CONVERGENCE OF FOURIER SERIES

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Abstract. The purpose of this paper is to explore the basic question of the convergence of Fourier series. This paper will not delve into the deeper questions of convergence that measure theory illuminates, but requires only the basic principles set out by introductory real and complex analysis.

Contents

1. Introduction

The Fourier series of an integrable function is defined, and with the definition of Cesàro means and Abel means, this allows the Fourier series to be recognized as an approximation of the function. We can define a sense of convergence for continuous periodic functions and a "mean-square convergence" over the entire function for Riemann integrable functions.

2. Basic Definition of the Fourier Series

We shall refer to all complex-valued 2π-periodic functions on $\mathbb{R}$ as functions on the circle. We will also assume functions on the the circle are Riemann integrable on some interval of $[a, b]$ with $b - a = 2\pi$. This assumption may be repeated by saying simply a function on the circle is integrable.

Definition 2.1. The N^{th} Fourier coefficient of f is defined to be

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx/}dx, \quad n \in \mathbb{Z}.$$

Note that since f is 2π-periodic, we may replace the endpoints in the above integral with any real $a < b$ such that $b - a = 2\pi$.

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Definition 2.2. The Fourier series associated with the function $f(x)$ is

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}.$$

We shall denote this association as

$$f(x) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi inx/2\pi}.$$

The N^{th} partial sum of the Fourier series or the truncated Fourier series of f is defined to be is

$$S_N f(x) := \sum_{n=-N}^{N} f(n)e^{2\hat{\pi} inx/2\pi}.$$

3. Summability and Kernels

It is useful to define certain notions of "means" which will aid in the question of the convergence of Fourier series. Specifically, convolutions and the notion of a family of "good kernels" are useful for questions of convergence.

Definition 3.1. The N^{th} Cesàro mean or the N^{th} Cesàro sum of the sequence of partial sums $\{\sigma_n\}$ of the series of complex numbers $\sum_{k=0}^{\infty} c_k$ is defined to be

$$\sigma_N = \frac{a_0 + a_1 + ... + a_{N-1}}{N}.$$

where the a_N are the N^{th} partial sums of the given series of complex numbers

It is well known that if $\sum c_k$ converges to a limit σ, then the Cesàro means converge to σ.

Definition 3.2. A series of complex numbers, $\sum_{k=0}^{\infty} c_k$, is said to be Abel summable to s if for every $0 \leq r < 1$, the series

$$A(r) = \sum_{k=0}^{\infty} c_k r^k$$

converges, and

$$\lim_{r \to 1} A(r) = s.$$

The Abel means of the series are defined to be the quantities $A(r)$.

Definition 3.3. The convolution of two functions f and g on the circle (denoted by $f * g$) is defined to be

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)g(x-y)dy.$$

And as before, since the convolution is a function on the circle we may choose limits of integration $[a, b]$ such that $b - a = 2\pi$.

Proposition 3.4. *Given that f, g, are h are integrable functions on the circle, we have*

(1) $f * (g + h) = (f * g) + (f * h)$,
(2) $(cf) * g = c(f * g) = f * (cg)$ *for any* $c \in \mathbb{C}$,
(3) $f * g = g * f$,
(4) $(f * g) * h = f * (g * h)$,

(5) $f * g$ is continuous (if either f or g is continuous,
(6) $\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n)$.

Proof. (1)-(4) follow easily from definitions

(5) Suppose g is continuous and given $\epsilon > 0$, there exists $\delta > 0$ so that if $|s-t| < \delta$, then $|g(s) - g(t)| < \epsilon$. We have $|x_1 - x_2| < \delta$ implies $|(x_1 - y) - (x_2 - y)| < \delta$ for any y.

So

$$\begin{aligned}|(f * g)(x_1) - (f * g)(x_2)| \leq& \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} f(y)[g(x_1 - y) - g(x_2 - y)]dy\right| \\ \leq& \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(y)||g(x_1 - y) - g(x_2 - y)|dy \\ \leq& \frac{\epsilon}{2\pi}\int_{-\pi}^{\pi} |f(y)|dy \leq \epsilon B\end{aligned}$$

where $|f(x)| \leq B$ for all $x \in [-\pi, \pi]$.

(6) This follows from Fubini's theorem

$$\begin{aligned}\widehat{f * g}(n) =& \frac{1}{2\pi}\int_{-\pi}^{\pi} (f * g)(x)e^{-inx}dx \\ =& \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} f(y)g(x-y)dy\right) e^{-inx}dx \\ =& \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iny}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{-inx}dx\right) dy \\ =& \hat{f}(n)\hat{g}(n).\end{aligned}$$

□

Definition 3.5. A family of integrable functions $\{K_n\}_{n=1}^{\infty}$ on the circle is said to be a family of good kernels if it satisfies the following three properties:

(a) For all $n \geq 1$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)dx = 1.$$

(b) There exists $M > 0$ such that for all $n \geq 1$,

$$\int_{-\pi}^{\pi} |K_n(x)|dx \leq M.$$

(c) For every $\delta > 0$,

$$\int_{\delta \leq |x| \leq \pi} |K_n(x)|dx \to 0$$

as $n \to \infty$.

The significance of good kernels in the context of Fourier analysis is seen in their connection with convolutions.

Theorem 3.6. *Let $\{K_n\}_{n=1}^{\infty}$ be a family of good kernels, and let f be an integrable function on the circle. Then whenever f is continuous at x,*

$$\lim_{n\to\infty} (f * K_n)(x) = f(x).$$

If f is continuous, then convergence is uniform on $[-\pi, \pi]$.

Proof. If $\epsilon > 0$ and f is continuous at x, choose δ such that if $|y| < \delta$, then $|f(x-y) - f(x)| < \epsilon$.

Using the first property of good kernels, we can write

$$\begin{aligned}(f * K_n)(x) - f(x) =& \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)f(x-y)dy - f(x)\\ =& \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)[f(x-y) - f(x)]dy.\end{aligned}$$

Taking the absolute value,

$$\begin{aligned}|(f * K_n)(x) - f(x)| =& |\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)[f(x-y) - f(x)]dy|\\ \leq& \frac{1}{2\pi}\int_{|y|<\delta} |K_n(y)||f(x-y) - f(x)|dy\\ +& \frac{1}{2\pi}\int_{\delta\leq|y|\leq\pi} |K_n(y)||f(x-y) - f(x)|dy\\ \leq& \frac{\epsilon}{2\pi}\int_{-\pi}^{\pi} |K_n(y)|dy + \frac{2B}{2\pi}\int_{\delta\leq|y|\leq\pi} |K_n(y)|dy,\end{aligned}$$

where B is a bound for $|f|$. The second property of good kernels shows that

$$\frac{\epsilon}{2\pi}\int_{-\pi}^{\pi} |K_n(y)|dy \leq \frac{\epsilon M}{2\pi},$$

for all $n \geq 1$. The third property of good kernels shows that

$$\frac{2B}{2\pi}\int_{\delta\leq|y|\leq\pi} |K_n(y)|dy \leq \epsilon,$$

for all $n \geq N(\delta)$. Therefore we have

$$|(f * K_n)(x) - f(x)| \leq C\epsilon.$$

This proves the first part of the theorem. For the second part, we note that if f is continuous on $[-\pi, \pi]$, then δ can be chosen independent of x. □

The N$^{\text{th}}$ Dirichlet kernel is defined to be

$$D_N(x) = \sum_{n=-N}^{N} e^{inx}$$

Its definition is motivated by writing the N$^{\text{th}}$ partial sum of the Fourier series of $f$as a convolution:

$$\begin{aligned}S_N(f)(x) =& \sum_{n=-N}^{N} \hat{f}(n)e^{inx}\\ =& \sum_{n=-N}^{N}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iny}d\right)e^{inx}\\ =& \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\left(\sum_{n=-N}^{N} e^{in(x-y)}\right)dy\\ =& (f * D_N)(x).\end{aligned}$$

If the Dirichlet kernels were a good family of kernels, then we could develop a sense of convergence from the partial sums of the Fourier series. However, we will see that the family of Dirichlet kernels is not a family of good kernels. In particular, the Dirichlet family of kernels does not satisfy the second property of good kernels. First, we have a closed form expression for the N^{th} Dirichlet kernel,

$$D_N(x) = \frac{\sin((N+1/2)x)}{\sin(x/2)}$$

We can derive this using the geometric series identity and trigonometric identities,

$$\begin{aligned} D_N(x) = &\sum_{n=-N}^{N} e^{inx} \\ =&\sum_{n=0}^{N} e^{inx} + \sum_{n=0}^{N-1} e^{-i(n+1)x} \\ =&\frac{1-e^{i(N+1)x}}{1-e^{ix}} + \frac{1-e^{-iNx}}{1-e^{-ix}} \\ =&\frac{\sin((N+1/2)x)}{\sin(x/2)} \end{aligned}$$

Now

$$\int_{-\pi}^{\pi} |D_N(x)| dx = \int_{-\pi}^{\pi} \left| \frac{\sin((N+1/2)x)}{\sin(x/2)} \right|$$

We can bound the integrand from below and change variables to obtain

$$\begin{aligned} \int_{-\pi}^{\pi} |D_N(x)| dx| \geq &c \int_{-\pi}^{\pi} \frac{|\sin((N+1/2)x)|}{|x|} dx \\ \geq &c \int_{\pi}^{N\pi} \frac{|\sin(x)|}{|x|} dx \\ =&c \sum_{k=1}^{N-1} \int_{k\pi}^{(k+1)\pi} \frac{1}{k} \\ \geq &c \log(n) \end{aligned}$$

where $c > 0$. This implies that the Dirichlet kernels are not a family of good kernels. However, using the summing techniques previously developed, we can define good families of kernels related to the Dirichlet kernels and the Fourier series.

We now present two families of kernels that are good kernels: (1) the Fejér family of kernels and (2) the Poisson family of kernels

(1) Our first example's definition is motivated by writing the N^{th} Cesàro mean of the Fourier series as a convolution.

$$\sigma_N(f)(x) = \frac{S_0(f)(x) + S_1(f)(x) + ... + S_{N-1}(f)(x)}{N}.$$

We have $S_n(f) = f * D_n$. Hence

$$\sigma_N(f)(x) = (f * F_N)(x)$$

The N^{th} Fejér kernel is defined to be

$$F_N(x) = \frac{D_0(x) + ... + D_{N-1}(x)}{N}.$$

A closed form expression for the Fejèr kernels is given by

$$F_N(x) = \frac{1}{N}\frac{sin^2(Nx/2)}{\sin^2(x/2)}.$$

We have shown previously that $D_N = \frac{\sin((N+1/2)x)}{\sin(x/2)}$. Using the first expression for the N$^{\text{th}}$ Fejér kernel,

$$\begin{aligned} NF_N(x) &= \sum_{n=0}^{N-1} D_n \\ &= \frac{\sin(x/2)}{\sin(x/2)} + ... + \frac{\sin((N-1/2)x)}{\sin(x/2)} \\ &= \frac{\sin^2(x/2)}{\sin^2(x/2)} + \frac{\sin(3x/2)\sin(x/2)}{\sin^2(x/2)} + ... + \frac{\sin((N-1/2)x)\sin(x/2)}{\sin^2(x/2)}. \end{aligned}$$

We have the identity

$$\sin((k+1/2)x)\sin(x/2) = 1/2(\cos(kx) - \cos((k+1)x).$$

Thus, we have telescoping, and our previous expression becomes

$$NF_N(x) = \frac{1/2(1-\cos(Nx))}{\sin^2(x/2)}$$

or

$$F_N(x) = \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)}.$$

(2) The definition of the second family is motivated by writing the Abel mean of the Fourier series as a convolution $f(x) \sim \sum_{n=-\infty}^{\infty} a_n e^{inx}$, which is

$$A_r(f)(x) = \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{inx}$$

This is defined to be

$$A_r(f)(x) = (f * P_r)(x)$$

The N$^{\text{th}}$ Poisson kernel is defined to be

$$P_r(x) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{inx}.$$

If $0 \le r < 1$, then we have a closed form expression

$$P_r(x) = \frac{1-r^2}{1-2r\cos(x)+r^2}$$

To see this, split the kernel into the expression:

$$\sum_{n=0}^{\infty} \left(re^{i\theta}\right)^n + \sum_{n=1}^{\infty} \left(re^{-i\theta}\right)^n$$

By DeMoivre's formula and by summing a geometric series, we obtain the desired closed form.

Proposition 3.7. *The Fejér family of kernels and the Poisson family of kernels are families of good kernels.*

Proof. We first study the Fejér family of kernels.

We obtain the first property by direct computation (the first holds for a similar identity for Dirichlet kernel)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x)dx = 1$$

Since F_N is positive, this also shows the second property. If $\delta > 0$, then we have a lower bound: $\sin^2(x/2) \geq c_\delta > 0$. For $|x| \geq \delta$, we have

$$\lim_{N\to\infty}\int_{\delta\leq|x|\leq\pi} |F_N(x)|dx \leq \frac{2\pi}{cN}$$

which proves the third property. Therefore the Fejér family of kernels is a family of good kernels.

We now proceed to study the Poisson family of kernels.

Similar to the Fejér kernels, we can see that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty} r^{|n|}e^{inx}dx = 1$$

since

$$\int_{-\pi}^{\pi}\left(e^{inx} + e^{-inx}\right) = 0, \quad \text{for all } n \neq 0$$

which proves the first property. If $0 \leq r < 1$, the Poisson kernels are positive which proves the second property.

To show the third property, assume $1/2 \leq r \leq 1$ and $\delta \leq |x| \leq \pi$, we have $1 - 2r\cos(x) + r^2 \geq c_\delta > 0$. This implies $P_r(x) \leq (1 - r^2)/c_\delta$. Therefore, the Poisson family of kernels is a family of good kernels.

□

4. The Fourier Series of a Continuous Function

Given the formal definition, one could ask whether a given function $f(x)$ is uniquely determined by its associated Fourier series. In particular, if f and g have the same Fourier coefficients does it follow $f = g$? Strictly speaking, no.

Example 4.1. Let

$$f(x) = \left\{\begin{array}{ll} 1 & : x \neq 0 \\ 0 & : x = 0 \end{array}\right\}$$
$$g(x) = 1$$

and extend f, g to 2π-periodic functions on $\mathbb{R}$.

Computing either set of Fourier coefficients we see $f(x) \sim 1$ and $g(x) \sim 1$.

Thus we can find two functions that differ at a finite set of points that have the same Fourier series. However, if f, g are continuous, the answer to the above question is yes.

Theorem 4.2. *If f is a continuous function on the circle and $\sum_{n=-\infty}^{\infty} \mid \hat{f(n)} \mid < \infty$, then the Fourier series converges uniformly to f.*

We state a lemma that essentially proves the above theorem.

Lemma 4.3. *Suppose that h is an integrable function on the circle with $\hat{h}(n) = 0$ for all $n \in \mathbb{Z}$. Then $h(x_0) = 0$ whenever h is continuous at the point x_0.*

With this lemma we first prove Theorem 4.2.

Proof of Theorem 4.2. Define

$$g(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}.$$

Note convergence is uniform on $[-\pi, \pi]$ by our hypothesis. Then

$$\begin{aligned} g(x) &\sim \sum_{m=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)e^{-imx} \right) e^{imx} \\ &= \sum_{m=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{n=-\infty}^{\infty} \hat{f}(m)e^{inx} \right) e^{-imx} \right) e^{imx} \\ &= \sum_{m=-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} \hat{f}(m) \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx}e^{-imx} \right) \right) e^{imx} \end{aligned}$$

By uniform convergence, the interchange of summation and integration is allowed. For all $n \neq m$, we have the above integrals are equal to 0. Thus,

$$g(x) \sim \sum_{m=-\infty}^{\infty} \hat{f}(m)e^{imx}.$$

Lemma 4.3 gives the result. □

Proof of Lemma 4.3. This is an application of Theorem 3.6. Recall that the Fejér kernels are good kernels. By Theorem 3.6, if h is continuous at x_0, then

$$\lim_{n\to\infty} (h * F_n)(x_0) = h(x_0).$$

Since all the Cesàro averages are zero, we are done. □

There is an immediate corollary that follows from the proof of the lemma.

Corollary 4.4. *Continuous functions on the circle can be uniformly approximated by trigonometric polynomials.*

We just note that the partial sums are trigonometric polynomials.

5. The Mean-Square Convergence

This section explores a more general idea of global convergence of the Fourier series. A continuous function may have a Fourier series that fails to converge at a point, but perhaps there may be a way of determining if the Fourier series converges to the function globally in a certain sense. The main result of this section is a type of "mean-square convergence".

The following result actually holds for merely integrable functions, but for simplicity we state and prove it for continuous functions.

Theorem 5.1. *Suppose f is continuous on the circle. Then*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - S_N(f)(x)|^2 dx \to 0 \qquad \text{as } N \to \infty$$

The proof requires some understanding of inner product spaces. Specifically the delicateness of the global convergence of Fourier series could be attributed to the difference between Hilbert-spaces and pre-Hilbert-spaces.

Definitions 5.2. A Hilbert space is a complex inner product space which is complete with respect to the norm induced by the inner product. If the inner product space is not complete, we say that it is a pre-Hilbert space.

Examples 5.3. It is trivial exercise in linear algebra to show that $\mathbb{C}^n$, with the usual inner product is a Hilbert space.

We define $\ell^2(\mathbb{Z})$, to be the set of all two-sided infinite sequences of complex numbers $\{a_n\}_{n=-\infty}^{\infty}$ such that

$$\sum_{n\in\mathbb{Z}} |a_n|^2 < \infty.$$

The inner product is defined to be

$$\langle A, B\rangle_{\ell^2(\mathbb{Z})} = \sum_{n\in\mathbb{Z}} a_n\overline{b_n}$$

where $A = \{a_n\}$ and $B = \{b_n\}$.

The norm is therefore

$$\|A\|_{\ell^2(\mathbb{Z})} = \langle A, A\rangle^{1/2} = \left(\sum_{n\in\mathbb{Z}} |a_n|^2\right)^{1/2}.$$

It is straightforward to show $\ell^2(\mathbb{Z})$ is a Hilbert space.

An important pre-Hilbert space is the space of integrable functions on the circle with inner product

$$\langle f, g\rangle_{L^2([-\pi,\pi])} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx$$

and norm

$$\|f\|^2_{L^2([-\pi,\pi])} = \langle f, f\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx$$

The proof that this is not a Hilbert-space is simple and we will omit it here.

The following lemma used to prove Theorem 5.1 is sometimes referred to as the "Best Approximation Lemma."

Lemma 5.4. *If f is continuous on the circle with Fourier coefficients $\hat{f}(n)$, then*

$$\|f - S_N(f)\|_{L^2([-\pi,\pi])} \le \|f - \sum_{|n|\le N} c_n e_n\|$$

for any complex numbers c_n. Moreover, equality holds if and only if $c_n = \hat{f}(n)$ for all $|n| \le N$.

Proof. For each integer n, let $e_n(x) = e^{inx}$. Note

$$\langle e_n, e_m\rangle = \begin{cases} 1 & : if \;\; n = m \\ 0 & : if \;\; n \ne m \end{cases}$$

If f is a continuous function on the circle, we may write $\hat{f}(n)$ using the $L^2([-\pi,\pi])$ inner product

$$\hat{f}(n) = \langle f, e_n\rangle$$

and the truncated Fourier series as $S_N(f) = \sum_{|n|\le N} \hat{f}(n)e_n$.

Since the set $\{e_n\}_{|n|\leq N}$ is an orthonormal set, we can apply the Pythagorean Theorem to the decomposition:

$$f - \sum_{|n|\leq N} c_n e_n = f - S_N(f) + \sum_{|n|\leq N} (\hat{f}(n) - c_n e_n)$$

We get

$$\|f - \sum_{|n|\leq N} c_n e_n\|^2_{L^2([-\pi,\pi])} = \|f - S_N(f)\|^2 + \sum_{|n|\leq N} |\hat{f}(n) - c_n|^2$$

thus demonstrating the lemma. □

Proof of Theorem 5.1. Suppose f is continuous on the circle. Let $\epsilon > 0$, There exists by Corollary 4.4 a trigonometric polynomial P of degree D such that

$$|f(x) - P(x)| < \epsilon$$

for all $x \in [-\pi, \pi]$. Then choose $\{c_n\}$ of complex numbers such that $\sum_{|n|\leq N} c_n e_n = P$ Then whenever we have $N \geq D$,

$$\|f - S_N(f)\| \leq \|f - P\| < \epsilon$$

And we are done. □

What we have shown for continuous functions on the circle in this paper may be extended in some sense to merely integrable functions and with the introduction of Lebesgue integration, one may see that the pre-Hilbert space with $L^2([-\pi, \pi])$ norm may be completed as the inner product space $\ell^2(\mathbb{Z})$, opening up deeper questions of convergence.

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