THE TOPOLOGY OF COMPLEX HYPERSURFACES

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Abstract. Let f be a polynomial in $n+1$ complex variables. In this paper, we study the topology of the set $V = f^{-1}(0)$. While V is a differentiable manifold in the neighborhood of simple points, the topology is more complicated at singular points. We use the following construction to study the topology at singular points. Let S_ϵ be a small sphere around a singular point, and let $K = S_\epsilon \cap V$. We will prove that $S_\epsilon - K$ is a smooth fiber bundle over S^1, with fibers homotopy equivalent to a CW complex of dimension $\leq n$, which allows us to obtain substantial information about the fibers. Finally, we show that K is a $(n-2)$-connected differentiable manifold.

Contents

1. Introduction and Overview

Let $f \in \mathbb{C}[x_1, \ldots, x_{n+1}]$ be a non-constant polynomial. In this paper we investigate the topology of $V = f^{-1}(0)$. We start off in the second section by studying a more general object called an affine variety. We establish some basic definitions and properties of affine varieties, focusing on how close varieties come to being manifolds. We observe that singular points, which we will define, are the obstructions to V being a differentiable manifold. To study the topology of an affine variety at a singular point z we take a small sphere S_ϵ around z and intersect it with V. When there are no other singular points in the neighborhood of z, the topology of $K = V \cap S_\epsilon$ and the way it is embedded in S_ϵ completely determines the topology of V. We then explore the properties of K and $S_\epsilon - K$. In particular, we prove the Fibration Theorem: $S_\epsilon - K$ is a smooth fiber bundle over S^1 with projection map $\phi = \frac{f(z)}{|f(z)|}$. Roughly, this means that $S_\epsilon - K$ is a circle with a copy of some differentiable manifold above each point. We then study the Morse theory of $|f|$ on $F_\theta = \phi^{-1}(e^{i\theta})$ and $S_\epsilon - K$. This enables us to show that F_θ is homotopy equivalent to a CW-complex of dimension $\leq n$ and K is $(n-2)$-connected. Our treatment will follow [2].

2. Affine Varieties

Before restricting to the case of a hypersurface, we first discuss a more general geometric object called an affine variety. These generalize curves and surfaces in the most obvious way.

Definition 2.1. Suppose $S \subseteq \Phi^m$ for a field Φ. Then we call S an affine variety if there exists a set of polynomials $E \in \Phi[x_1, \dots, x_m]$ such that

$$S = \{z \in \Phi^m : f(z) = 0 \text{ for all } f \in E\}$$

If $S = f^{-1}(0)$ for a single non-constant polynomial f, we call S a hypersurface.

Given a set of polynomials E, we denote the associated affine variety by $V(E)$. Similarly, given an affine variety V, we denote the set of polynomials vanishing on V by $I(V)$. We claim that $I(V)$ is an ideal. For $z \in V$, polynomials $f, g \in I(V)$ and $h \in \Phi[x_1, \dots, x_m]$, it is clear that

$$0 = f(z) + g(z) = -f(z) = f(z)h(z)$$

and the zero polynomial vanishes at z, so $I(V)$ is an ideal.

Affine varieties are a natural geometric object to consider, and we could hope that they are differentiable manifolds. It turns out that this is not true since the variety may behave poorly at certain points. A typical example in $\mathbb{R}^2$ is the zero set of the polynomial $f(x, y) = xy$. This curve has two branches which intersect at $(0, 0)$, and thus fails to be a manifold around $(0, 0)$.

Before we can define singular points, we need one basic theorem from commutative algebra.

Theorem 2.2 (Hilbert's Basis Theorem)**.** *Every ideal in a polynomial ring in finitely many variables over a field is finitely generated.*

See [5, Section 9.6, Theorem 21] for the proof in the more general case of polynomial rings over Noetherian rings. Every field has only two ideals: the trivial ideal $\{0\}$ and the entire field. The first ideal is generated by 0, and the second is generated by 1. Hence every field is automatically Noetherian.

Using Hilbert's Basis Theorem, we can define singular points. Let V be an affine variety and let $I(V)$ be generated by $f_1, \dots, f_l$. Let ρ be the maximum rank that the $l \times m$ matrix $(\frac{\partial f_i}{\partial x_j})$ achieves on V.

Definition 2.3. A point $x \in V$ is a simple point if $\operatorname{rank}(\frac{\partial f_i}{\partial x_j})$ evaluated at x is ρ. Otherwise x is said to be a singular point. We denote the set of singular points of V by $\Sigma(V)$.

We claim that the rank of the matrix of partial derivatives is independent of the polynomials chosen to span $I(V)$. If we add $f = g_1 f_1 + \ldots + g_l f_l$ to the generating set of the ideal, then we add the row with i-th entry

$$\begin{aligned}\frac{\partial f}{\partial x_i} &= g_1 \frac{\partial f_1}{\partial x_i} + \ldots + g_l \frac{\partial f_l}{\partial x_i} \\ &\quad + f_1 \frac{\partial g_1}{\partial x_i} + \ldots + f_l \frac{\partial g_l}{\partial x_i} \\ &= g_1 \frac{\partial f_1}{\partial x_i} + \ldots + g_l \frac{\partial f_l}{\partial x_i}\end{aligned}$$

to the matrix of partial derivatives (the second half of the summation disappears since we are evaluating at a zero point of the f_i). However, this is a linear combination of the rows already present so it doesn't affect the rank. Therefore if E_1 and E_2 are two sets of generators for $I(V)$, a simple induction argument shows that the rank of the matrix of partial derivatives is the same for E_1, E_2, and $E_1 \cup E_2$. Therefore the singular points of V are well-defined. Note that in the case of a hypersurface V with $I(V) = (f)$ for a non-constant polynomial f, a point $x \in V$ is singular iff $\frac{\partial f}{\partial x_j} = 0$ for $j = 1, 2, \ldots, m$. We now give some examples of singular points.

Example 2.4.

(1) Referring to our earlier example, $f(x, y) = xy$, we can see that the $(0, 0)$ is the only singular point of $V(f)$. Two branches of the variety intersect at $(0, 0)$. It is also clear that $(0, 0)$ is the only point at which the variety fails to be a differentiable manifold.
(2) The variety $V = V(y^2 - x^3)$ in $\mathbb{R}^2$ also has an isolated singular point at $(0, 0)$. In this case the variety has a sharp cusp where the curve above the x-axis meets the curve below the x-axis.
(3) Consider $V = V(z_1^2 - z_2^2)$ in $\mathbb{C}^2$. This is a complex hypersurface with an isolated singularity at $(0, 0)$. This singularity is where the surfaces $z_1 = z_2$ and $z_1 = -z_2$ meet.

As these examples demonstrate, the singular points of a variety can be thought of the points at which the topology of the manifold is "bad." The following two theorems make this idea more precise. Let Φ denote either the complex field or the real field.

Theorem 2.5. *If V is an affine variety in Φ^m, then $V - \Sigma(V)$ is a differentiable manifold of dimension $m - \rho$ over Φ, where ρ is defined as in Definition 2.3.*

See [4] for the proof. This theorem shows that V is topologically "nice," except at singular points. The next result says that we really do need to remove the singular points in the case of the complex numbers.

Theorem 2.6. *An affine variety in $\mathbb{C}^m$ fails to be a manifold in any neighborhood of a singular point.*

See [2] for the proof. The analogous statement for $\mathbb{R}$ is false, as the zero set of $y^3 + 2x^2y - x^4$ is a differentiable manifold with a singularity at the origin. We will not prove this but it is clear from looking at the graph.

Let $z \in V$ be a singular point. Ideally we would want to understand the topology of $V \cap D_\epsilon$ where D_ϵ is the open ball of radius ϵ centered around z. Rather than doing this directly, we take a sphere S_ϵ of radius ϵ centered around z and study the intersection $K = S_\epsilon \cap V$. Though V may not be a manifold in a neighborhood of z, K is a manifold. See [2] for a proof. Though it is impossible for the pair (S_ϵ, K) to be homeomorphic to the pair $(D_\epsilon, D_\epsilon \cap V)$, the next best thing happens. The following theorem relates our construction to the local topology of V.

Theorem 2.7. *Let x^0 be a simple point or isolated singular point in V. Let S_ϵ and K be as above. Then $(D_\epsilon, V \cap D_\epsilon)$ is homeomorphic to* $(\mathrm{Cone}(S_\epsilon), \mathrm{Cone}(K))$.

Therefore by understanding K and the way K is embedded in S_ϵ, we can understand the topology of V near x^0. We only sketch the proof. A complete proof can be found in [2].

Proof. First note that an affine variety in $\mathbb{C}^m$ is homeomorphic to some affine variety in $\mathbb{R}^{2m}$ by splitting up the real and complex parts of the polynomials defining the complex variety. Hence it suffices to prove the real case. Since we assumed x^0 is a simple or isolated singular point, there exists an $\epsilon > 0$ such that D_ϵ contains no singular points of V except possibly x^0. We will give an outline before going into the technical details. We will first construct a smooth vector field on $D_\epsilon - x^0$ that points outward from x^0. For each point $z \in D_\epsilon - x^0$ we look for a curve $p_z(t)$ starting at z with velocity given by the vector field. The correspondence between the pair (z, t) and the end point of the curve (which we will force to belong to S_ϵ) gives the desired homeomorphism.

We want a smooth vector field $v(x)$ on $D_\epsilon - x^0$ such that $\langle v(x), x - x^0\rangle > 0$ for every x and $v(x)$ is tangent to $V - \Sigma(V)$ whenever $x \in V - \Sigma(V)$. We construct the vector field locally and then use a partition of unity to find a global solution. Let $x \in D_\epsilon - x^0$, let U be a small neighborhood containing x, and let $y \in U$. First suppose $x \notin V$. Then we locally define $v(y) = y - x^0$, and this clearly works for U small enough. Now suppose $x \in V$. Let $x_1, \dots, x_n$ be the standard coordinate functions on $\mathbb{R}^n$. For some local coordinates $u_1, \dots, u_n$ and choice of h, it can be shown that the vector

$$\pm(\frac{\partial x_1}{\partial u_h}, \dots, \frac{\partial x_n}{\partial u_h})$$

evaluated at y satisfies the requirements.

Now we normalize to get

$$w(x) = \frac{v(x)}{\langle 2(x - x^0), v(x)\rangle}$$

which is also defined on $D_\epsilon - x^0$. By the existence of solutions to differential equations, there is a curve $p(t)$ satisfying

$$\frac{dp(t)}{dt} = w(p(t))$$

Possibly after translating the domain of p, and using $\langle v(x), x - x^0\rangle > 0$, we can show

$$(*) \qquad ||p(t) - x^0||^2 = t$$

and p is defined on some maximal interval (α, β). Suppose $\beta \leq \epsilon^2$. Since D_ϵ is compact, we have

$$p(t) \to x' \in D_\epsilon \text{ as } t \to \epsilon^2$$

By (*) we have $x' \in D_\epsilon - x^0$, so using the local uniqueness and existence of solutions to differential equations, we can extend $p(t)$ to a larger domain, contradicting the maximality of β. We can similarly show that we can take $\alpha = 0$. Then we assume p has domain $(0, \epsilon^2]$. Then $||p(t) - x^0||^2 = t$ implies $p(\epsilon^2) \in S_\epsilon$. Furthermore, $p(t)$ is uniquely determined by $p(\epsilon^2)$, so we can define

$$P(a, t)\colon S_\epsilon \times (0, \epsilon^2] \to D_\epsilon - x^0$$

by $P(a, t) = p(t)$ where $p(t)$ is the unique smooth curve satisfying $p(\epsilon^2) = a$ and having derivative $w(p(t))$. Since $P(a, t) \to x^0$ as $t \to 0$, the function

$$f(ta + (1 - t)x^0) = P(a, t\epsilon^2)$$

is a homeomorphism $\text{Cone}(S_\epsilon) \to D_\epsilon$. Finally, the fact that $v(x)$ is tangent to $V - \Sigma(V)$ for $x \in V - \Sigma(V)$ shows that f restricts to a homeomorphism $\text{Cone}(K) \to V \cap D_\epsilon$. □

This theorem will motivate the major results of the paper. By understanding the pair (S_ϵ, K), we can learn about the topology of the variety even at points where V may fail to be a manifold. We conclude the section by finding S_ϵ and K for the examples given in Example 2.4.

Example 2.8.

(1) When $V = V(xy)$, the origin is the only singular point so we can take $\epsilon = 1$. Then S_ϵ is the unit sphere around the origin. It is clear that

$$K = \{(0,1), (0,-1), (1,0), (-1,0)\}$$

It is also clear that

$$D_\epsilon \cap V = (\{0\} \times [-1,1]) \cup ([-1,1] \times \{0\})$$

Thinking of $\text{Cone}(S_\epsilon)$ as being points on line segments between S_ϵ and the origin, it is obvious that $(\text{Cone}(S_\epsilon), \text{Cone}(K))$ is homeomorphic to $(D_\epsilon, D_\epsilon \cap V)$.

(2) When $V = V(y^2 - x^3)$, we can again take $\epsilon = 1$. The analysis is similar to the previous example, but K consists of two points and $D_\epsilon \cap V$ is two curved lines meeting at the origin.

(3) We now look at a complex variety. Letting $V = V(z_1^2 - z_2^2)$, we take $\epsilon = 1$ and so S_ϵ is a three-dimensional sphere. Then we have

$$K = \{(z,z) : |z| = 1\} \cup \{(z,-z) : |z| = 1\}$$

and

$$D_\epsilon \cap V = \{(z,z) : |z| \leq 1\} \cup \{(z,-z) : |z| \leq 1\}$$

3. The Fibration Theorem

Let f be a polynomial in $n+1$ complex variables which vanishes at the origin, and let V, K, S_ϵ be as before. In this section, we will show that $S_\epsilon - K$ is a smooth fiber bundle over S^1. Informally, this means that $S_\epsilon - K$ can be decomposed into S^1 copies of some differentiable manifold. A formal definition is given below.

Definition 3.1. Let E, B, and F be differentiable manifolds and let π be a smooth surjection $E \to B$. The tuple (E, B, π, F) is called a smooth fiber bundle if for any $x \in B$, there exists an open set $U \subseteq B$ containing x and a diffeomorphism $\psi : \pi^{-1}(U) \to U \times F$ such that

$$\pi|_{\pi^{-1}(U)} = \text{proj}_U \circ \psi$$

where $\text{proj}_U : U \times F \to U$ is the projection onto the first component.

We call F the fiber and π the projection map of the fiber bundle. We will prove the following theorem over the course of this section.

Theorem 3.2 (The Fibration Theorem)**.** *For sufficiently small ϵ, we have that $S_\epsilon - K$ is a smooth fiber bundle over S^1 with projection map $\phi(z) = f(z)/|f(z)|$.*

We first give a simple example of the fiber bundle.

Example 3.3. Let $n \geq 1$ and let $V = V(z^n) = \{0\}$. Then for $\epsilon = 1$, we have $S_\epsilon - K = S_\epsilon$ and $\phi(e^{i\theta}) = e^{in\theta}$. It is easy to see that the fibers are 0-dimensional differentiable manifolds with n points. Also note that even though the varieties are the same for different n, the fiber bundles obtained are different.

We will let $F_\theta = \phi^{-1}(e^{i\theta})$. We will first show that the F_θ are diffentiable manifolds of dimension $2n$. By the following theorem, it suffices to prove that every $y \in S^1$ is a regular value of ϕ.

Theorem 3.4 (The Preimage Theorem)**.** *Suppose X and Y are smooth manifolds, $f : X \to Y$ is a smooth map, and $y \in Y$ is a regular value. Then $f^{-1}(y)$ is a submanifold of X of dimension* $\dim(X) - \dim(Y)$*.*

See [1, Section 1.4] for the easy proof. We will show ϕ has no critical points, and thus no critical values. From the Preimage Theorem, it follows that the F_θ are differentiable manifolds of dimension

$$\dim(S_\epsilon) - \dim(S^1) = 2n + 1 - 1 = 2n$$

We will now give a definition which allows us to give an alternate characterization of the critical points of ϕ.

Definition 3.5. Suppose $f : \mathbb{C}^m \to \mathbb{C}$ is analytic. Then define the gradient of f to be

$$\nabla f = \left(\overline{\frac{\partial f}{\partial z_1}}, \ldots, \overline{\frac{\partial f}{\partial z_m}} \right)$$

We take complex conjugates so that the directional derivative of f along v at z is $\langle v, \nabla f(z) \rangle$. Also note that even though the logarithm function is only defined up to multiples of $2\pi i$, $\nabla \log$ is well-defined. Then the following lemma gives a easily manipulated characterization of the critical points of ϕ.

Lemma 3.6. *A point $z \in S_\epsilon$ is a critical point of ϕ iff $i\nabla \log f(z)$ is a real multiple of z.*

Proof. Let $\theta(z)$ be a real-valued function satisfying $\phi(z) = f(z)/|f(z)| = e^{i\theta(z)}$. Like the complex logarithm, $\theta(z)$ is really only defined up to multiples of 2π, but we will only care about the derivative of $\theta(z)$, which doesn't depend on the branch chosen. Then

$$i\theta = log(f/|f|) = log(f) - log(|f|)$$

So we have

$$\theta = \mathrm{Re}(\theta) = \mathrm{Re}(-ilog(f)) - \mathrm{Re}(-ilog(|f|)) = \mathrm{Re}(-ilog(f))$$

Since S^1 has dimension 1, z is a critical point of ϕ iff $\phi^{'}(z) = 0$, which happens precisely when the directional derivative of θ at z along every v tangent to S_ϵ vanishes. To compute the directional derivative of θ along v, first pick a curve $p(t)$ in $S_\epsilon - K$ such that $dp/dt(0) = v$. Then we have

$$\begin{aligned} \frac{d\theta(p(t))}{dt} &= \mathrm{Re}\left(\frac{d(-i\log(f(p(t))))}{dt} \right) \\ &= \mathrm{Re}\langle v,\, \nabla(-i\log(f))\rangle \\ &= \mathrm{Re}\langle v, i\nabla(\log(f))\rangle \end{aligned}$$

By thinking of $\mathbb{C}^{n+1}$ as a real vector space of dimension $2n+2$, we claim that a vector v is tangent to S_ϵ at z iff $\mathrm{Re}(\langle v, z\rangle) = 0$. To see this, suppose

$$v = (v_1 + iv_1', \ldots, v_{n+1} + iv_{n+1}')$$

and

$$z = (z_1 + iz_1', \ldots, z_{n+1} + iz_{n+1}')$$

Then

$$\begin{aligned}\mathrm{Re}\langle v, z\rangle &= \mathrm{Re}(v_1z_1 + iv_1'z_1 - iv_1z_1' + v_1'z_1' \\ &\quad + \ldots + v_{n+1}z_{n+1} + iv_{n+1}'z_{n+1} - iv_{n+1}z_{n+1}' + v_{n+1}'z_{n+1}') \\ &= v_1z_1 + v_1'z_1' + \ldots + v_{n+1}z_{n+1} + v_{n+1}'z_{n+1}'\end{aligned}$$

Therefore the claim follows from the fact that a vector $v \in \mathbb{R}^{2n+2}$ is tangent to a $(2n+1)$-sphere at z iff $\langle v, z\rangle = 0$.

We now complete the proof of the lemma. Suppose $i\nabla(\log(f(z)))$ is a real multiple of z. Then $i\nabla(\log(f(z)))$ is normal to S_ϵ, so $Re\langle v, i\nabla\log(f(z))\rangle = 0$ for any vector v tangent to S_ϵ. So z is a critical point of ϕ.

Conversely, suppose $i\nabla(\log(f(z)))$ and z are linearly independent over $\mathbb{R}$. Thinking of $\mathbb{C}^{n+1}$ as a $(2n+2)$-dimensional real vector space, we conclude that there exists a vector v such that

$$\begin{aligned}\mathrm{Re}\langle v, z\rangle &= 0 \\ \mathrm{Re}\langle v, i\nabla\log f(z)\rangle &= 1\end{aligned}$$

By the criterion proved earlier in the proof, it follows that z is not a critical point of ϕ. □

Then to show that ϕ has no critical points, it suffices to prove the following lemma.

Lemma 3.7. *There exists an $\epsilon > 0$ such that for every $z \in \mathbb{C}^{n+1}$ with $|z| < \epsilon$, we have that z and $i\nabla\log(f(z))$ are linearly independent over $\mathbb{R}$.*

We will assume the following lemma without proof and use it to prove Lemma 3.7 and the Fibration Theorem. The proof is technical and unenlightening. It can be found in [2].

Lemma 3.8. *There exists an $\epsilon_0 > 0$ such that for any $z \in \mathbb{C}^m - V$ with magnitude $\leq \epsilon_0$, one of the following holds:*

(1) z and $\nabla\log(f(z))$ are linearly independent over $\mathbb{C}$, or
(2) $\nabla\log(f(z)) = \lambda z$ where λ is a non-zero complex number with $\arg(\lambda) \in (-\pi/4, \pi/4)$

Lemma 3.7 follows easily by the following argument: suppose $|z| < \epsilon_0$. If z and $\nabla\log(f(z))$ are linearly independent over $\mathbb{C}$, then z and $i\nabla\log(f(z))$ are certainly linearly independent over $\mathbb{R}$. In the other case, $\nabla\log(f(z)) = \lambda z$ with $\lambda \neq 0$ and $arg(\lambda) \in (-\pi/4, \pi, 4)$. Then it is clear that $Re(\lambda) > 0$, so the complex part of $i\lambda$ is non-zero and again z and $i\nabla log(f(z))$ are linearly independent over $\mathbb{R}$. Then we have proved Lemma 3.7. From this and Lemma 3.6, we have that ϕ has no critical points for small enough ϵ. So we know that the fibers F_θ are $2n$-dimensional differentiable manifolds.

We are now ready to start proving the Fibration Theorem. As in the proof of 2.7, we start by constructing a smooth vector field with suitable properties. Then we construct a family of smooth curves with tangent vectors given by the vector field. This will give a diffeomorphism $U \times F_\theta \to \phi^{-1}(e^{i\theta})$.

Lemma 3.9. *For sufficiently small ϵ, there is a smooth vector field $v(z)$ on $S_\epsilon - K$ of tangent vectors such that for all $z \in S_\epsilon - K$*

$$\langle v(z), i\nabla \log(f(z))\rangle \neq 0$$

and has argument in $(-\pi/4, \pi/4)$.

Proof. We construct the vector field locally and use a partition of unity to construct it globally. Let $z \in S_\epsilon - K$.

If z and $\nabla \log(f(z))$ are linearly independent over $\mathbb{C}$, then, by continuity, for y in a neighborhood of z, we have y and $\nabla \log(f(y))$ are linearly independent. Then define $v(y)$ to be a solution to the equations

$$\begin{aligned} &\langle v(y), y\rangle = 0 \\ &\langle v(y), i\nabla \log(f(y))\rangle = 1 \end{aligned}$$

Then $v(y)$ is tangent to $S_\epsilon - K$ at y by the first equation.

If z and $\nabla \log(f(z))$ are not linearly independent, then $\nabla \log(f(z)) = \lambda z$ for some complex λ with argument in $(-\pi/4, \pi/4)$ by Lemma 3.8. In this case, take $v(z) = iz$. Since $\langle z, z\rangle$ is real, it is clear that $Re\langle iz, z\rangle = 0$ so iz is tangent to $S_\epsilon - K$ at z. Finally,

$$\begin{aligned} \langle iz, i\nabla \log(f(z))\rangle &= \langle z, \nabla \log(f(z))\rangle \\ &= \langle z, \lambda z\rangle = \bar{\lambda}||z||^2 \end{aligned}$$

so $v(z)$ satisfies the requirements and can be extended to a suitable smooth tangent field in a small neighborhood of z. $\square$

Now we normalize and let

$$w(z) = \frac{v(z)}{\mathrm{Re}\langle v(z), i\nabla \log(f(z))\rangle}$$

Since $\langle v(z), i\nabla \log(f(z))\rangle$ is non-zero and has argument less than $\pi/4$ in absolute value, the denominator is nonzero. Then we have

$$\mathrm{Re}(\langle w(z), i\nabla \log(f(z))\rangle) = 1$$

Furthermore, since

$$-\frac{\pi}{4} < \arg\langle v(z), i\nabla \log(f(z))\rangle < \frac{\pi}{4}$$

it is clear that

$$|\mathrm{Im}\langle v(z), i\nabla \log(f(z))\rangle| < |\mathrm{Re}\langle v(z), i\nabla \log(f(z))\rangle|$$

Thus we have

$$\begin{aligned} |\mathrm{Re}\langle w(z), \nabla \log(f(z))\rangle| &= \left|\frac{\mathrm{Re}\langle v(z), \nabla \log(f(z))\rangle}{\mathrm{Re}\langle v(z), i\nabla \log(f(z))\rangle}\right| \\ &= \left|\frac{\mathrm{Im}\langle v(z), i\nabla \log(f(z))\rangle}{\mathrm{Re}\langle v(z), i\nabla \log(f(z))\rangle}\right| \\ &< 1 \end{aligned}$$

Lemma 3.10. *For any $z \in S_\epsilon - K$, there is a unique smooth curve $p\colon \mathbb{R} \to S_\epsilon - K$ with $p(0) = z$ and derivative $w(p(t))$.*

Proof. We can obviously construct p locally since $w(x)$ is a smooth tangent vector field on $S_\epsilon - K$. Since $S_\epsilon - K$ is not compact, we may not be able to extend the domain of a curve p to be the entire real numbers since p may reach K in finite time. Using the fact derived above that

$$|\mathrm{Re}\langle w(z), \nabla \log(f(z))\rangle| < 1$$

we have

$$\left|\frac{\mathrm{d}(\mathrm{Re}(\log f(p)))}{\mathrm{dt}}\right| < 1$$

It follows that $\mathrm{Re}(\log(f(p(t))))$ does not approach $-\infty$ as $t \to T$ for any finite T. Hence $f(p(t))$ does not approach 0 as $t \to T$. Then $p(t)$ does not approach K as $t \to T$. □

We can finally prove the Fibration Theorem.

Proof of 3.2. As in the proof of 3.6, let θ satisfy $\phi(z) = e^{i\theta(z)}$. Using the computation in that lemma, we have

$$\frac{d\theta(p(t))}{dt} = \mathrm{Re}\left\langle \frac{dp}{dt}, i\nabla \log(f)\right\rangle = 1$$

Then $\theta(p(t)) = t + c$ for some constant c, i.e. $\phi(p(t))$ wraps around S^1 with constant velocity. Now $p(t)$ is a smooth function both of t and of its starting point $z \in S_\epsilon - K$, so we can define a smooth function $h_t(z) = p(t)$ where $p(t)$ is the unique smooth curve with starting point z. For each t, the map $h_t(z)$ is a diffeomorphism of $S_\epsilon - K$ onto itself that maps $F_\theta = \phi^{-1}(e^{i\theta})$ diffeomorphically onto $F_{\theta+t}$. This proves that the fibers are all diffeomorphic. Now fix some $\theta \in (-\pi, \pi]$ and take a small neighborhood $U \subseteq S^1$ containing $e^{i\theta}$. Then we have a diffeomorphism

$$\begin{aligned} \psi\colon U \times F_\theta &\to \phi^{-1}(U) \\ (e^{i(t+\theta)}, z) &\mapsto h_t(z) \end{aligned}$$

Since $h_t(F_\theta) = F_{\theta+t}$, we have $\phi(h_t(z)) = e^{i(t+\theta)}$. Finally we conclude that $\phi|_{\phi^{-1}(U)} = \mathrm{proj}_U \circ \psi^{-1}$. □

4. The Topology of the Fibers

In this section we apply Morse theory to study the topology of the fibers. First note that $|f|$ is a smooth, real-valued function. By showing that we can approximate $|f|$ with a Morse function whose Morse index at any critical point is $\le n$, we will be able to prove the following theorem.

Theorem 4.1. *Each fiber F_θ has the homotopy type of a CW complex of dimension $\le n$.*

We can also use the Morse theory of $|f|$ to study $S_\epsilon - K$. This will give us the following theorem about K.

Theorem 4.2. *K is $(n-2)$-connected.*

Rather than using $|f|$ directly, we first study the Morse theory of $a\colon S_\epsilon - K \to \mathbb{R}$ defined by $a(z) = \log|f(z)|$. We denote its restriction to the fiber F_θ by a_θ. Since $|f|$ is positive-valued and smooth on $S_\epsilon - K$, and the real logarithm is smooth and always has non-zero derivative, a is smooth and has the same critical points as $|f|$.

Similarly, the function a_θ has the same critical points as $|f|$ restricted to F_θ. As in Lemma 3.6, we can give an alternate characterization of the critical points of a_θ.

Lemma 4.3. *A point $z \in F_\theta$ is a critical point of a_θ iff $\nabla \log(f(z))$ is a complex multiple of z.*

The proof is similar to the proof of Lemma 3.6 and is omitted. This lemma gives the following useful corollary.

Corollary 4.4. *The tangent space of F_θ at a critical point z of a_θ is a complex vector space.*

Now we want to determine the Morse index of a_θ at the critical points. We define the Hessian as follows: let v be a tangent vector at a critical point z of a_θ, and let p be a smooth curve in F_θ with initial value z and initial velocity v. Then the Hessian is the function

$$H(v) = \frac{d^2 a_\theta(p(t))}{dt^2}$$

where the derivative is evaluated at $t = 0$. It's not difficult to see that this is equivalent to the usual definition. We compute the Hessian at a critical point z in the next lemma.

Lemma 4.5. *There is a matrix of complex numbers (b_{ij}) and a positive real number c such that*

$$H(v) = \sum_{\substack{i=1,\dots,n+1 \\ j=1,\dots,n+1}} \mathrm{Re}(b_{ij} v_i v_k) - c||v||^2$$

The proof is a straightforward computation, so we only sketch it.

Proof. Since z is a critical point of a_θ, by Lemma 4.3 there is a complex number λ so that

$$\nabla \log(f(z)) = \lambda z$$

Then a simple calculation of the second derivative of $a_\theta(p(t))$ shows

$$(*) \qquad H(v) = \left\langle \frac{d^2 p(t)}{dt^2}, \lambda z \right\rangle + \sum_{\substack{i=1,\dots,n+1 \\ j=1,\dots,n+1}} D_{ij} v_i v_j$$

where

$$D_{ij} = \frac{\partial^2 \log f}{\partial z_i \, \partial z_j}$$

Then it can be easily verified that

$$\mathrm{Re} \left\langle \frac{d^2 p(t)}{dt^2}, z \right\rangle = -||v||^2$$

Using the fact that $\mathrm{Re}(\lambda) \neq 0$, we can multiply both sides of (*) by λ, take the real part of both sides, and divide out by $\mathrm{Re}(\lambda)$ to obtain the desired form for $H(v)$. $\square$

Now we can finally prove the following proposition.

Proposition 4.6. *The Morse index of a_θ at any critical point is $\geq n$. Therefore the Morse index of a at any critical point is at least n.*

Proof. We want to know the dimension of the largest subspace of the tangent space at z for which $H(v)$ is negative definite. Note that if $H(v) \geq 0$, Lemma 4.5 tells us that $H(iv) < 0$. Moreover, by Corollary 4.4, if v is tangent to F_θ at z, then iv is tangent to F_θ at z. Since the Hessian of a_θ is real and symmetric, we can decompose the tangent space at z as $V = T_0 \oplus T_1$, where H is negative definite on T_0 and positive semi-definite on T_1. Then the index of H is the dimension of T_0. But H is negative definite on iT_1, and $\dim(iT_1) = \dim(T_1)$, so $\dim(T_1) \leq \dim(T_0)$. Finally since $\dim(T_0) + \dim(T_1) = 2n$, the result for a_θ follows.

We now prove the second statement. A critical point of a is a critical point of some a_θ, and the index of a at a point z is at least the index of the a_θ at z since the tangent space of $S_\epsilon - K$ at z contains the tangent space of F_θ at z. This completes the proof. □

Now we need to find an approximation $s_\theta \colon F_\theta \to \mathbb{R}_+$ of $|f|$ with no non-degenerate critical points.

The following lemma gives us such a function.

Lemma 4.7. *There is a smooth function $s_\theta \colon F_\theta \to \mathbb{R}_+$ whose critical points are non-degenerate and have Morse index at least n. Furthermore, $s_\theta(z) = |f(z)|$ whenever $|f(z)|$ is sufficiently small. Likewise, there is a function $s \colon S_\epsilon - K \to \mathbb{R}$ with the corresponding properties.*

The proof uses basic Morse theory and the results we've established about a_θ. It can be found in [2]. We are finally ready to prove Theorem 4.1.

Proof of 4.1. We need a Morse function $g \colon F_\theta \to \mathbb{R}$ such that $g^{-1}((-\infty, c])$ is compact for every c. We claim that

$$g(z) = -\log(s_\theta(z))$$

satisfies these conditions. It is clear that g has no non-degenerate critical points since s_θ doesn't. Let $c \in \mathbb{R}$ and let $E = g^{-1}((-\infty, c])$. Since g is continuous, E is a closed subset of F_θ. Since F_θ is closed in $S_\epsilon - K$, it follows that E is closed in $S_\epsilon - K$. We know S_ϵ is compact by Heine-Borel. So to show that E is compact, it suffices to show that E is closed in S_ϵ. Since E is closed in $S_\epsilon - K$, it suffices to show that E has no limit points in K. Note that $g(z) \leq c$ iff $s_\theta(z) \geq e^{-c}$. Suppose there is a sequence of points $(a_i)_{i \in \mathbb{N}}$ in E which converges to a point $a \in K$. Then $|f(a)| = 0$. Since $s_\theta(z) = |f(z)|$ when $|f(z)|$ is sufficiently small, and $s_\theta(a_i) \geq e^{-c}$, it follows that $|f(a_i)|$ does not approach 0. This contradicts the continuity of $|f|$, so E is compact. Then we can apply standard Morse theory to g.

The index of s_θ and $\log(s_\theta)$ at any critical point is at least n. Therefore the index of g at any critical point is at most $2n - n = n$. Finally, standard Morse theory shows that F_θ has the homotopy type of a CW complex with a cell of dimension I for each critical point of index I (see [3, Theorem 3.5]). So F_θ has the homotopy type of a CW complex of dimension at most n. □

A similar argument can be used to show that K is $(n-2)$-connected. We sketch it here. See [2] for more detail.

Proof of Theorem 4.2. For $\delta > 0$, let N_δ be the set of $z \in S_\epsilon$ with $|f(z)| \leq \delta$. Clearly, N_δ contains K. It can be shown that for small enough δ, we have that N_δ is a differentiable manifold with boundary. The Morse theory of s on $S_\epsilon - \mathrm{Int}(N_\delta)$ shows that S_ϵ has the homotopy type of N_δ with some cells of dimension $\geq n$

attached. Attaching cells of dimension $\geq n$ does not change the homotopy groups in dimensions $\leq n-2$, so we have

$$\pi_i(N_\delta) = \pi_i(S_\epsilon) = 0$$

for $i = 1, \ldots, n-2$. Finally, it can be shown that K is a deformation retract of N_δ, so K is $(n-2)$-connected. $\square$

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