rank and degree and we get an immediate result just considering the projection maps. If $O_2(G)$ is infinite, we need to do a little more work since it is not necessarily definable, and so we cannot simply quotient out and expect to get a group with a finite Morley rank.

In this case, we let $Q = d(O_2(G))$ be the definable closure. Since $O_2(G)$ is solvable, so is Q. Now if S and R are arbitrary 2-subgroups of G, then $\overline{S} = SQ/Q$ and $\overline{R} = RQ/Q$ are 2-subgroups of $\overline{G} = G/Q$. Now, $\overline{G}$ is a group of lower Morley rank and degree than G, and so $\overline{S}$ and $\overline{R}$ are conjugate to subgroups of some Sylow 2-subgroup $\overline{P}$ of $\overline{G}$. Then, we can consider the full preimage P of $\overline{P}$ under the projection map. Since P is an extension of the solvable group Q by the solvable group $\overline{P}$, P is solvable. Hence, by the preceding theorem, if S and R are maximal, they are conjugate in $d(P)$.

In either case, we see that all Sylow 2-subgroups of G must be conjugate, unless $O_2(G)$ is trivial. $\square$

Next, we will rule out the possibility of finite Sylow 2-subgroups.

Theorem 3.10. *If G is a group of finite rank with a finite Sylow 2-subgroup S, then all Sylow 2-subgroups are conjugate.*

Proof. Let G be a counterexample of minimal rank and degree. Among the Sylow 2-subgroups that are not conjugate to S, pick T so that $|S \cap T|$ is maximal.

First, suppose that $S \cap T$ is trivial and pick central involutions $s \in Z(S)$, $t \in Z(T)$ – note that the centers are nontrivial because p-groups of finite Morley rank are nilpotent-by-finite. By Proposition 3.3, these are either conjugate or commute with a third involution u. If s and t are conjugate, say $s = x^{-1}tx$, then note that $x^{-1}Tx$ is a Sylow 2-subgroup with nontrivial intersection with S. Hence, S and $x^{-1}Tx$ are conjugate, since any nonconjugate Sylow 2-subgroup can only have trivial intersection with S, and so S and T must be conjugate.

If s and t commute with a third involution u, we consider the subgroups $\langle s, u\rangle$ and $\langle t, u\rangle$. Since s, t commute with u, we see that $\langle s, u\rangle$ and $\langle t, u\rangle$ are 2-groups, and so can be extended to Sylow 2-subgroups S_1 and T_1 of G. Now, S_1 has nontrivial intersection with S, and is therefore conjugate to S, say $g^{-1}Sg = S_1$. Since conjugation by g is an automorphism of G, it is easy to see that no Sylow 2-subgroup of G that is not conjugate to S_1 can have nontrivial intersection with S_1. So T_1 must be conjugate to S_1, and the same argument applied to T_1 shows that T is conjugate to S. Hence, S and T are conjugate, a contradiction.

So we can assume that $D = S \cap T$ is nontrivial. Since D is a nontrivial 2-subgroup, it cannot be normal in G as $O_2(G)$ was shown to be trivial. So $N = N_G(D)$ is a proper subgroup of G. Moreover, it should be clear that D is a proper subgroup of both S and T since Sylow groups are maximal with respect to inclusion. Recall that S and T are then nilpotent-by-finite being 2-subgroups of a group of finite Morley rank, and hence we have the strict inclusions $D \subsetneq N_S(D) = S_1 \subset N$ and $D \subsetneq N_T(D) = T_1 \subset N$.

Now let S_2 and T_2 be Sylow 2-subgroups of N containing S_1 and T_1 respectively. Since N is a proper subgroup of G, and is definable as the normalizer of a finite group, its Sylow 2-subgroups are conjugate. Hence, we can write $S_2 = x^{-1}T_2x$ for some $x \in N$. Now, let U be a Sylow 2-subgroup of G containing S_2. Then we see that $D \subsetneq S_1 \subset S_2 \cap S \subset U \cap S$, where we have the second inclusion because $S_1 \subset S$ by definition. Hence, since $U \cap S \supsetneq D$, we see that U must be conjugate to S. Similarly, we see that $U \cap x^{-1}Tx \supsetneq D$ and, exactly as before, we observe that conjugation is an automorphism of G, and so U must be conjugate to $x^{-1}Tx$. Hence, S is conjugate to T, which is a contradiction, completing the proof. □

To deal with infinite Sylow 2-subgroups, we will need a rather different approach. Henceforth, when we speak of 'nonconjugate' subgroups or a 'nonconjugate pair', we will be invariably referring to nonconjugate Sylow 2-subgroups.

Proposition 3.11. *If (P, Q) is a pair of nonconjugate Sylow 2-subgroups of G and H is a proper definable subgroup of G such that $P \cap H$ and $Q \cap H$ properly contain $P \cap Q$, then there is a nonconjugate pair (P_1, Q_1) such that P and P_1 are conjugate and $P_1 \cap Q_1$ properly contains $P \cap Q$. Moreover, either $P \cap H$ or $Q \cap H$ is contained in $P_1 \cap Q_1$.*

Proof. Since H is a proper subgroup of G, all Sylow 2-subgroups of H are conjugate. Let R be a Sylow 2-subgroup of H containing $P \cap H$ (since $P \cap H$ is a 2-subgroup of H, we can choose such an R). Since $Q \cap H$ can also be extended to a Sylow 2-subgroup of H, some conjugate $x^{-1}Rx$ of R must be a Sylow 2-subgroup of H containing $Q \cap H$. Now, let U be a Sylow 2-subgroup of G containing R.

If P is not conjugate to U, let $(P_1, Q_1) = (P, U)$, in which case we have $P \cap U \supset P \cap (P \cap H) \supsetneq P \cap Q$. If P is conjugate to U, let $(P_1, Q_1) = (x^{-1}Ux, Q)$, and so we have $x^{-1}Ux \cap Q \supset (Q \cap H) \cap Q \supsetneq P \cap Q$. □

We say that the pair P_1, Q_1 above is obtained by *gluing* P and Q in H. This process will be the driving force for our argument. The central difficulty now is that the Sylow 2-subgroups need not be definable. However, we can show that this must be the case in a counterexample by proving that all 2-tori in G are trivial.

To start, say we had a counterexample G to the theorem, with nonconjugate Sylow 2-subgroups P and Q. Using the preceding proposition, we can show that we may take the intersection to be nontrivial.

Proposition 3.12. *If G is a group of finite rank with nonconjugate Sylow 2-subgroups P and Q, we can find nonconjugate Sylow 2-subgroups P_1 and Q_1 with P_1 conjugate to P and $P_1 \cap Q_1$ nontrivial.*

Proof. Since P and Q are nilpotent-by-finite, and hence have nontrivial centers, we can pick involutions $x \in Z(P)$ and $y \in Z(Q)$. If x and y are conjugate, we have $x = g^{-1}yg$ for some $g \in G$. We certainly have that $P \subset C_G(x)$ and since $x \in Z(g^{-1}Qg)$, we also have $g^{-1}Qg \subset C_G(x)$. But $C_G(x)$ must be a proper definable subgroup of G since $O_2(G)$ is trivial, and so P and $g^{-1}Qg$ are conjugate since G was minimal with nonconjugate Sylow 2-subgroups. So P and Q are conjugate, a contradiction.

So x and y must commute with a third involution z, and we can let $H = C_G(z)$. For exactly the same reasons as above, H is a proper subgroup of G. If $P \cap Q$ is nontrivial, there is nothing to prove - so we can take $P \cap Q$ to be trivial. In this case, note that $x \in P \cap H$ and $y \in Q \cap H$ since x and y commute with z. So $P \cap H \supsetneq P \cap Q$ and $Q \cap H \supsetneq P \cap Q$ since x and y are nontrivial, allowing us to apply the previous proposition to get nonconjugate P_1, Q_1 with P_1 conjugate to P and $P_1 \cap Q_1 \supsetneq P \cap Q$, as required. □

Once we have a nonconjugate pair with nontrivial intersection, we can continuously increase the intersection.

Proposition 3.13. *Let P_1 and Q_1 be a nonconjugate pair with nontrivial intersection in a group G of finite rank. Then there are nonconjugate P_2 and Q_2 where $P_2 \cap Q_2 \supsetneq P_1 \cap Q_2$.*

Proof. Recall that P_1 and Q_1 are 2-subgroups of finite rank, and so are also nilpotent-by-finite. Hence, we have the normalizer condition: $N_{P_1}(P_1 \cap Q_1) > P_1 \cap Q_1$ and $N_{Q_1}(P_1 \cap Q_1) > P_1 \cap Q_1$. Now, let H' be the subgroup generated by all involutions in the center $Z(P_1 \cap Q_1)$, and let $H = N_G(d(H'))$. Clearly, H' is abelian of exponent 2, and we argue that $d(H')$ is therefore also abelian of exponent 2. To see this, note that the subgroup of G formed by commuting elements of order 2 is a definable subgroup containing H', and so also contains $d(H')$. But then, if $d(H')$ is abelian of exponent 2, we must have $H \subsetneq G$ since we assumed the maximal normal 2-subgroup of G was trivial.

Now, if $p \in P_1$ normalizes $P_1 \cap Q_1$, we see that it must also normalize the center $Z(P_1 \cap Q_1)$, and therefore the subgroup generated by involutions in this center. So $d(N_{P_1}(P_1 \cap Q_1)$ normalizes $d(H')$, from which it follows that H contains $N_{P_1}(P_1 \cap Q_1)$. So we have $P_1 \cap H \supset N_{P_1}(P_1 \cap Q_1) \supsetneq P_1 \cap Q_1$ and, by an identical argument, $Q_1 \cap H \supset N_{Q_1}(P_1 \cap Q_1) \supsetneq P_1 \cap Q_1$. So all the conditions to glue P_1 and Q_1 in H are satisfied, and the resulting pair is precisely the pair P_2, Q_2 desired. $\square$

And with this, we can force a contradiction if we assume that G contains a nontrivial 2-torus.

Proposition 3.14. *If G is a group of finite Morley rank with nonconjugate Sylow 2-subgroups P and Q, then G cannot contain a nontrivial 2-torus.*

Proof. Suppose G contained nontrivial 2-tori. Then we can choose a nonconjugate pair with nontrivial intersection P, Q where the maximal torus $T(P)$ of P is nontrivial. Recall from the discussion following Definition 3.6 that there is a constant n so that $[N_G(T) : C_G(T)] < n$ for any 2-torus T of G.

Note that, in the previous proposition, we get P_1 conjugate to P, so P_1 will contain a nontrivial 2-torus if P does. Hence, we can apply the preceding proposition to get P_1, Q_1 with $P_1 \cap Q_1 \supsetneq P \cap Q$. We can keep applying this theorem until we get, say, $|P_2 \cap Q_2| > n^2$. Call this latter intersection D, and let $T = T(P_2)$ and $R = T(Q_2)$ be the maximal 2-tori of P_2 and Q_2 respectively.

Now, note that $D \subset N_G(T), N_G(R)$ since conjugation by an element of D is an automorphism of P_2 and Q_2, and so must fix their (unique) maximal 2-tori. Moreover, $C_D(T) = D \cap C_G(T)$ and $C_D(R) = D \cap C_G(R)$ have index at most n in D since $D/C_D(T)$ and $D/C_D(R)$ imbed in $N_G(T)/C_G(T)$ and $N_G(R)/C_G(R)$ respectively. Since D has order greater than n^2, these cannot be disjoint and so must have nontrivial intersection, say $X = C_D(T) \cap C_D(R)$.

Since D normalizes T and R, and $C_D(T)$ and $C_D(R)$ are normal in the normalizers of T and R, they must be normal in D. Hence, their intersection is normal in D. Since X is nontrivial, we see that $X \cap Z(D)$ is also nontrivial by an earlier lemma. This intersection is a 2-group, so it contains an involution z. Let $H = C_G(z)$ and note that $T, R, D \subset H$.

First, suppose that R is nontrivial. Since $O_2(G)$ is trivial, H must be a proper subgroup of G. Since H is definable, all of its Sylow 2-subgroups are then conjugate – in particular, the Sylow 2-subgroups $P_2 \cap H$ and $Q_2 \cap H$, and so their maximal 2-tori, must be conjugate in H. So we can write $R = h^{-1}Th$ for some $h \in H$. Since we then have $R \subset h^{-1}P_2h$ as a maximal 2-torus, we see that $h^{-1}P_2h$ normalizes R. But the normalizer of a 2-torus is definable, so $h^{-1}P_2h$ and Q_2 are contained in the group $N_G(R)$ of finite rank. Moreover, $N_G(R)$ is a proper subgroup of G since R is a 2-group and therefore cannot be normal in G. Since G was minimal with nonconjugate Sylow 2-subgroups, this shows that Q_2 and $h^{-1}P_2h$, and hence Q_2 and P_2 are conjugate, a contradiction.

So we can take R to be trivial, in which case Q_2 is a definable nilpotent subgroup of G. Let $Z = Z(Q_2)^\circ$, and note that since $z \in Q_2$, we easily have that $Z \subset C_G(z) = H$. Since Q_2 is infinite, so is $Z(Q_2)$ and hence Z is nontrivial. Now let S be a Sylow 2-subgroup of H containing Z. Again, since Sylow 2-subgroups of H are conjugate, we see by extending T to a Sylow 2-subgroup of H that T is conjugate to some subgroup of S, say $x^{-1}Tx \subset S$. Since Z is connected in S, it must be contained in S°, the maximal connected subgroup of S. Moreover, every element of $x^{-1}Tx$ commutes with every element of Z.

Since Z is a nontrivial 2-group, $C_G(Z)$ must be a proper subgroup of G as $O_2(G)$ was trivial. So the Sylow 2-subgroups of $C_G(Z)$ are conjugate, and clearly $Q_2 \subset C_G(Z)$. But we have just shown that $x^{-1}Tx \subset C_G(Z)$, so Q_2 must contain a

2-torus conjugate to $x^{-1}Tx$, which contradicts our assumption that R was trivial. Thus, G cannot contain nontrivial 2-tori. □

The power we have gained in this last proposition is enormous: now, the Sylow 2-subgroups of G are definable, have bounded exponent and are nilpotent – with this, we are almost done. Recall that a proper definable subgroup R of infinite index in a nilpotent group P satisfies $rk(N_P(R)) > rk(R)$. So, say we have a nonconjugate pair P_2, Q_2 of Sylow 2-subgroups of G and suppose that $P_2 \cap Q_2$ had infinite index in P_2 and Q_2. Then, we must have $rk(N_{P_2}(P_2 \cap Q_2)) > rk(P_2 \cap Q_2)$, and similarly $rk(N_{Q_2}(P_2 \cap Q_2)) > rk(P_2 \cap Q_2)$. In particular, $N_G(P_2 \cap Q_2) \cap P_2$ and $N_G(P_2 \cap Q_2) \cap Q_2$ both properly contain $P_2 \cap Q_2$. So we can repeatedly glue P_2 and Q_2 in $N_G(P_2 \cap Q_2)$ until the resulting $P_2' \cap Q_2'$ has finite index in, say, P_2'.

So, we see that we may as well assume that $P_2 \cap Q_2$ has finite index in P_2. Hence, $P_2^\circ = (P_2 \cap Q_2)^\circ$. Now suppose $[P_2 \cap Q_2 : P_2^\circ] < [P_2 : P_2^\circ]$; then, using the same process, we can increase this index by increasing the intersection $P_2 \cap Q_2$ in P_2, until we have $[P_2 \cap Q_2 : P_2^\circ] = [P_2 : P_2^\circ]$. Since $P_2 \cap Q_2 \subset P_2$, this gives us equality and so $P_2 \subset Q_2$, which contradicts the maximality of P_2 as a Sylow subgroup, and proves the theorem.

Acknowledgements. Thanks to my mentor, Jonny Stephenson, for actually working through this paper and to Prof. May for the REU. I would also be remiss not to mention that most of the proofs follow Borovik and Nesin's book, listed below.

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