THE ERDŐS-SZEKERES THEOREM: A GEOMETRIC APPLICATION OF RAMSEY'S THEOREM

YUJIA PAN

Abstract. In this paper, we examine Ramsey's theorem, originally a combinatorial result, and use it to prove a result of a geometric nature, the Erdős-Szekeres theorem on convex polygons: given any positive integer k, it is possible to find a least positive integer, $ES(k)$, such that any set of at least $ES(k)$ points is guaranteed to contain the vertex set of a convex k-gon. We generalize the problem to higher dimensions and examine a better upper bound on the the Erdős-Szekeres numbers $ES(k)$.

Contents

1. A Party Problem: An Introduction to Ramsey's Theorem

Ramsey theory is a fascinating field within combinatorics concerned with the preservation of a certain property in a structure after it is partitioned. Although it is a beautiful field of mathematics in its own right, its versatility has made it useful in other areas of mathematics, including topology, ergodic theory, and, as we will see in the course of this paper, geometry.

Specifically, the study of Ramsey theory is motivated by the following question: how large must a structure be so that after it is partitioned into (finitely) many substructures, at least one of the substructures is guaranteed to have a certain property of interest? While this motivating question may seem relatively straightforward, it is not always so simple to answer, as evidenced by the many Ramsey-style problems that remain open.

One central result is Ramsey's theorem, a powerful generalization of the pigeonhole principle. We introduce Ramsey's theorem by asking the reader to ponder a classic "party problem".

Date: August 30, 2013.

Party Problem 1.1. *Suppose there are six people at a party. Is it true that we are we guaranteed to find either: (1) a group of three people who are mutually acquainted or (2) a group of three people who are mutually unacquainted?*

Note that this is clearly a Ramsey-style problem; the property of interest is pairwise acquaintanceship, and the underlying aim is to find out how large the group needs to be to guarantee pairwise acquaintanceship in (at least) one of its parts if the group is partitioned into 3-element parts. The answer is yes, which is easy to see once one reinterprets the problem in the language of graph theory. To do this, we first introduce several useful definitions and some notation.

Definition 1.2. A *simple undirected graph* $G(V, E)$ is a pair of sets such that V denotes the vertex set of G and E, a set of two-element subsets of V, denotes the edge set of G.

Definition 1.3. A *complete graph* on n vertices, denoted K_n, is a simple undirected graph such that there exists an edge between i and j for all $i, j \in V$, provided that $i \neq j$.

Definition 1.4. An *r-coloring* of a graph G is a function $\chi : E \to C$ where $|C| = r$.

Notation 1.5. Given a complete graph K_N, we write

$$K_N \to K_{p_1}, \ldots, K_{p_r}$$

if it is true that any r-coloring of K_N is guaranteed to contain a monochromatic subgraph K_{p_i} for some $i \in \{1, 2, \ldots, r\}$.

Now we can reinterpret the problem. We represent each person by a vertex, and consider the complete graph on these vertices. We color the edges in the following way: if two people know each other, color the edge between their representative vertices red; if they do not know each other, color the edge between their representative vertices blue. The party problem, then, asks us whether we can find a a complete graph on three vertices, all of whose edges are red (a red triangle) or a complete graph on three vertices, all of whose edges are blue (a blue triangle) given a complete graph on six vertices. That is, we must show

$$K_6 \to K_3, K_3.$$

Let us be given a complete graph on six vertices. Choose one vertex and call it x. The point x has five edges emanating from it, so by the pigeonhole principle, it must be that three of these edges are the same color. Without loss of generality, suppose that this color is red. Now label the other endpoints of these red edges A, B, and C and consider the edges AB, BC, and AC. If at least one of these edges is red, then will have a red triangle. Otherwise, all of these edges are blue and we have a blue triangle. This proves that we may answer the party problem in the affirmative.

Technically, we have only shown that a group of six people is *sufficient* to guarantee this property. However, we have not shown that it is *necessary*. That is, we have shown that $R(3, 3) \leq 6$, but not that $R(3, 3) = 6$. To prove the equality, it simply remains to be shown that $R(3, 3) > 5$. This is easily done by exhibiting a two-coloring of K_5 that does not contain a monochromatic K_3 (see Figure 1).

Here we have shown that $R(3, 3) = 6$ without much difficulty. In general, however, it is difficult to compute the exact value of Ramsey numbers, even when we

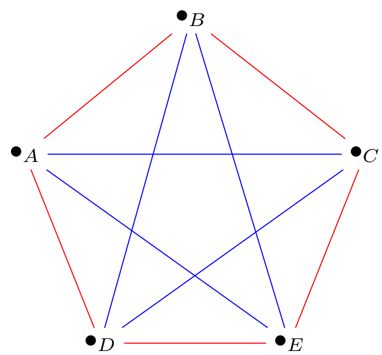


Figure 1.

ignore hypergraphs (which we will cover in the next section) and restrict ourselves to colorings of graphs. It is still an open problem to determine better bounds (or better yet, a formula, if one exists) for these numbers.

2. Ramsey's Theorem

Now that the reader has been exposed to the "flavor" of Ramsey-style problems, we can examine Ramsey's theorem in its original graph-theoretic terms.

While it is unecessary to prove the following theorem in order to prove Ramsey's theorem for hypergraphs in r colors (which is the form of the theorem we use in the proof of the Erdős-Szekeres theorem), we present this proof to familiarize the reader with Ramsey's theorem in a simpler case.

Theorem 2.1. *Ramsey's theorem (for graphs, two colors). For integers $p, q \geq 2$, there exists a least positive integer $r(p, q)$ such that any two-coloring of K_N is guaranteed to contain either a monochromatic subgraph, K_p, or a monochromatic subgraph, K_q.*

Proof. We proceed by double induction on $p \geq 2$ and $q \geq 2$.

It is easy to check that $R(p, 2) = p$. Consider a two-coloring of K_p with red and blue. If one of the edges is red, then we obtain a monochromatic K_2 and we are done. If not, then all the edges are blue and we have a monochromatic K_p, so we are also done. This shows that $R(p, 2) \leq p$. Now consider a two-coloring of the graph K_{p-1} in which all edges are colored blue. Then this graph does not admit a blue K_p, nor does it admit a red K_2. Therefore $R(p, 2) > p$, which, with the previous inequality, establishes equality. An analagous argument establishes that $R(2, q) = q$.

As our induction hypothesis, we assume that the numbers $R(p, q-1)$ and $R(p-1, q)$ exist for $q, p \geq 3$. To show that the number $r(p, q)$ exists, it is sufficient to bound $R(p, q)$ from above. We claim that $R(p, q) \leq R(p, q-1) + R(p-1, q)$.

Let $N = R(p, q-1) + R(p-1, q)$. We will show that $K_N \to K_p, K_q$. Consider one vertex of K_N and call it x. Note that the edges emanating from x are colored either red or blue. Let R denote the set of vertices connected to x by a red edge and let B denote the set of vertices connected to x by a blue edge.

It must be true that x is connected to $N-1$ vertices, so it must be true that

$$|R| + |B| = N - 1.$$

Thus, it is true that

$$|R| + |B| = R(p, q-1) + R(p-1, q) - 1.$$

Note that it must be the case that either $|R| \geq r(p, q-1)$ or $|B| \geq R(p-1, q)$. If this is not the case, then $|R| \leq R(p, q-1) - 1$ and $|B| \leq R(p-1, q) - 1$, which gives us

$$|R| + |B| \leq R(p, q-1) + R(p-1, q) - 2 = N - 2$$

which contradicts the fact that x must be connected to $N-1$ vertices.

Without loss of generality, suppose $|R| \geq R(p, q-1)$. Consider $K_{|R|}$ on the vertices of R, which is a subgraph of our original K_N.

Then $K_{|R|} \to K_p, K_{q-1}$. If our complete graph on $|R|$ vertices contains a subgraph K_p for which every edge is red, then we are done. If instead, our complete graph on $|R|$ vertices contains a subgraph K_{q-1} for which every edge is blue, then we are also done; since x is connected to every vertex in B by a blue edge, we can append x to the subgraph K_{q-1} to obtain a blue K_q. $\square$

Having shown this result for graphs, it is natural to ask whether such numbers exist for hypergraphs, in which the edges are no longer subsets of vertices of size two but rather subsets of arbitrary size.

Definition 2.2. [7] A *k-uniform hypergraph* on n vertices, denoted K_n^k is a pair $H(V, E)$ such that V denotes the set of vertices and E denotes the set of k-element subsets of V, which we call *hyperedges* of the k-uniform hypergraph. Henceforth, we will refer to "k-uniform hypergraphs" simply as "hypergraphs".

One method of proof for proving Ramsey's theorem for hypergraphs in r colors involves proving the infinite case, then applying the compactness principle to derive the finite case [3]. However, we present a different proof.

Theorem 2.3. *Ramsey's theorem (for hypergraphs, two colors). Given $k > 0$ and $p, q \geq k$, there exists $N = R^k(p, q)$ such that for all $n \geq N$ and for all two-colorings of size-k subsets of $\{1, \ldots, n\}$ there exists either a subset T of size p for which every subset of size k is red or a subset T of size q for which every subset of size k is blue.*

Proof. It is trivial to check that $R^1(p, q)$ exists. We assume, as our induction hypothesis, that $R^{k-1}(p, q)$ exists for all p and q.

Let $N = R^{k-1}(R^k(p-1, q), R^k(p, q-1)) + 1$. Consider K_N^k and color all k-subsets either red or blue. Let us denote this coloring of K_N^k by χ. Now choose some vertex x and consider all subsets of size $k-1$ not containing x. Call this set S. S has the structure of K_{N-1}^{k-1} and has a coloring, χ', induced in the following way: $\chi'(J) = \chi(J \cup \{x\})$ for all $J \in S$.

By Ramsey's theorem, we are guaranteed one of the following:

(1) S has a subset B, $|B| = R^k(p-1, q)$, and all $(k-1)$-subsets of B are blue.
(2) S has a subset R, $|R| = R^k(p, q-1)$, and all $(k-1)$-subsets of R are red.

Without loss of generality, suppose we have the first case. By induction, we may assume that $R^k(p-1, q)$ exists, thus guaranteeing one of the following:

(1) B has a subset T of size $p-1$ for which all k-subsets are blue.
(2) B has a subset T' of size q for which all k-subsets are red.

If we have the second case, then we are done. If we have the first case, then we can adjoin x to T to obtain a set of size p for which all k-subsets are blue. To see this, let $\widetilde{T} = T \cup \{x\}$. There are two kinds of k-subsets to consider. If a k-subset of $\widetilde{T}$ contains x, then it is blue by the induced coloration of $(k-1)$-subsets of B. If the k-subset does not contain x, then it is a k-subset of T and is therefore blue by assumption. □

Theorem 2.4. *Ramsey's theorem (for hypergraphs, r colors). Let $n_1, n_2, \ldots, n_r$ be positive integers. Given $r \geq 2$ colors and a "size" $k > 0$, there exists N such that for all $n \geq N$ and for all r-colorings of k-subsets of $\{1, \ldots, n\}$ there exists $1 \leq i \leq r$ such that there is a subset $T \subset \{1, \ldots, n\}$ of size n_i such that every subset $S \subseteq T$ where $|S| = k$ is colored with color i.*

Proof. By (2.3), we know $R^k(n_1, \ldots, n_{r-1})$ exists for $r = 3$, our base case. To prove the theorem, we proceed by induction on r, the number of colors. Let

$$N = R^k(n_1, \ldots, n_{r-2}, R^k(n_{r-1}, n_r)).$$

By our induction hypothesis, we have one of the following:

(1) We have a subset T of size n_i for which all k-subsets are color i for $i \in \{1, \ldots, r-2\}$.
(2) We have a subset T of size $R^k(n_{r-1}, n_r)$ for which all k-subsets are either color $r-1$ or color r. By definition, we have either a set $S \subset T$ of size n_{r-1} for which all k-subsets are color $r-1$ or a set $S' \subset T$ of size n_r for which all k-subsets are color r.

This proves the theorem. □

3. The Erdős-Szekeres Theorem

We now use Ramsey's theorem to prove the Erdős-Szekeres theorem on convex polygons, a striking application of combinatorics to prove a result in geometry.

Let us first introduce some necessary definitions.

Definition 3.1. A set of points in the plane is in *general position* if no three points are colinear.

Definition 3.2. We say that a set is *convex* if for any two points in the set, the line segment connecting the points is entirely contained in the set.

Definition 3.3. The *convex hull* of a set of points S in the plane is the smallest convex set that contains S. Equivalently, the convex hull of a set of points S in the plane is the union of all triangles with vertices in S.

Definition 3.4. We say that a set of points S is in *convex position* if no point $x \in S$ lies in the interior of the convex hull formed by the points of $S \backslash \{x\}$.

Definition 3.5. Let P be a polygon formed by k vertices. We say that P is a *convex k-gon* if and only if none of the k points lies in the interior of P.

Definition 3.6. [6] Let P be a polygon. A *triangulation of* P is the decomposition of P into triangles, such that the union of the triangles is P and the intersection of any two triangles is a common edge. This edge may be empty.

Theorem 3.7. *(Erdős-Szekeres) Given a positive integer k, there exists a least positive integer $ES(k)$ so that any set of $n \geq ES(k)$ points in general position contains the vertex set of a convex k-gon.*

To prove this theorem, we make use of the following lemma.

Lemma 3.8. *Any set of five points in general position contains the vertex set of a convex quadrilateral. That is, $ES(4) = 5$.*

Proof. (Lemma) It is clear that $ES(4) > 4$. We will show that $ES(4) \leq 5$. Consider the convex hull of the set of five points. If the convex hull is a pentagon or a quadrilateral, then we are done. If the convex hull is a triangle, then two points must lie in the interior of the triangle. Draw a line through both of these points. By the pigeonhole principle, two vertices of the triangle must lie on one side of the line. Connect those vertices to the two points through which the line was originally drawn to obtain a convex quadrilateral. Therefore $ES(4) \leq 5$, which, with the previous inequality, establishes equality. □

Proof. (Erdős-Szekeres) Let us be given at least $R^4(k, 5)$ points in general position. For every subset of 4 points, consider the polygon formed by their convex hull. If it is a quadrilateral, color it red. If it is a triangle, color it blue. We claim that $ES(k) \leq R^4(k, 5)$. By Ramsey's theorem, we are guaranteed one of the following:

(1) A subset of size k such that all 4-subsets are red.
(2) A subset of size 5 such that all 4-subsets are blue.

However, the second case is impossible by our lemma. Therefore we are guaranteed a subset of k points such that any subset of four points is red (i.e. is the vertex set of a convex quadrilateral).

Consider this set of k points. Let H be the convex hull that they determine. If we can show that H is an k-gon, then we will be done. Suppose towards a contradiction that H is not an k-gon. Then there exists some point x in the interior of H. Now triangulate H. It must be that x lies in the interior of one of the triangular partitions. (If it did not, then we would have three colinear points, which contradicts the condition that these points are in general position.) Since all 4-subsets are red (i.e. in convex position), x along with the three vertices of the triangular partition containing it are red. But these four points are not the vertex set of a convex 4-gon, a contradiction. Therefore we have that H is an k-gon.

□

4. Better Upper Bounds on $ES(k)$

In proving that the Erdős-Szekeres numbers exist in two dimensions, we obtained an upper bound on $ES(k)$; namely, we have that $ES(k) \leq R^4(k, 5)$. However, this bound not tight. Ramsey's theorem allows us to create fairly elegant existential proofs, but typically yields very large bounds.

As is the case for the classical Ramsey numbers $R(p, q)$, it is still an open problem to determine better bounds (or a formula, if one exists) for these numbers. A wide variety of techniques have been employed in an attempt to solve this problem. We begin this section by following another interesting proof of the Erdős-Szekeres theorem from Bòna [1], which uses a different coloring technique and yields a slightly better upper bound.

Theorem 4.1. $ES(k) \leq R^3(k,k)$.

Proof. Let us be given a set of $R^3(k,k)$ points in the plane. Enumerate these points from 1 to $R^3(k,k)$. Now consider the complete graph on these points, $K_{R^3(k,k)}$. Let $\{x_i, x_j, x_k\}$, where $i < j < k$, be a 3-subset of our $R^3(k,k)$ points. These points are the vertices of a triangle in $K_{R^3(k,k)}$. Color the 3-subset red if the path through x_i, x_j, x_k, in that order, runs clockwise. Color it blue otherwise.

By Ramsey's theorem, we are guaranteed a subset of points, S, of size k for which all 3-subsets are the same color. If we can show that the points of S form a convex k-gon, then we will be done. Suppose towards a contradiction that the points of S do not form the vertex set of a convex k-gon. Then there exists a point x that lies inside the convex hull of $S \backslash \{x\}$. Now triangulate the convex hull. It must be that x lies in one of the triangular partitions, as shown in Figure 2.

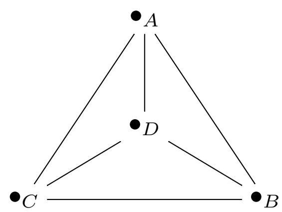


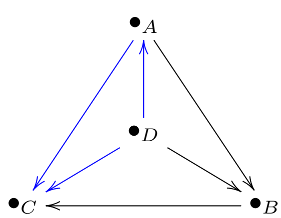
Figure 2.

Without loss of generality, let $A < B < C$ and suppose all triangles are colored red. Since triangle ABD is red, it must be true that either: (1) $D < A < B$ or (2) $A < B < D$.

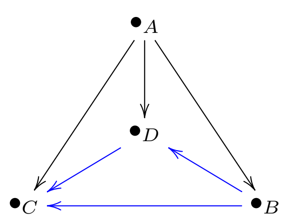
(1) If $D < A < B$, then it must be that $D < A < C$ and triangle DAC is blue, which is a contradiction.

(2) If $A < B < D$, then it could be that either $D < C$ or $C < D$. If we have the first case, then $B < D < C$ and triangle DBC is blue, which is a contradiction. If we have the second case, then $A < C < D$ and triangle ADC is blue, which is again a contradiction. Therefore, the points of S determine a convex k-gon, and this proves the theorem.

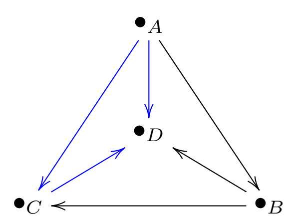
The figures below illustrate the contradictory triangles one obtains in each of the cases. A directed edge between two vertices x and y such that the arrow emmanates from x and points towards y means "x is less than y".



Case (1).



Case (2), $D < C$



Case (2), $D < C$

□

Up to this point, we have used coloring strategies to achieve upper bounds on the Erdős-Szekeres numbers. However, it is not the only technique at our disposal.

Here we follow a proof, originally by Erdős and Szekeres, that uses a geometric technique to obtain an even better upper bound on $ES(k)$.

We first introduce some useful notation and definitions.

Notation 4.2. We denote the slope between two points x and y by $m(x, y)$.

Definition 4.3. Let there be a coordinate system (x, y) in the plane. Let $\{a_{i_1}, a_{i_2}, \dots, a_{i_r}\}$ be a set of points in general position in the plane, such that the x-coordinate of a_{i_k} is less than the x-coordinate of $a_{i_{k+1}}$ for all $k \in \{1, 2, \dots, r-1\}$. (Note that by rotating the plane, we can ensure that no two points share the same x-coordinate.) We say that $\{a_{i_1}, a_{i_2}, \dots, a_{i_r}\}$ is an *r-cup* if

$$m(a_{i_1}, a_{i_2}) < m(a_{i_2}, a_{i_3}) < \cdots < m(a_{i_{r-1}}, a_{i_r}).$$

Definition 4.4. Under the same conditions above, we say that $\{a_1, a_2, \dots, a_r\}$ is an *r-cap* if

$$m(a_{i_1}, a_{i_2}) > m(a_{i_2}, a_{i_3}) > \cdots > m(a_{i_{r-1}}, a_{i_r}).$$

Following the notation used by Morris in [1], we let $f(k, l)$ be the least positive integer such that any set of at least $f(k, l)$ points contains either a k-cup or an l-cap.

Theorem 4.5. $f(k, l) \leq \binom{k+l-4}{k-2} + 1$.

Proof. It is easy to check that $f(k, 3) = k$. Let $S = \{a_{i_1}, a_{i_2}, \dots, a_{i_k}\}$ be a set of k points such that the x-coordinate of a_{i_t} is less than the x-coordinate of $a_{i_{t+1}}$ for all $t \in \{1, 2, \dots, k-1\}$. If this set contains a 3-cap, then we are done. If not, then

$$m(a_{i_1}, a_{i_2}) \leq m(a_{i_2}, a_{i_3}) \leq \cdots \leq m(a_{i_{k-1}}, a_{i_k}).$$

However, no two quantities in the expression above can be equal, since having two line segments with the same slope would contradict the assumption that our points are in general position. Therefore we have the following:

$$m(a_{i_1}, a_{i_2}) < m(a_{i_2}, a_{i_3}) < \cdots < m(a_{i_{k-1}}, a_{i_k}).$$

That is, we have a k-cup. By symmetry, we also have $f(k, 3) = f(3, k)$.

We now prove the recurrence, $f(k, l) \leq f(k, l-1) + f(k-1, l) + 1$. Let us be given a set X of $f(k, l-1) + f(k-1, l) + 1$ points in the plane. Let L be the set of left endpoints of $(k-1)$-cups of X.

If $X \backslash L$ contains $f(k-1, l)$ points, then it must contain an l-cap (since it cannot contain a $k-1$ cup).

If $X \backslash L$ does not contain $f(k-1, l)$ points, then it contains at most $f(k-1, l)-1$ points, which implies that L contains at least $f(k, l-1)$ points. If L contains a k-cup, then we are done. Now suppose L contains a $l-1$-cap $\{a_{i_1}, a_{i_2}, \dots, a_{i_{l-1}}\}$. Since $a_{i_{l-1}}$ belongs to L, we know that it is the left endpoint of some $k-1$ cup in X. Let $\{a_{j_1}, a_{j_2}, \dots, a_{j_{k-1}}\}$ be a $k-1$ cup such that $i_{l-1} = j_1$.

Now consider $m(a_{i_{l-2}}, a_{i_{l-1}})$ and $m(a_{i_{l-1}}, a_{j_2})$. If $m(a_{i_{l-2}}, a_{i_{l-1}}) < m(a_{i_{l-1}}, a_{j_2})$, then we can add $a_{i_{l-2}}$ to the $k-1$ cup to obtain a k-cup. If $m(a_{i_{l-2}}, a_{i_{l-1}}) > m(a_{i_{l-1}}, a_{j_2})$, then we can add a_{j_2} to the $l-1$ cap to obtain an l-cap.

Having proven the recurrence, we proceed to prove the theorem by induction. We assume as our inductive hypothesis that $f(k, l) \leq \binom{k+l-4}{k-2} + 1$ for all $k + l < s$.

To prove the statement for $k + l = s$, we apply the inductive hypothesis to the recurrence.

$$f(k,l) \leq f(k-1,l) + f(k,l-1) - 1$$

$$f(k,l) \leq \binom{k+l-5}{k-3} + \binom{k+l-5}{k-2} + 1$$

Appealing to Pascal's identity gives us the desired upper bound.

$$f(k,l) \leq \binom{k+l-4}{k-2} + 1.$$

□

Now let us observe that $ES(k) \leq f(k,k)$. Given $f(k,k)$ points, we are guaranteed either a k-cup or a k-cap. If we are given a k-cup $\{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\}$, we can connect a_{i_1} to a_{i_k} to obtain a convex k-gon. Similarly, if we are given a k-cap, we can obtain a convex k-gon.

With this observation as well as Theorem 4.5, we obtain the following upper bound for the Erdős-Szekeres numbers:

$$ES(k) \leq \binom{k+l-4}{k-2} + 1.$$

5. Higher Dimensions

Having shown the Erdős-Szekeres theorem in two dimensions, it is natural to ask whether a similar statement holds in higher dimensions. We can, in fact, prove the theorem for $\mathbb{R}^d$. However, this requires that we first generalize our concepts of convex hulls, points in general position, convexity, and triangulation.

Definition 5.1. [2] The *convex hull* of a set of points $S = \{x_1, x_2, \ldots, x_n\}$ in $\mathbb{R}^d$ is the set of all convex combinations of points in S. That is, the convex hull of S is the set:

$$\{\sum_{i=1}^{n} \alpha_i x_i : \sum_{i=1}^{n} \alpha_i = 1, \alpha_i \geq 0\}$$

for all i.

Definition 5.2. [2] A set of points $S \subset \mathbb{R}^d$ is in *general position* if for all $T \subset S$ such that $|T| = d+1$, no $x \in T$ lies in the convex hull of $T \backslash \{x\}$.

Definition 5.3. We say that a set of points S is in *convex position* if no point $x \in S$ lies in the interior of the convex hull formed by the points of $S \backslash \{x\}$. Equivalently, a set of points S is in convex position if and only if they form the vertex set of a convex polytope in $\mathbb{R}^d$.

Definition 5.4. A *convex k-polytope* is the convex hull of k points in convex position.

Definition 5.5. A *d-simplex* is a d-dimensional polytope that coincides with its convex hull on its $d+1$ vertices.

Definition 5.6. [6] A *triangulation* of a polytope P in $\mathbb{R}^d$ is the decomposition of P into d-simplices, such that the union of the d-simplices is P and the intersection of any two d-simplices is a shared face. Note that this face may be empty.

We also make use of the following lemma. For a proof, the interested reader is encouraged to consult [5].

Lemma 5.7. *Any set of $d+3$ points in general position in general position in $\mathbb{R}^d$ contains some subset of $d+2$ points in convex position.*

Now we are able to prove the following generalization.

Theorem 5.8. *Given a positive integer, k, there exists a least positive integer $ES^d(k)$ so that any set of at least $ES^d(k)$ points in general position in $\mathbb{R}^d$ contains the vertex set of a convex k-polytope.*

Proof. Let us be given $R^{d+2}(k, d+3)$ points in general position. For every subset of $d+2$ points, consider whether or not the points are in convex position. If they are in convex position, color the subset red. Otherwise, color the subset blue. We claim that $N \leq R^{d+2}(k, d+3)$.

By Ramsey's theorem, we are guaranteed one of the following:

(1) A subset of size k such that all subsets of size $d+2$ are red.
(2) A subset of size $d+3$ such that all subsets of size $d+2$ are blue.

However, the second case is impossible by (3.7). Therefore we are guaranteed a subset of k points such that all subsets of size $d+2$ are red.

Let H be the convex hull determined by these k points. If we can show that these k points are in convex position, then we will be done. Suppose towards a contradiction that these k points are not in convex position. Then there must exist some point x that lies in the interior of H. Now triangulate H. It must be that x lies in one of the d-simplices. Let S be the set consisting of x and the $d+1$ vertices of the d-simplex containing x.

Since all subsets of size $d+2$ are red, it must be true that S is red. But S is not the vertex set of a convex polytope on $d+2$ vertices, therefore these points are blue. This is a contradiction; therefore we have that our k points are in convex position. □

Acknowledgments. It is a pleasure to thank my mentor, Chris Skalit, for his patient guidance and unwavering support. His helpful references and willingness to delve into the world of combinatorics made this paper a possibility. Ultimately, I thank him for showing me how to be a better mathematical reader and thinker.

I would also like to thank Peter May for organizing the REU. To spend a summer learning just how diverse and beautiful mathematics can be has been a truly enjoyable and rewarding experience.

References

[1] M. Bòna. *A Walk Through Combinatorics*, World Scientific, Second Edition, Singapore, 2006.
[2] A. Brøndsted, *An Introduction to Convex Polytopes*, Springer, New York, 1983.
[3] P. Cameron, *Sets, Logic and Categories*, Springer, London, 1998.

[4] W. Morris and V. Soltan, *The Erdős-Szekeres Problem on Points in Convex Position - A Survey*, Bull. Amer. Math. Soc. (N.S). 37 (2000), 437–458.
[5] T. Motzkin, *Cooperative Classes of Finite Sets in One and More Dimensions*, J. Combin. Theory, 3 (1967), 244–251.
[6] F. Santos, *Triangulations of Polytopes*, http://personales.unican.es/santosf/Talks/icm2006.pdf.
[7] D. B. West, *Introduction to Graph Theory*, Second Edition, Pearson, 2000.