A PRIMER ON DE RHAM COHOMOLOGY AND SPECTRAL SEQUENCES

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Abstract. Here we present a computation of the de Rham cohomology of various circle bundles over S^2, making use of the machinery of spectral sequences. In the spirit of exposition, to do so we develop the machinery of $\mathbb{R}$-valued de Rham cohomology, spectral sequences, and the Euler class. Some familiarity with the basics of algebraic topology and differential geometry are assumed, but not entirely required, and appropriate citations are given to any outside material. In general the latter half of the paper is less accessible and self-contained, but the first half should be readily accessible to a student with only background in the basics of analysis.

Contents

1. Introduction to Differential Forms and de Rham Cohomology

We begin by developing the de Rham complex on $\mathbb{R}^n$. We then move on to compute a few examples that serve to illustrate how studying the differential forms on a subset of $\mathbb{R}^n$ can convey data about the topology of the subset and discuss the relevance of the computations to de Rham cohomology. It is necessary to get a few definitions out of the way before motivation can become clear, so we ask that the reader bears with us as we develop the dryer earlier parts of the theory.

Let $x_1, x_2, \ldots, x_n$ be the standard choice of coordinates on $\mathbb{R}^n$.

Definition 1.1. We define the algebra Ω^* over $\mathbb{R}$ to be the algebra generated by $1, dx_1, dx_2, \ldots, dx_n$ and subject to the relation: $dx_i dx_j = -dx_i dx_j$.

Note that the anti-commutativity above implies that $(dx_i)^2 = 0$.

Date: August 19, 2013.

Definition 1.2. We define the algebra of C^∞ differential forms on $\mathbb{R}^n$ to be the space: $\{C^\infty$ functions on $\mathbb{R}^n\} \otimes_{\mathbb{R}} \Omega^*$.

For notational convenience we will refer to this space as $\Omega^*(\mathbb{R}^n)$. When, in the subsequent parts of this section, we generalize the notion of differential forms to smooth manifolds, we will refer to the space of differential forms on M by $\Omega^*(M)$. Further, it is useful to endow this algebra with a grading. We denote by $\Omega^q(\mathbb{R}^n)$ the space of q-forms on $\mathbb{R}^n$, or the forms of the form

$$\sum_{i=1}^{n} f_i dx_{i1} dx_{i2} \dots dx_{iq}$$

so that

$$\Omega^*(\mathbb{R}^n) = \bigoplus_{q=0}^{n} \Omega^q(\mathbb{R}^n)$$

Finally we define the differential operator on this complex. We define the differential $d : \Omega^q(\mathbb{R}^n) \to \Omega^{q+1}(\mathbb{R}^n)$ on monic forms as

$$d\left(f dx_{i1} dx_{i2} dx_{i3} \dots dx_{iq}\right) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j \wedge (dx_{i1} dx_{i2} \dots dx_{iq})$$

and extended linearly on general forms. Above we use the index i only to indicate that the Ω^* component may not be sequentially ordered. Henceforth we shall adapt the convention that $\sum\limits_{i=1}^{n} f_i dx_{i1} dx_{i2} \dots dx_{iq}$ may be denoted by $\sum f_I dx_I$ for convenience. We also will freely make use the notion of wedge product that grants a ring structure to the forms. Let $\alpha = \sum f_I dx_I$, $\beta = \sum g_J dx_J$, then

$$\alpha \wedge \beta = \sum_{I,J} f_I g_J dx_I \wedge dx_J = (-1)^{\deg(\alpha)\deg(\beta)} \beta \wedge \alpha$$

Here the final equality is from the commutation relation on the elements of Ω^*.

One thing to note here is that d is an antiderivation with respect to the wedge product, that is to say:

Proposition 1.3. $d(\alpha \wedge \beta) = d(\alpha) \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge d(\beta)$

Proof. On monic forms we have that, taking $\alpha = f_I dx_I, \beta = g_J dx_J$,

$$\begin{aligned} d(\alpha \wedge \beta) &= d(f_I g_J dx_I \wedge dx_J) = d(f_I g_J) dx_I dx_J \\ &= d(f_I) g_J dx_I dx_J + f_I d(g_J) dx_I dx_J \\ &= d(\alpha) \wedge \beta + (-1)^{deg(\alpha)} \alpha \wedge d(\beta) \end{aligned}$$

□

Finally we have perhaps the most important elementary property of the differential operator:

Proposition 1.4. $d^2 = 0$

Proof. Let α be as above:

$$d(\alpha) = d(f_I) dx_I + f_I d(dx_I) = d(f_I) dx_I = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \wedge dx_I$$

The next step makes use of the equality of mixed partials

$$d^2(\alpha) = d(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \wedge dx_I) = \sum_{j=1}^{n}\sum_{i=1}^{n} \frac{\partial^2 f_I}{\partial x_i \partial x_j}(dx_j dx_i) \wedge dx_I$$

Commuting dx_i, dx_j we have our result.

□

It is important to make note of the situation we now find ourselves in. We have a graded algebra with a linear differential operator that increments the grading and is such that $d^2 = 0$. We are left with a diagram of the form

$$0 \xrightarrow{d} \Omega^0(\mathbb{R}^n) \xrightarrow{d} \Omega^1(\mathbb{R}^n) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n(\mathbb{R}^n) \xrightarrow{d} 0$$

such that the image of each map $d : \Omega^q(\mathbb{R}^n) \to \Omega^{q+1}(\mathbb{R}^n)$ is contained within the kernel of its successor $d : \Omega^{q+1}(\mathbb{R}^n) \to \Omega^{q+2}(\mathbb{R}^n)$. At this point it is likely very tempting to ask why we have created this rather strange (at first glance at least) object, and what its physical relevance is. Fortunately, we are also at the stage where we can give some kind of answer to that question.

We call elements of $\Omega^q(\mathbb{R}^n)$ that are in the kernel of d the *closed* q-$forms$ on $\mathbb{R}^n$ and the elements that are in the image of d the *exact* q-$forms$ on $\mathbb{R}^n$. To make the exposition easier, we consider the case of $\mathbb{R}^2$, and examine what closed and exact forms represent on this space. The closed 0-forms on $\mathbb{R}^2$ are constant functions (there are obviously no exact 0-forms), and for dimensional reasons all 2-forms on $\mathbb{R}^2$ are of course closed (and it is fairly easy to show that they are all exact as well). But interesting concerns come up when one looks at the meaning behind a closed one form on $\mathbb{R}^2$. Such a form is given by $\alpha = fdx + gdy$ such that: $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 0$. The exact forms satisfy this condition by the equality of mixed partials, and every form which satisfies this equation is exact as we define

$$h = \int_0^x f(u,0)du + \int_0^y g(x,v)dv$$

and we have:

$$dh = f(x,0)dx + g(x,y)dy + \left(\int_0^y \frac{\partial g}{\partial x}(x,u)du\right) dx$$

$$dh = f(x,y)dx + g(x,y)dy$$

$$dh = \alpha$$

as desired.

Now repeating this experiment in the punctured plane leads to a surprisingly different result. In $\mathbb{R}^2 - \{0\}$ we have the form

$$d\theta = \frac{ydx - xdy}{x^2 + y^2}$$

which is closed but not exact. This is of course the derivative of $arctan(\frac{y}{x})$, which measures the angle of a vector (x, y) in the upper or lower half planes (but is not globally defined). One might notice at this point that there is nothing special about removing the origin from $\mathbb{R}^2$, one might remove any point, or any collection of points, and around each hole one would have such a non-trivial form. So we see that at least in this crude way, studying the classes solutions of these "god-given differential equations" gives us an understanding of the topology of the region in question. Because we are uninterested in the difference between two solutions that differ by adding a trivial solution (a q-form that solves the equation because it is already the exterior derivative of a q-1 form), we quotient by the exact forms and form equivalence classes of forms.

This gives rise to the notion of the de Rham cohomology groups of $U \subseteq \mathbb{R}^n$, denoted $H^q_{DR}(U)$.

Definition 1.5. The qth de Rham cohomology group of $U \subseteq \mathbb{R}^n$ is given by

$$H^q_{DR}(U) = \ker(d : \Omega^q(U) \to \Omega^{q+1}(U))/\operatorname{im}(d : \Omega^{q-1}(U) \to \Omega^q(U))$$

As such, an element of $H^q_{DR}(U)$ is an equivalence class of closed q-forms on U that differ by exact q-forms on U. If we have some closed form ω we will reference its associated equivalence class in $H^q_{DR}(U)$ by $[\omega]$.

2. Cohomology of Manifolds and Homotopy Invariance

In this section we generalize the notion of de Rham cohomology on $\mathbb{R}^n$ to C^∞ manifolds. We also present an introductory look at the Mayer-Vietoris sequence, and finally prove the homotopy invariance of de Rham cohomology. We end the section by using the notion of homotopy invariance to compute the de Rham cohomology groups of a few manifolds.

As a general principle, we would like our cohomology theory to interact 'nicely' with maps in the category appropriate to the objects being studied. In this case, given a smooth map $f : \mathbb{R}^n \to \mathbb{R}^m$, we have a pullback map on C^∞ functions $f^* : \Omega^0(\mathbb{R}^m) \to \Omega^0(\mathbb{R}^n)$ that is precomposition by f:

$$f^*(g) = g \circ f$$

So this appears to be a good place to start, but of course we want some way to extend this notion to forms of any degree. Additionally, we want $f^* \circ d = d \circ f^*$ such that f^* preserves closedness and exactness of forms. These conditions force the definition:

Definition 2.1. The map on cohomology f^* induced by a smooth function f is given by:

$$f^*(g_I \, dy_{i_1} \dots dy_{i_q}) = g_I \circ f \; df_{i_1} \dots df_{i_q}$$

Where $f_{i_j} = y_{i_j} \circ f$. That with this definition f^* commutes with d is a clear application of the chain rule.

We now extend our theory to smooth manifolds. For the basics of manifold theory we refer the reader to [2]. We define a differential form ω on a manifold M to be a collection of forms ω_{A_i} on each of the open sets in the atlas of M, such that if $A_i \cap A_j$ includes into A_i by the map i, and A_j by the map j, then $i^*\omega_{A_i} = j^*\omega_{A_j}$. With this definition the construction of differential forms on the manifold proceeds as in section one, keeping the appropriate definition of "partial derivative" in mind. For more detail we refer the reader to the above reference, and [1].

It is now time to construct the Mayer-Vietoris sequence, a useful bit of theory that facilitates the computation of the de Rham cohomology of a manifold by breaking it into open sets. Here we quickly run through a proof that relies only on the observations we have made thus far concerning the elementary properties of the de Rham complex. In the next chapter we develop a little bit more algebraic sophistication and give a more general proof of Mayer-Vietoris.

Given a decomposition of a manifold M into open sets U and V we have the chain of maps

$$M \leftarrow U \coprod V \leftleftarrows U \cap V$$

where the maps are the appropriate inclusions. We denote the inclusion of V into $U \coprod V$ by j and the appropriate inclusion of U by i. We also denote by k the inclusion of $U \coprod V$ into M. Changing the spaces to their de Rham complexes and the appropriate maps to pullbacks we end up with the chain

$$\Omega^*(M) \to \Omega^*(U) \oplus \Omega^*(V) \rightrightarrows \Omega^*(U \cap V)$$

Augmenting the diagram by changing the final map i^*, j^* to the map $\delta = j^* - i^*$ we propose that one obtains a short exact sequence of chain complexes

$$0 \to \Omega^*(M) \to \Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V) \to 0$$

which is the eponymous Mayer-Vietoris sequence.

Proposition 2.2. *The above sequence is exact.*

Proof. This is obvious for the first map; for the second map, we note that a global form on M would be such that $k^*\omega$ would of course vanish under difference taking along the boundary. In the last step, suppose we have some $\omega \in \Omega^*(U \cap V)$. Take a partition of unity subordinate to the cover U, V and call the two components τ_U, τ_V. Then $-\tau_V \omega$ is a form on U and $\tau_U \omega$ is a form on V such that: $\alpha = (-\tau_V \omega, \tau_U \omega)$ is a form on $U \coprod V$ such that $\delta(\alpha) = \omega$.

□

In general it is true that a short exact sequence of chain maps gives rise to a long exact sequence in cohomology. We show that the Mayer-Vietoris sequence gives rise to a long exact sequence in cohomology in the following proposition.

Proposition 2.3. *The Mayer-Vietoris sequence gives rise to a long exact sequence in cohomology of the same name, as indicated by the diagram below:*

$$\begin{tikzcd}
\cdots \arrow[r] & H^i(M) \arrow[r] & H^i(U) \bigoplus H^i(V) \arrow[r] & H^i(U \cap V) \arrow[dll, "d^*"] & \\
& H^{i+1}(M) \arrow[r] & H^{i+1}(U) \bigoplus H^{i+1}(V) \arrow[r] & H^{i+1}(U \cap V) \arrow[r] & \dots
\end{tikzcd}$$

Proof. Consider the commutative diagram:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \Omega^{q+1}(M) \arrow[r, "k^*"] \arrow[u, "d"] & \Omega^{q+1}(U) \bigoplus \Omega^{q+1}(V) \arrow[r, "\delta"] \arrow[u, "d"] & \Omega^{q+1}(U \cap V) \arrow[r] \arrow[u, "d"] & 0 \\
0 \arrow[r] & \Omega^{q}(M) \arrow[r, "k^*"] \arrow[u, "d"] & \Omega^{q}(U) \bigoplus \Omega^{q}(V) \arrow[r, "\delta"] \arrow[u, "d"] & \Omega^{q}(U \cap V) \arrow[r] \arrow[u, "d"] & 0 \\
 & \vdots \arrow[u, "d"] & \vdots \arrow[u, "d"] & \vdots \arrow[u, "d"] &
\end{tikzcd}$$

Take some $\omega \in \Omega^q(U \cap V)$ such that $d\omega = 0$. By exactness there exists some $\alpha \in \Omega^q(U) \bigoplus \Omega^q(V)$ such that $\delta(\alpha) = \omega$. By commutativity, $\delta(d\alpha) = 0$ so there exists some $\beta \in \Omega^{q+1}(M)$ such that $k^*\beta = d\alpha$, and by the commutativity of the diagram this β is closed. A bit more diagram chasing shows that this β is independent of the choice of α, but that is left to the reader. So we define this β to be $d^*(\omega)$.

□

Example 2.4. We briefly digress here to show one example use of the Mayer-Vietoris sequence, a computation of: $H^*(\mathbb{R}^2 - \{0\})$. Assume the results of the sample computations that we did earlier. Recall that we have shown that: $H^1(\mathbb{R}^2), H^2(\mathbb{R}^2) = 0$. Note that in general for some manifold M, as there are no exact 0-forms and the only closed 0-forms are functions that are constant on each connected component of M, we have that: $H^0(M) = \mathbb{R}^k$ where k is the number of connected components of M. Returning to the punctured plane, let U be the region parameterized by $r \in (0, \infty)$, $\theta \in (\frac{\pi}{4}, \frac{7\pi}{4})$ and V be parameterized by $r \in (0, \infty)$, $\theta \in (\frac{3\pi}{4}, -\frac{3\pi}{4})$. Note that U, V are diffeomorphic to $\mathbb{R}^2$, and that $U \cap V$ has two connected components. Let $\phi : X \to Y$ be a diffeomorphism, then if $H^i(Y)$ is nontrivial then $H^i(X)$ is as well as ϕ^* preserves the closedness and dimension of forms. Therefore this set up gives us the diagram

$$\begin{tikzcd}
 & H^2(\mathbb{R}^2 - \{0\}) \arrow[r] & H^2(U) \bigoplus H^2(V) \arrow[r] & H^2(U \cap V) \arrow[r] & 0 \\
 & H^1(\mathbb{R}^2 - \{0\}) \arrow[r] & H^1(U) \bigoplus H^1(V) \arrow[r] & H^1(U \cap V) \arrow[ull, "d^*"] & \\
0 \arrow[r] & H^0(\mathbb{R}^2 - \{0\}) \arrow[r] & H^0(U) \bigoplus H^0(V) \arrow[r] & H^0(U \cap V) \arrow[ull, "d^*"] &
\end{tikzcd}$$

With the above facts in mind this is just:

$$\begin{tikzcd}
 & H^2(\mathbb{R}^2 - \{0\}) \arrow[r] & 0 \bigoplus 0 \arrow[r] & H^2(U \cap V) \arrow[r] & 0 \\
 & H^1(\mathbb{R}^2 - \{0\}) \arrow[r] & 0 \bigoplus 0 \arrow[r] & 0 \arrow[ull, "d^*"'] & \\
0 \arrow[r] & \mathbb{R} \arrow[r] & \mathbb{R} \bigoplus \mathbb{R} \arrow[r] & \mathbb{R}^2 \arrow[ull, "d^*"'] &
\end{tikzcd}$$

So we have that $H^1(\mathbb{R}^2 - \{0\}) = \mathbb{R}$, $H^0(\mathbb{R}^2 - \{0\}) = \mathbb{R}$, $H^2(\mathbb{R}^2 - \{0\}) = 0$.

We close this section with a proof of the homotopy invariance of de Rham cohomology. To accomplish this, we prove a generalization of the Poincare Lemma:

Theorem 2.5. *If M is a differentiable manifold:*

$$H^*(M \times \mathbb{R}^1) \cong H^*(M)$$

The Poincare Lemma states the less powerful result that:

$$H^*(\mathbb{R}^n \times \mathbb{R}^1) \cong H^*(\mathbb{R}^n)$$

Proof. Consider the diagram:

$$\begin{tikzcd}
M \times \mathbb{R}^1 \arrow[d, shift left, "\pi"] \\
M \arrow[u, shift left, "s"]
\end{tikzcd}$$

Where s is the zero section. From this we get the diagram:

$$\begin{tikzcd}
\Omega^*(M \times \mathbb{R}^1) \arrow[d, shift left, "s^*"] \\
\Omega^*(M) \arrow[u, shift left, "\pi^*"]
\end{tikzcd}$$

We want to prove that these induced maps are isomorphisms. As $\pi \circ s$ is already the identity, we know that: $(\pi \circ s)^* = s^* \circ \pi^* = \mathbb{1}$; however $(s \circ \pi)^*$ is not the identity, at least looking only at forms, as it annihilates all of the fiber information of a form. It is thus our hope that it proves to be the identity at least in cohomology.

We thus want to prove that $\pi^* \circ s^*$ is chain-homotopic to the identity, or that there exists some map $\pi_* : \Omega^*(M \times \mathbb{R}^1) \to \Omega^{*-1}(M \times \mathbb{R}^1)$ such that

$$\mathbb{1} - \pi^* s^* = d\pi_* - \pi_* d$$

So that $\pi^* s^*$ induces the identity in cohomology. Let α denote a form on M. We note that all forms on $M \times \mathbb{R}$ are a combination of

$$(\pi^* \alpha) f(x, t)$$

which we call forms of type one, and

$$(\pi^* \alpha) f(x, t) dt$$

which we call forms of type two. We define π_* by

$$\pi_*\psi = \begin{cases} 0 & : \psi \text{ is of type one.} \\ (\pi^*\alpha)\int\limits_0^t f(x,t)dt & : \psi \text{ is of type two.} \end{cases}$$

We now preform a routine verification to ensure that π_* is in fact the desired homotopy.

On forms of type one:

$$\psi = (\pi^*\alpha)f(x,t)$$

$$(\mathbb{1} - \pi^*s^*)(\psi) = (\pi^*\alpha)f(x,t) - (\pi^*\alpha)f(x,0)$$

$$\begin{align} (d\pi_* - \pi_*d)(\psi) &= (d\pi_* - \pi_*d)((\pi^*\alpha)f(x,t)) \tag{2.6} \\ &= -\pi_*(d(\pi^*\alpha) + (-1)^q(\pi^*\alpha)d(f(x,t))) \tag{2.7} \\ &= -\pi_*((\pi^*d\alpha) + (-1)^q(\pi^*\alpha)\left(\frac{\partial f(x,t)}{\partial x}dx + \frac{\partial f(x,t)}{\partial t}dt\right) \tag{2.8} \\ &= (-1)^{q+1}(\pi^*\alpha)\left(\int_0^t \frac{\partial f(x,t)}{\partial t}dt\right) \tag{2.9} \\ &= (-1)^{q+1}(\pi^*\alpha)(f(x,t) - f(x,0)) \tag{2.10} \end{align}$$

as desired.

On forms of type two:

$$\psi = (\pi^*\alpha)f(x,t)dt$$

And as $ds_t = 0$ as $s_t = 0$:

$$(\mathbb{1} - \pi^*s^*)(\psi) = (\pi^*\alpha)f(x,t)dt - \pi^*(\alpha f(x,0)ds_t) = \psi$$

$$\begin{align} & \tag{2.11} \\ (d\pi_* - \pi_*d)\psi &= (d\pi_* - \pi_*d)(\pi^*\alpha)f(x,t)dt \notag \\ &= d\left((\pi^*\alpha)\int_0^t f(x,t)dt\right) \tag{2.12} \\ &- \pi_*\left(\pi^*(d\alpha)f(x,t)dt + (-1)^{q+1}(\pi^*\alpha)\frac{\partial f}{\partial x}dxdt\right) \tag{2.13} \\ &= (\pi^*d\alpha)\int_0^t f(x,t)dt + (-1)^{q+1}\left(\pi^*\alpha)(f(x,t)dt + dx\int_0^t \frac{\partial f}{\partial x}dt\right) \tag{2.14} \\ &- (\pi^*d\alpha)\int_0^t f(x,t)dt + (-1)^q(\pi^*\alpha)dx\int_0^t \frac{\partial f}{\partial x}dt \tag{2.15} \\ &= (-1)^{q+1}(\pi^*\alpha)f(x,t)dt \tag{2.16} \end{align}$$

and so we are done.

□

Corollary 2.17. *Homotopic maps induce the same map on cohomology.*

Proof. Let $f, g : M \to N$ be maps and $F : M \times \mathbb{R} \to N$ be the homotopy between them. Let s_0 be the 0-section and s_1 be the 1-section (the map taking $(x) \in M$ to $(x, 1) \in M \times \mathbb{R}$). Then

$$f = F \circ s_0$$

$$g = F \circ s_1$$

so:

$$f^* = s_0^* \circ F^*$$

$$g^* = s_1^* \circ F^*$$

The proof above holds to show that s_1^* inverts π^* with only a small modification of π_*. As π^* is an isomorphism, we have that $s_1^* = s_0^*$ and thus

$$f^* = g^*$$

so we are done.

□

So finally we have that:

Corollary 2.18. *Two manifolds of the same homotopy type have the same de Rham cohomology.*

Amongst a plethora of other results, one should note that this implies that $H^*(\mathbb{R}^n) = H^*(\{0\})$ and therefore contractible spaces have the cohomology of a point, that is to say $\mathbb{R}$ in dimension 0 and 0 in all other dimensions. More generally, that if a q-dimensional manifold M does not have $H^i(M) = 0$ for $i > p$ then M cannot be homotopic to a manifold of dimension p. This allows us to immediately conclude that a higher cohomology group of a manifold is trivial by demonstrating a deformation retract onto an appropriate subspace.

3. Spectral Sequences: Exact Couples and Filtered Complexes

Spectral sequences are an incredibly powerful tool that aid in the computation of many important invariants from algebraic topology, including homology, cohomology, and homotopy groups. Here we develop the machinery of spectral sequences on double complexes, and demonstrate applications to a generalization of Mayer-Vietoris sequence that establishes an isomorphism between the de Rham cohomology of a manifold, and another invariant of the manifold, called the Cech cohomology.

Following the lead of Bott and Tu [1], the homological algebra involved is done through exact couples.

Definition 3.1. Exact couples are exact sequences of abelian groups of the form

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & B \arrow[ul, "k"] & \end{tikzcd}$$

where i, j, k are group homomorphisms, and $ik = ji = kj = 0$.

From an exact couple we can obtain what we call a derived couple through the following procedure. We define: $d : B \to B$ given by $d = j \circ k$ to be a differential operator on B, such that we can take $H(B)$ with respect to this operator. We define the derived couple to be

$$\begin{tikzcd}
i(A) \arrow[rr, "i'"] & & i(A) \arrow[dl, "j'"] \\
& H(B) \arrow[ul, "k'"] &
\end{tikzcd}$$

Where $i'a' = i'(ia) = i^2(a)$ for $a' \in i(A)$ and $a \in A$. If $a' \in i(A)$, then $j'a' = [ja]$, where $a' = ia$, and $[ja]$ is the homology class of ja. Finally, k' is defined as follows: if $[b] \in H(B)$ then we have that $db = jkb = 0$ so by exactness $kb = ia$ for some $a \in A$. So we define $k'[b] = kb = ia \in i(A)$. It is a routine verification to show that this derived couple is exact and that the map j is well-defined. We leave these to the reader.

In general we treat a differential complex C with a differential D that increments the grading of the complex C, $C = \bigoplus_{k \in \mathbb{Z}} C^k$, $D : C^k \to C^{k+1}$ (just as we have in the de Rham complex). We naturally define a sub complex of C to be some subgroup K of C such that $DK \subseteq K$.

Definition 3.2. We define a sequence of subcomplexes $C = K_0 \supset K_1 \supset K_2 \supset \ldots$ to be a filtration on the complex C.

Defining $A = \bigoplus_{p \in \mathbb{Z}} K_P$, $i : A \to A$ to be the inclusion that decrements grading, and we define B by the short exact sequence

$$0 \longrightarrow A \xrightarrow{i} A \xrightarrow{j} B \longrightarrow 0$$

which of course yields a long exact sequence in cohomology

$$\ldots \longrightarrow H^*(A) \xrightarrow{i_*} H^*(A) \xrightarrow{j_*} H^*(B) \xrightarrow{k_*} H^{*+1}(A) \longrightarrow \ldots$$

which gives us our exact couple.

A filtered complex is said to have finite length if for some $l \in \mathbb{N}$, $K_l \neq 0$ but $K_n = 0$ for $n > l$. It can be shown that if we have a filtered complex of length l, then the associated groups A_i stabilize for $i = l + 1$ as do the corresponding B_i.

Before we move to the next section we would like to establish some terminology. We call a sequence of differential groups E_i where each E_{i+1} is the cohomology of E_i a *spectral sequence*. In this sequence the E_i are called the *pages* of the sequence, and are equal the derived B_i from the couple in which $B_1 = H(B)$ and $d = j \circ k$ as the differential operator.

4. Spectral Sequences: Double Complexes

Suppose we have a double complex of the form $K = \bigoplus_i A_i = \bigoplus_{p,q} K^{p,q}$ with the filtration

$$A_i = \oplus_{p \geq i} \oplus_{q \geq 0} K^{p,q}$$

and differential operators:

$$\delta : K^{p,q} \to K^{p+1,q},$$

$$d : K^{p,q} \to K^{p,q+1}$$

on the rows and columns respectively, and on the single complex

$$K = \oplus_i K^i = \oplus_i \oplus_{p+q=i} K^{p,q}$$

given by

$$D = \delta + (-1)^p d$$

We note that the above single complex is obtained by summing along the diagonals of the double complex. This gives us the double complex A where the degrees of an element of A are given by an element's filtration degree plus its degree in the single complex $K = \bigoplus K^i$. Finally, we make A into a single complex by summing along the diagonals as above to give rise to: $\bigoplus_k A^k = A$. Where elements of each A^k have bidegree k (filtration degree plus bidegree in K) in A. We have the natural inclusion map

$$i : A^k \cap K_{p+1} \to A^k \cap K_p$$

This construction of course gives us the associated graded group $B = \bigoplus K_p/K_{p+1}$, where K_p/K_{p+1} represents a column of the double complex, which of course inherits the differential $D = (-1)^p d$ from the differential on the complex K. But this means that taking cohomology as we did in the general filtered case leaves us with

$$E_1 = H_D(B) = H_d(K),$$

the d-cohomology of the columns.

Now we attempt to determine concretely what the differential on this E_1 page is. We know that $d_1 = j_1 \circ k_1$. Further we have from the machinery of exact couples the sequence of short exact sequences

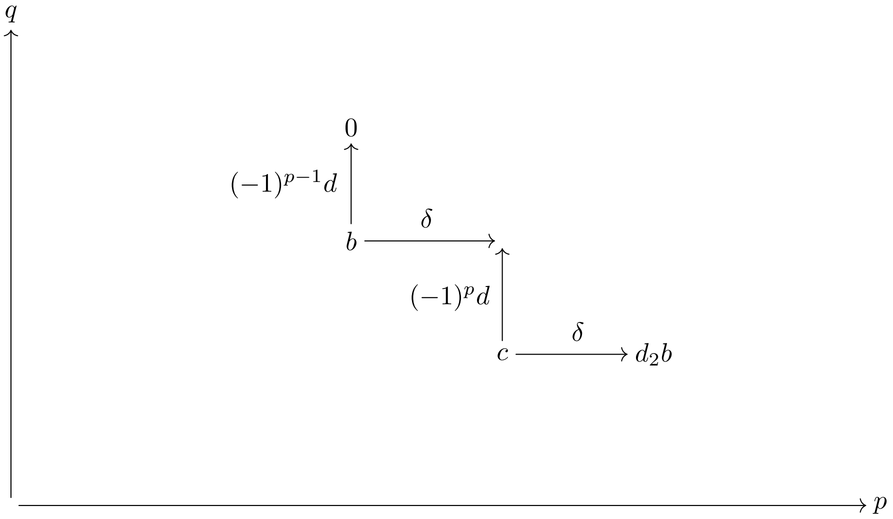
$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & A^{k+1} \cap K_{p+1} \arrow[r] \arrow[u] & A^{k+1} \cap K_p \arrow[r] \arrow[u] & B^{k+1} \cap K_p/K_{p+1} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & A^k \cap K_{p+1} \arrow[r] \arrow[u] & A^k \cap K_p \arrow[r] \arrow[u] & B^k \cap K_p/K_{p+1} \arrow[r] \arrow[u] & 0 \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

And further we know that k_1 is the connecting homomorphism that gives us our long exact sequence, which we compute by picking a representative $a \in A^k \cap K_p$ for the cohomology class $[a] \in B^k \cap K_p/K_{p+1}$, then computing $Da = \delta(a)$ as $[a]$ lives under H_d, then we invert the map i which we are guaranteed by the exactness and commutativity of the diagram. This implies that $k_1([a]) = [\delta(a)]$, and so $(j_1 \circ k_1) = \delta$.

So we have that $E_2 = H_\delta H_d(K)$. Now we wish to find d_2 in a way that generalizes to the further pages of the complex. As on the further pages of the complex we are dealing with anywhere from one to three different notions of equivalence class, we denote the equivalence class generated by ω on the E_n page by $[\omega]_n$. We note that if $b \in E_2$ then $db = 0$ and $\delta(b) = (-1)^p dc$ for some c in column p. Now we know that we should have $d_2 = j_2 k_2$ and further: $j_2 k_2 [b]_2 = j_2 k_1 [b]_1$. If we can find some a such that $i(a) = Db$ for some representative b of $[b]_1$, then we have that $d_2 b = [j_1 a]$ (by the definitions of k_2 and j_2). In our inversion, we choose to represent $[b]_1$ by $b \pm c$ (here the choice of sign is dependent on the column in which c lies) where c is as above. In this case we take the next step to compute k_1 and take D of this element

$$k_1(b \pm c) = D(b \pm c) = \delta b \pm db \pm ((-1)^p dc + \delta c) = \delta b - dc \pm \delta c = \delta c$$

where we choose the sign such that we acquire this result. We may pick such a representative as when we project back to the column containing b by applying j, we will lose the contribution from the component in the c column. Therefore we have that $d_2[b]_2 = [\delta c]_2$. This process is demonstrated in the diagram below.



We note that this process easily generalizes to higher pages. Say $d_2[b] = 0$ then we have that there exists some c_2 such that $dc_2 = \delta c_1$ and if we represent $[b]_3$ by $b+c_1+c_2$ (with the appropriate signs) then taking $D(b+c_1+c_2)$ we end up canceling all components except for δc_2, and therefore $d_3[b] = [\delta c_2]$. In general if b is on page r then we can extend it to a similar chain of maps of length r (from b to c_{r-1}), and by the process above we obtain that $d_r[b] = [\delta c_{r-1}]$. So in general we have that the bigrading on K survives in the spectral sequence: $E_r = \bigoplus_{p,q} E_r^{p,q}$, and that $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$. We state, but do not prove the following theorem, which comes from the theorem stated in the previous section about general filtered complexes:

Theorem 4.1. *Given a double complex $K = \bigoplus_{p,q} K^{p,q}$ there is a spectral sequence $\{E_r, d_r\}$ converging to the total cohomology $H_D(K)$ such that each E_r has the above bigrading and differential, and moreover if it is given the filtration discussed above:*

$$E_1^{p,q} = H_d^{p,q}(K)$$
$$E_2^{p,q} = H_\delta^{p,q}(H_d(K))$$

and the graded complex of the total cohomology of the complex in dimension n is given by:

$$\oplus_{p+q=n} E_\infty^{p,q}(K)$$

for all n. For a proof of this statement, we refer the reader to [2].

5. The Cech-de Rham Complex

Earlier we discussed the Mayer-Vietoris principle in the simple case of two open sets on a smooth manifold. To obtain a more useful generalization to a countable open cover of a smooth manifold, we need to develop the notion of Cech cohomology, and more generally the Cech-de Rham complex.

Given a general open cover $\mathfrak{U} = \{U_\alpha\}$ where α is in some countable ordered index set, we can consider the complex $C^p(\mathfrak{U}, \mathbb{R})$.

Definition 5.1. We define the *Cech Complex* $C^p(\mathfrak{U}, \mathbb{R})$ of a countable open cover $\mathfrak{U} = \{U_\alpha\}$ of a manifold M to be given by the space of locally constant functions on $\prod_\alpha U_{\alpha_0,\alpha_1,\dots,\alpha_p}$ with values in $\mathbb{R}$, where alpha denotes all chains of indices in increasing order corresponding to appropriate intersections of open sets. $U_{\alpha_0,\alpha_1,\dots,\alpha_p}$ denotes: $U_{\alpha_0} \cap U_{\alpha_1} \cap \dots \cap U_{\alpha_p}$ where $U_{\alpha_n} \in \mathfrak{U}$.

This complex can be endowed with a differential δ. An element $\omega \in C^p(\mathfrak{U}, \mathbb{R}) = \prod_\alpha U_{\alpha_0,\alpha_1,\dots,\alpha_p}$ has components on each $U_{\alpha_0,\alpha_1,\dots,\alpha_p}$ given by its restriction to that intersection, which we refer to by $\omega_{\alpha_0,\alpha_1,\dots,\alpha_p}$. Given ω we define $\delta : C^p(\mathfrak{U}, \mathbb{R}) \to C^{p+1}(\mathfrak{U}, \mathbb{R})$ to be

$$\delta(\omega)_{\alpha_0,\alpha_1,\dots,\alpha_{p+1}} = \sum_{n=0}^{p+1} (-1)^n \omega_{\alpha_0,\alpha_1,\dots,\widehat{\alpha_n},\dots,\alpha_p}$$

where here the hat denotes the omission of that index, and therefore the restriction of the component of ω on the intersection given by all of the sets excluding that index to the set $U_{\alpha_0,\alpha_1,\dots,\alpha_{p+1}}$. We have that $\delta^2 = 0$ essentially because we omit each pair of indices twice, but with opposite signs (if we omit α_i then α_j, if $i < j$ then that term will have the sign $(-1)^i(-1)^{j-1}$ as the set in the jth position is shifted over by the removal of the ith set. So when we remove j then i we get the same result but with the opposite sign). Given this, we can take the cohomology of this complex and so obtain the *Cech Cohomology* of the cover.

Finally, this notion of the Cech complex generalizes beyond locally constant $\mathbb{R}$-valued functions, in fact by replacing "locally constant function on U_α" with an element of $\Omega^p(U_\alpha)$ we can develop a new complex, the double complex known as the Cech-de Rham complex. This complex looks like

$$\begin{tikzcd}
q & \vdots & & & & \\
& \prod \Omega^3(U_{\alpha_0}) \arrow[r, "\delta"] & \prod \Omega^3(U_{\alpha_0\alpha_1}) \arrow[r, "\delta"] & \prod \Omega^3(U_{\alpha_0\cdots\alpha_2}) \arrow[r, "\delta"] & \prod \Omega^3(U_{\alpha_0\cdots\alpha_3}) & \\
& \prod \Omega^2(U_{\alpha_0}) \arrow[r, "\delta"] \arrow[u, "d"] & \prod \Omega^2(U_{\alpha_0\alpha_1}) \arrow[r, "\delta"] \arrow[u, "d"] & \prod \Omega^2(U_{\alpha_0\cdots\alpha_2}) \arrow[r, "\delta"] \arrow[u, "d"] & \prod \Omega^2(U_{\alpha_0\cdots\alpha_3}) \arrow[u, "d"] & \\
& \prod \Omega^1(U_{\alpha_0}) \arrow[r, "\delta"] \arrow[u, "d"] & \prod \Omega^1(U_{\alpha_0\alpha_1}) \arrow[r, "\delta"] \arrow[u, "d"] & \prod \Omega^1(U_{\alpha_0\cdots\alpha_2}) \arrow[r, "\delta"] \arrow[u, "d"] & \prod \Omega^1(U_{\alpha_0\cdots\alpha_3}) \arrow[u, "d"] & \\
& \prod \Omega^0(U_{\alpha_0}) \arrow[r, "\delta"] \arrow[u, "d"] & \prod \Omega^0(U_{\alpha_0\alpha_1}) \arrow[r, "\delta"] \arrow[u, "d"] & \prod \Omega^0(U_{\alpha_0\cdots\alpha_2}) \arrow[r, "\delta"] \arrow[u, "d"] & \prod \Omega^0(U_{\alpha_0\cdots\alpha_3}) \arrow[u, "d"] & \cdots \\
{} \arrow[uuuuu] \arrow[rrrrr] & & & & & p
\end{tikzcd}$$

Where in the $p-th$ column the product is being taken over all p-fold intersections of elements of the open cover. We claim that all of the rows in the complex are exact.

Proposition 5.2. *The rows of the Cech-de Rham complex are exact (after the first column), and therefore the complex collapses to the first column upon taking δ-cohomology.*

Proof. If $\omega \in \Omega^*(U_{\alpha_0,\alpha_1,\ldots,\alpha_p})$ is a cocycle we have that:

$$\delta(\omega)_{\alpha,\alpha_0,\alpha_1,\ldots,\alpha_p} = \omega_{\alpha_0,\alpha_1,\ldots,\alpha_p} + \sum_i (-1)^{i+1} \omega_{\alpha,\alpha_1,\ldots,\widehat{\alpha_i},\ldots,\alpha_p} = 0$$

and so:

$$\omega_{\alpha_0,\alpha_1,\ldots,\alpha_p} = \sum_i (-1)^i \omega_{\alpha,\alpha_0,\ldots,\widehat{\alpha_i},\ldots,\alpha_p}$$

we use this condition, first taking ρ_α to be a partition of unity subordinate to the cover $\{U_\alpha\}$, then defining:

$$\psi_{\alpha_0,\alpha_1,\ldots,\alpha_{p-1}} = \sum_\alpha \rho_\alpha \omega_{\alpha,\alpha_0,\ldots,\alpha_{p-1}}$$

$$\delta(\psi)_{\alpha_0,\alpha_1,\ldots,\alpha_p} = \sum_i (-1)^i \psi_{\alpha_0,\ldots,\widehat{\alpha_i},\ldots,\alpha_p} = \sum_{i,\alpha} (-1)^i \rho_\alpha \omega_{\alpha,\alpha_0,\ldots,\widehat{\alpha_i},\ldots,\alpha_p}$$

$$= \sum_\alpha \rho_\alpha \sum_i (-1)^i \omega_{\alpha,\alpha_0,\ldots,\alpha_p} = \sum_\alpha \rho_\alpha \omega_{\alpha_0,\alpha_1,\ldots,\alpha_p} = \omega_{\alpha_0,\alpha_1,\ldots,\alpha_p}$$

□

A cover $\mathfrak{U} = \{U_\alpha\}$ of a manifold M is called a *good cover* if for all finite subsets of the cover $U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}$ we have that $U_{\alpha_1} \cap U_{\alpha_2} \cap \cdots \cap U_{\alpha_n}$ is diffeomorphic to $\mathbb{R}^n$. We now show that every manifold admits a good cover.

Proposition 5.3. *Every manifold permits a good cover.*

Proof. It is well known that every manifold permits a Riemannian structure. Endow a manifold M with a Riemannian metric. Now note that from differential geometry we have that every point on a Riemannian manifold is contained within a geodesically convex neighborhood. The intersection of any finite collection of geodesically convex neighborhoods is geodesically convex. Finally also recall that any geodesically convex neighborhood of a Riemannian manifold of dimension n is diffeomorphic to $\mathbb{R}^n$, and so taking as a cover the appropriate collection of geodesically convex neighborhoods we have our result. □

The differential geometry here is, as is necessary, cited rather then done out. We refer a reader who is curious about the details of the results mentioned to [2], however it should be noted that an understanding of this material is unnecessary for the rest of the paper. We will, however make frequent use of the fact that every manifold permits a good cover in the latter parts of the paper, and one should keep this fact in mind.

So from the Poincare lemma and our computation of the cohomology of $\mathbb{R}^n$, we see that if the cover $\mathfrak{U}$ is a good cover, then the d-cohomology of the columns collapses similarly onto the first row. As a final note, we point out that the functions under consideration in Cech cohomology are precisely elements of the zeroth de Rham cohomology of the cover. We are now ready to prove the key theorem of this section:

Theorem 5.4. *If $\mathfrak{U}$ is a good cover of a smooth manifold M, then $H_{DR}(M) \cong H(\mathfrak{U}, \mathbb{R})$.*

Proof. The proof is by way of spectral sequences. Taking the filtration as discussed in the previous section on the Cech-de Rham complex $K^{p,q} = H^p(\mathfrak{U}, \Omega^q)$. Then taking the E_1 term we get that $E_1 = H_d(K)$:

$$\begin{tikzcd}
q & & & & & \\
& \vdots & \vdots & \vdots & \vdots & \\
& 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & \cdots \\
& C^0(\mathfrak{U}, \mathbb{R}) \arrow[u] \arrow[r] & C^1(\mathfrak{U}, \mathbb{R}) \arrow[u] \arrow[r] & C^2(\mathfrak{U}, \mathbb{R}) \arrow[u] \arrow[r] & C^3(\mathfrak{U}, \mathbb{R}) \arrow[u] \arrow[r] & \cdots \\
{} \arrow[uuuu] \arrow[rrrrr] & & & & & p
\end{tikzcd}$$

And we have $E_2 = H_\delta H_d(K)$ so we get

$$\begin{tikzcd}
q & & & & & \\
& \vdots & \vdots & \vdots & \vdots & \\
& 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & \cdots \\
& H^0(\mathfrak{U}, \mathbb{R}) \arrow[u] \arrow[r] & H^1(\mathfrak{U}, \mathbb{R}) \arrow[u] \arrow[r] & H^2(\mathfrak{U}, \mathbb{R}) \arrow[u] \arrow[r] & H^3(\mathfrak{U}, \mathbb{R}) \arrow[u] \arrow[r] & \cdots \\
{} \arrow[uuuu] \arrow[rrrrr] & & & & & p
\end{tikzcd}$$

It is clear from the definitions of the differentials that the spectral sequence stabilizes or is "degenerate" on this page as all of the differentials on the E_2 page will be trivial. So we have that $H(\mathfrak{U}, \mathbb{R}) \cong H_D(K)$.

However if instead we filter as: $K_i = \bigoplus_{q \geq i, p \geq 0} K^{p,q}$ then we get the dual situation to what we had before, and our differential on the E_0 page is now δ. So the E_1 page is

$$\begin{tikzcd}
q & & & \\
& \vdots & \vdots & \\
& \Omega^3(M) \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & \cdots \\
& \Omega^2(M) \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & \cdots \\
& \Omega^1(M) \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & \cdots \\
& \Omega^0(M) \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & \cdots \\
{} \arrow[uuuuuu] \arrow[rrr] & & & p
\end{tikzcd}$$

and moving to the E_2 page:

$$\begin{tikzcd}
q & & & \\
 & \vdots & \vdots & \\
 & H^3(M) \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & \cdots \\
 & H^2(M) \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & \cdots \\
 & H^1(M) \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & \cdots \\
 & H^0(M) \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & \cdots \\
 \arrow[uuuuuu] \arrow[rrrr] & & & & p
\end{tikzcd}$$

and this degenerates here as well so we get that $H(M) \cong H_D(K)$. Therefore we have our result.

$\square$

6. Fiber Bundles

Fiber bundles are a crucial object in topology that generalize the notion of a product space by only requiring that the space in question look "locally" like a product. In particular, the cohomology of fiber bundles can often be well understood by understanding the cohomology of the base manifold of the bundle and that of the fiber component. In this section we briefly delve into the definitions and basic results concerning fiber bundles.

Definition 6.1. Given C^∞ manifolds F, M, E and a surjection $\pi : E \to M$ we define an F-bundle over M to consist of the manifolds F, M, E, together with the projection π, the open cover $\{U_i\}$ of M, and F-preserving diffeomorphisms $\phi_i : \pi^{-1}(U_i) \to U_i \times F$. We refer to these diffeomorphisms as a *trivialization* of the total space.

We refer to M as the *base-space*, F as the $fiber$ of the bundle, and E as the *total space*. For any $x \in M$, $\pi^{-1}(x)$ is called the fiber over x. Given the collection of diffeomorphisms $\{\phi_i\}$ it is natural to ask how they behave at intersections, as it is of course this behavior that determines the structure of a fiber bundle. Given a point x such that $x \in U_i, U_j$, we can consider: $\phi_i\phi_j^{-1}|_{\{x\}\times F} \in \text{Diff}(F)$, and as such we have some functions on intersections $g_{i,j} : U_i \cap U_j \hookrightarrow \text{Diff}(F)$. We call the subgroup of Diff(F) that is mapped to by the g functions the *structure group* of the fiber bundle. We note that these functions that map to the structure group satisfy the condition: $g_{i,j} \cdot g_{j,k} = g_{i,k}$.

We call a fiber bundle with fiber $\mathbb{R}^n$ and structure group GL($n, \mathbb{R}$) a rank n *vector bundle*. Such a structure assigns to every point of the base manifold M a real vector space of dimension n. Given a vector bundle E and some open set U

in M, one might naturally be curious about whether or not π is invertible over U, and as it turns out the study of this question sheds an enormous amount of light onto the structure of a fiber bundle.

Definition 6.2. Given a rank n vector bundle with total space E, and an open subset $U \subseteq M$ a section of U is a smooth map $s : U \to E$ such that $s \circ \pi = \mathbb{1}$ on U.

It is an easy task to define a section locally everywhere, as on small scales the total space resembles a product space. The trouble comes when we would like to find a global section of all of M. This is not always possible, and when it is we know something more about the structure of the vector bundle. Note that there is always one trivial global section in any vector bundle: the zero section.

Given a map between manifolds $f : M_1 \to M_2$, and a rank n vector bundle $\pi : E \to M_2$, we obtain a vector bundle over M_1 by taking: $f^{-1}E = \{(x, e)|\pi(e) = f(x)\}$. As this takes product bundles back to product bundles, this induced bundle is in fact a vector bundle. We call this induced map the *pullback* of E under f. The topological relevance of this object is the following; first, it is the maximal subset of $E \times M_1$ such that the diagram:

$$\begin{tikzcd}
f^{-1}(E) \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\
M_1 \arrow[r, "f"] & M_2
\end{tikzcd}$$

commutes. Second of all, the following fact holds true.

Theorem 6.3. *Let B be a manifold that is the base of a rank n vector bundle $\pi : E \to B$. Let M be a compact manifold. If $f : M \to B$ is homotopic to $g : M \to B$, then $f^{-1}E \cong g^{-1}E$.*

So we can classify the space of possible bundles on a space M_1 induced by pullback from another bundle by studying the homotopy classes of maps: $f : M_1 \to M_2$. Which leads us to the refinement:

Corollary 6.4. *If M is contractible then the vector bundle $\pi : E \to M$ must be trivial, as the retraction of M onto a point is homotopic to the identity on M.*

Finally, we note that as for any rank n vector bundle $\pi : E \to M$ we have that, as the zero section embeds M diffeomorphically into E, we have, by shrinking every fiber to 0, that E retracts onto M, and therefore $H^*(E) = H^*(M)$. This should indicate that the standard de Rham cohomology is a bit too weak to discern much meaningful information about the finer points of the topology and geometry of these fiber bundles. As such we define a slight variation on the de Rham cohomology theory that is much more discerning with respect to the details of the structure of a vector bundle.

Definition 6.5. We define $\Omega^*_{CV}(E)$ to be the space of smooth $*$-forms ω on E such that for every compact set K in M, we have that $\pi^{-1}(K) \cap \operatorname{supp}(\omega)$ is compact. This is called the space of forms with *compact support in the vertical direction*.

This is the space of forms on E that are compactly supported when considering only movement along the fiber. The associated cohomology theory is called compact vertical cohomology.

7. The Cohomology of Sphere Bundles

Suppose we have that $F \hookrightarrow E \to M$ is a fiber bundle. To avoid concerns about monodromy we take M to be simply connected, and our fiber such that the cohomology groups of the fiber are all finite dimensional. Take $\{U_\alpha\}$ to be a good cover of M, then we have that $\pi^{-1}\{U_\alpha\}$ as a cover on E, the total space. From this we can derive the double complex

$$K^{p,q} = C^p(\pi^{-1}\{U_\alpha\}, \Omega^q) = \prod_{\alpha_0 < \cdots < \alpha_p} \Omega^q(\pi^{-1}U_{\alpha_0 \ldots \alpha_p})$$

Taking E_1 of this we get:

$$E_1^{p,q} = H_d^{p,q} = \prod_{\alpha_0 < \cdots < \alpha_p} \Omega^q(\pi^{-1}U_{\alpha_0 \ldots \alpha_p})$$

Taking the E_2 page:

$$E_2^{p,q} = H_\delta^p(\{U_\alpha\}, \mathbb{R}^m)$$

Where here m is the dimension of $H^q(F)$, and $H(X, Y)$ is meant to represent the cohomology of X with coefficients in Y. Now we know that the spectral sequence given converges to $H_D(K)$ from our discussion of spectral sequences on double complexes and we also know that this is equal to the de Rham cohomology of the total space, by the discussion on the Cech-de Rham complex at the end of the previous section. Further we know that the terms $E_2^{p,q}$ given above are isomorphic to $H^p(\{U_\alpha\}, \mathbb{R}) \otimes H^q(F) \cong H^p(M) \otimes H^q(F)$ and therefore we have the following theorem:

Theorem 7.1. *Given a fiber bundle $F \hookrightarrow E \to M$ where M is simply connected and the cohomology groups of F are finite dimensional, we have a spectral sequence E_i that converges to the cohomology of E with the $E_2^{p,q}$ terms given by: $H^p(\{U_\alpha\}, \mathbb{R}) \otimes H^q(F)$*

We refer a reader who is curious about the details of the above discussion to [1].

We now restrict our focus to the structure of circle bundles over surfaces so we can head for our eventual goal of being able to compute via spectral sequences the cohomology of a few simple circle bundles over the 2-sphere. We start with some oriented S^1 bundle over a manifold M, which we call E. We call the unique generator of $H^1(S^1)$ the "angular form" on S^1. It is our aim to construct a form Ψ on E such that its restriction to the fibers of the bundle gives us the angular forms on those fibers. We give E a Riemannian structure to make its geometry more apparent.

Now suppose we have some $\{U_\alpha\}$ a cover of M that trivializes E (endowed with sets of coordinates). Then on E_{U_α} we have coordinates: $r_\alpha, \theta_\alpha, \pi^*x_1, \pi^*x_2$ where x_1, x_2 are our coordinates on U_α. Crossing from E_{U_α} to E_{U_β} on $E_{U_\alpha \cap U_\beta}$ we get that the radial coordinates agree as the E is endowed with a Riemannian structure, but the θ coordinates on the respective open sets differ by some rotation. Because that E is orientable we can take some privileged notion of "positive direction" on each fiber. So we associate to each $U_\alpha \cap U_\beta$ some $\phi_{\alpha,\beta}$ unambiguously (on the unit circle), some $\phi_{\alpha,\beta} : U_\alpha \cap U_\beta \to \mathbb{R}$ representing the rotation required to change coordinates at a point on this intersection, such that $\theta_\beta = \theta_\alpha + \pi^*\phi_{\alpha,\beta}$. While the twist required to transfer from the set of coordinates induced by one component of the trivialization to that of another is in fact transitive, it is not necessarily true that $\phi_{\alpha,\beta} + \phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$; all we can say is that $\phi_{\alpha,\beta} + \phi_{\beta,\gamma} - \phi_{\alpha,\gamma} \in 2\pi\mathbb{Z}$. But this implies that the forms given by $\{d\phi_{\alpha,\beta}\}$ do in fact satisfy this condition.

Further we observe that the $\{d\phi_{\alpha,\beta}\} \in C^1(\pi^{-1}\mathfrak{U}, \Omega^1)$ are actually δ of some element of $C^0(\pi^{-1}\mathfrak{U}, \Omega^1)$ (we will discuss this in more detail in the next section), i.e.:

$$\frac{1}{2\pi} d\phi_{\alpha,\beta} = \chi_\beta - \chi_\alpha$$

Which we get by defining, with $\{\rho_\gamma\}$ a partition of unity subordinate to our cover

$$\chi_\alpha = \frac{1}{2\pi} \sum_\gamma \rho_\gamma d\phi_{\gamma,\alpha}$$

and noting that

$$\chi_\beta - \chi_\alpha = \frac{1}{2\pi} \sum_\gamma \rho_\gamma (d\phi_{\alpha,\gamma} + d\phi_{\gamma,\beta}) = \frac{1}{2\pi} d\phi_{\alpha,\beta}$$

This implies that the forms in $\{d\phi_{\alpha,\beta}\}$ piece together along the intersections of the cover to give a global 2-form on our manifold M, which we call the *Euler class* of the manifold and denote by e or $e(E)$. To show that this class is independent of the choice of χ forms chosen is a trivial verification. Combining this with our notes from before we see that

$$\frac{d\theta_\alpha}{2\pi} - \pi^* \chi_\alpha = \frac{d\theta_\beta}{2\pi} - \pi^* \chi_\beta$$

And so finally we have that $\{d\theta_\alpha - \pi^*\chi_\alpha\}$ can be combined to give a global 1-form on the total space, which is our desired angular form. We note that in general the form is not closed and in fact

$$d\psi = d\left(\frac{d\theta_\alpha}{2\pi} - \pi^*\chi_\alpha\right) = -\pi^* d\chi_\alpha = -\pi^* d\chi_\beta$$

Which gives us that $d\psi = -\pi^* e$. From this perspective we can understand the Euler class as the degree of obstruction to the construction of a global angular form on E.

There is an analogous construction in Cech cohomology, also called the Euler class, which we define by the diagram

$$\begin{tikzcd}
q & & & & \\
& 0 & & & \\
& \frac{d\theta_\alpha}{2\pi} \arrow[u, "d"] \arrow[r, "\delta"] & \frac{\pi^* d\phi_{\alpha,\beta}}{2\pi} & & \\
& & \frac{\pi^* \phi_{\alpha,\beta}}{2\pi} \arrow[u, "-d"] \arrow[r, "\delta"] & -\pi^*\epsilon & \\
{} \arrow[uuuu] \arrow[rrrr] & & & & p
\end{tikzcd}$$

Where the forms given are as above, and exist in the appropriate portion of the Cech-de Rham complex, while the element ϵ is the Euler class in Cech cohomology. Under the isomorphism between de Rham and Cech cohomology, it turns out that these two constructions map to each other under the isomorphism.

8. Some Computations

Finally we apply our result to computing the de Rham cohomology of a few important topological spaces making use of the information gleaned from knowledge of their contribution in some bundle structure. In particular we close with a computation of the de Rham cohomology of $\mathbb{C}P^n$ and the cohomology with integer coefficients of three circle bundles over the 2-sphere, namely: the trivial bundle, the unit tangent bundle, and the Hopf fibration.

$S^{2n+1} = \{(z_0, z_1, \ldots, z_n) — |z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 = 1\}$ and of course $z_i \in \mathbb{C}$ for all i. S^1 acts on this space by scalar multiplication, $(z_0, z_1, \ldots, z_n) \to (\lambda z_0, \ldots, \lambda z_n)$ for $\lambda \in S^1$. The quotient of S^{2n+1} by this action is of course $\mathbb{C}P^n$. This gives S^{2n+1} the structure of a circle bundle over complex projective space. As $\mathbb{C}P^n$ is simply connected for all n, we have that there is a spectral sequence converging to the cohomology of S^{2n+1} with $E_2^{p,q} = H^p(\mathbb{C}P^n) \otimes H^q(S^1)$. Now as we know from above that $H^*(S^1) = \mathbb{R}$, for * = 0, 1 and $H^*(S^1) = 0$ else, we have that the E_2 page looks like

$$\begin{tikzcd}
q & & & & & & \\
& \mathbb{R} \arrow[r, "\delta"] & G_1 \arrow[r, "\delta"] & \cdots \arrow[r, "\delta"] & G_{2n} & 0 & \\
& \mathbb{R} \arrow[u, "d"] \arrow[r, "\delta"] & G_1 \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \arrow[u, "d"] \arrow[r, "\delta"] & G_{2n} \arrow[u, "d"] & 0 & \\
{} \arrow[uuu] \arrow[rrrrrr] & & & & & & p
\end{tikzcd}$$

Where our G_i are $H^*(S^1) \otimes H^i\mathbb{C}P^n = \mathbb{R} \otimes H^i\mathbb{C}P^n = H^i(\mathbb{C}P^n)$. The row is 0 after column 2n because $\mathbb{C}P^n$ is 2n-dimensional. Now as from before we know that $d_3 : E_3^{p,q} \to E_3^{p+3,q-3+1}$ we have that d_3 is the zero map. Therefore the spectral sequence is degenerate at the third page and therefore must stabilize there. So we have $E_\infty = E_3$ given by the diagram below (it must be the cohomology of S^{2n+1}).

$$\begin{tikzcd}
q & & & & & & \\
& 0 \arrow[r, "\delta"] & 0 \arrow[r, "\delta"] & \cdots \arrow[r, "\delta"] & \mathbb{R} & 0 & \\
& \mathbb{R} \arrow[u, "d"] \arrow[r, "\delta"] & 0 \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \arrow[u, "d"] \arrow[r, "\delta"] & 0 \arrow[u, "d"] & 0 & \\
{} \arrow[uuu] \arrow[rrrrrr] & & & & & & p
\end{tikzcd}$$

First note that this implies that $G_{2n} = \mathbb{R}$ as d_2 is the zero map on G_{2n}. So as we know that the copy of G_{2n} in the spot $(2n, 0)$ must vanish, we have that $d_2 : G_{2n-2} \to G_{2n}$ is an isomorphism. Further this process propagates backwards so that we have in general

$$G_0 \cong G_2 \cong \ldots \cong G_{2n} \cong \mathbb{R}$$

And of course the same logic applies to show that $G_{2i+1} \cong 0$ for all $i \in \mathbb{N}$. Therefore we have that the cohomology of $\mathbb{C}P^n$ is $\mathbb{R}$ in even dimensions up to $2n$ and 0 otherwise.

Finally we compute the cohomology with integer coefficients of the three aforementioned circle bundles over the two sphere. We begin with the cohomology of the product bundle $S^1 \times S^2$ in such a way that will make the path to the solutions for the next two spaces quite clear. Arguing as above, in all three cases we get the E_2 page of a spectral sequence converging to the cohomology of our total space given by $E_2^{p,q} = H^p(S^2) \otimes H^q(S^1)$.

$$\begin{tikzcd}
q & & & & & \\
& \mathbb{Z} \arrow[r, "\delta"] & 0 \arrow[r, "\delta"] & \mathbb{Z} \arrow[r] & 0 & \\
& \mathbb{Z} \arrow[u, "d"] \arrow[r, "\delta"] & 0 \arrow[u, "d"] \arrow[r, "\delta"] & \mathbb{Z} \arrow[u, "d"] \arrow[r] & 0 \arrow[u] & \\
{} \arrow[uuu] \arrow[rrrrr] & & & & & p
\end{tikzcd}$$

And once again we know that this spectral sequence must stabilize on the next page, so we compute cohomology here and have the cohomology of our total space. Now the way d_2 acts on every group save one is entirely trivial. But we really need to understand its action on $(0, 1)$'s copy of $\mathbb{Z}$ to compute the cohomology of the bundle.

If we take $\mathfrak{U}$ to be the good cover of S^2 that trivializes the bundle and gives us this Cech-de Rham complex, we get that we can take the generator of $C^0(\mathfrak{U}, \Omega^1) = E_0^{0,1}$ to be the local angular forms on the elements of the trivialization. These survive to E_1 as they are by definition closed forms. Now if the bundle is orientable, we have that for $d\theta_\alpha$ the generator of the cohomology of S^1 over U_α and $d\theta_\beta$ the analog for U_β, $[d\theta_\alpha] = [d\theta_\beta]$. This is of course precisely the orientability condition for a fiber bundle, but it is also the condition for a form to survive to E_2, so these generators do. Since $[d\theta_\beta] - [d\theta_\alpha] = 0$, we have that $d\theta_\beta - d\theta_\alpha = d\phi$ for some form ϕ, and we have that d_2's action on these local angular forms can be understood by taking $\delta(\phi)$. Fortunately this process is not just reminiscent of, but identical to what we did in the previous section in our definition of the Cech cohomology's Euler class. This process is given by the diagram

$$\begin{tikzcd}
q & & & & \\
 & 0 & & & \\
 & \frac{d\theta_\alpha}{2\pi} \arrow[u, "d"] \arrow[r, "\delta"] & \frac{\pi^* d\phi_{\alpha,\beta}}{2\pi} & & \\
 & & \frac{\pi^* \phi_{\alpha,\beta}}{2\pi} \arrow[u, "-d"] \arrow[r, "\delta"] & -\pi^*\epsilon & \\
{} \arrow[uuuu] \arrow[rrrr] & & & & p
\end{tikzcd}$$

So as the Euler class in Cech cohomology is mapped under the Cech-de Rham isomorphism to the Euler class in de Rham cohomology, we only have to know the Euler class of our circle bundle to understand the action of d_2. In fact d_2 takes a generator of the group in the (0, 1) spot to the Euler class in $H^2(S^2)$ in the (2, 0) spot.

As the Euler class of the product bundle $S^1 \times S^2$ is 0, we have that the map is the zero map, and therefore d_2 is the zero map. So we have that $H^*(S^1 \times S^2) = \mathbb{Z}$ in dimensions 0-3 and is 0 elsewhere. As the Euler class of the Hopf fibration is a generator of $H^2(S^2)$, we have that $H^*(S^3) = \mathbb{Z}$ in dimensions 0, 3 and 0 otherwise (as expected from our knowledge of the cohomology groups of spheres). Finally, as the Euler class of the unit tangent bundle is twice a generator of $H^2(S^2)$ we have that $H^*(T^1S^2) = \mathbb{Z}$ in dimensions 0, 3 and $\mathbb{Z}_2$ in dimension 2 (as expected as it is known to be diffeomorphic to $\mathbb{R}P^3$).

Acknowledgments. I would like to thank Bena Tshishiku, my mentor, for his support and guidance in helping me learn a very large bit of material over a very short time span, suggestion of the idea that eventually became my project, and the hours spent staring at a board trying to figure out the finer points of various proofs. In a larger sense I would like to thank Bena for being willing to spend hours of his time helping me learn some material which we both (I assume) really thought was cool, it made for quite a wonderful summer. I would also like to thank Nick Salter for the recommendation of the book Bott and Tu and some initial guidance with respect to a choice of project, and of course Peter May for reviewing this paper.

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