THE BLACK-SCHOLES-MERTON EQUATIONS IN PRACTICE

TEDDY NIEMIEC

Abstract. This expository paper will investigate results of the most famous model in financial mathematics: the Black-Scholes-Merton equations. We will begin by defining some relatively basic mathematical terms and introducing core concepts. After establishing this groundwork, we will encounter the realm of finance. In this part, we will present various financial definitions that are crucial to understanding markets. Then we will investigate our chosen application of financial mathematics: the Black-Scholes-Merton equations. The assumptions for the model will be stated, and we will provide a proof for the derivation of the equations. Finally, we will conclude by reflecting upon the practical implications of this model.

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1. Basic Mathematical Vocabulary and Concepts

Probability and Random Variables. First let us discuss some foundational probability concepts [1].

Definition 1.1. A *sample space* Ω is a set comprised of elements ω. Each ω in Ω is an *outcome*.

Definition 1.2. A σ-algebra S of subsets Y of a set X is a collection of subsets that satisfies the following requirements:

- S contains X.
- S is closed under complementation, meaning for any element $Y \in S$, it is true that its complement Y^C is an element of S.
- S is closed under countable union.

Remark 1.3. For our purposes, we define the σ-algebra $\mathcal{F}$ of Ω as comprised of elements E, where each E is an event.

Definition 1.4. The *probability function* $\mathbb{P}\colon \mathcal{F} \to [0,1]$ is a function that satisfies the following:

- $\mathbb{P}(\Omega) = 1$
- Consider events $E_1, E_2, \ldots \in \mathcal{F}$ such that the events are all disjoint. Then
$$\mathbb{P}\left[\bigcup_{j=1}^{\infty} E_j\right] = \sum_{j=1}^{\infty} \mathbb{P}\left[E_j\right]$$

As mathematicians, we use $\mathbb{P}(E)$ to express the *probability of an event* E *occurring.*

These terms help bring us to our notion of a probability space:

Definition 1.5. A *probability space* is a sample space Ω taken with the σ-algebra $\mathcal{F}$ and the function $\mathbb{P}$, each with the respective properties outlined above. It is written as $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 1.6. A *Borel set*, denoted B, is any set in an arbitrary topological space that can be formed through the countable union, countable intersection, or relative complement of open sets.

Definition 1.7. A *random variable* is a measurable function $X\colon \Omega \to \mathbb{R}$ such that for every Borel set B on the standard topology on $\mathbb{R}$, we may write
$$X^{-1}(B) = \{\omega \in \Omega \mid X(\omega) \in B\} \in \mathcal{F}$$

Probability Density Functions and Expected Value. Expected value is a concept that builds upon the foundations laid above. It figures strongly in making predictions about the future and provides some motivation for our assumptions.

Definition 1.8. The random variable X is *continuous* if there exists a nonnegative function $f : \mathbb{R} \to \mathbb{R}$ such that for any $B \subset \mathbb{R}$, we have
$$P\{X \in B\} = \int_B f(x)dx$$
We call $f(x)$ the *probability density function* of X.

Definition 1.9. Consider some nonnegative random variable X. We define the function

$$\mathbb{E}(X) = \int X d\mathbb{P}$$

expressed with the Lebesgue integral, as the *expected value function.* If X is a random variable and $\mathbb{E}(|X|) < \infty$, then we define $\mathbb{E}(X)$ as above. If X takes positive and negative values and $\mathbb{E}(|X|) = \infty$, however, $\mathbb{E}(X)$ is undefined.

Definition 1.10. The nth *moment* of a continuous variable X is calculated as follows:

$$\mathbb{E}[X^n] = \int_{-\infty}^{\infty} x^n f(x) dx$$

Remark 1.11. We often call $\mathbb{E}(X)$ the *mean* of the random variable X. It is also the first moment of X.

Definition 1.12. The *variance* of a random variable X is given by

$$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

The *standard deviation* of X is simply the square root of the variance.

Remark 1.13. Note that the variance of X is the second moment of X.

The Normal Distribution and Wiener Processes. Now we shall introduce a special distribution known as the normal (Gaussian) distribution, which is popularly employed in statistics. We will also define a number of stochastic processes, which is important in modeling the seemingly random nature of the markets.

Definition 1.14. The *normal (Gaussian) distribution* of a random variable X is given by

$$f(X) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(X-\mu)^2}{2\sigma^2}\right)$$

We write it as $N(\mu, \sigma)$ or $\phi(\mu, \sigma)$ or something similar, where μ is its mean and σ is its standard deviation.

Definition 1.15. A stochastic (random) process is said to be *Markov* if its future value depends only on the present value and not the past. We write it formally as

$$\mathbb{P}[X_{n+1} = x_{n+1} \mid X_1 = x_1, \ldots, X_n = x_n] = \mathbb{P}[X_{n+1} = x_{n+1} \mid X_n = x_n]$$

Remark 1.16. Even if something follows a Markov process, it is still possible that past behavior is important. For example, volatility (discussed later) is an important concept that is gathered from the past history of a stock price; however this does not affect the future expected value.

Definition 1.17. A process of a variable z is said to be *Wiener* if the following properties hold:

- The change Δz during a small period of time Δt is

  $$\Delta z = \epsilon\sqrt{\Delta t}$$

  where ϵ follows a standardized normal distribution $\phi(0, 1)$.

- For any two different short intervals of time Δt, the resultant values of Δz are independent.

Remark 1.18. Note that a Wiener process is a special type of Markov process. Also, Δz has a mean of zero, a variance of Δt, and a standard deviation of $\sqrt{\Delta t}$ because we are simply transforming our normal distribution.

Ito's Lemma. This final subsection will introduce Ito's Lemma, a powerful differential result that has many applications, especially in financial mathematics. The proof is detailed and will not be provided. Only the result will be included.

Proposition 1.19 (Ito's Lemma)**.** *Given a differentiable equation G of variables x and t, where x follows the Ito's process given by*

$$dx = a(x,t)dt + b(x,t)dz$$

we may write

$$dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2(x,t)dt$$

Corollary 1.20. *If the above criteria are met, we can easily substitute and write*

$$dG = \left(\frac{\partial G}{\partial x}a(x,t) + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2(x,t)\right)dt + \frac{\partial G}{\partial x}b(x,t)dz$$

2. Basic Financial Vocabulary and Concepts

Definition 2.1. A *security* is a negotiable financial instrument that represents a type of financial value. A *derivative* is a kind of security that is so named because it derives its price from one or more underlying assets.

Definition 2.2. An individual can go *long* a security by buying it and hoping that the price goes up. Conversely, an individual can *short* sell a security by borrowing it, selling it immediately, and hoping that the price goes down to be bought back later.

Definition 2.3. An *options contract* is a type of derivative. It is sold by the option writer to the option holder. An options contract may be either a call or a put. A *call* provides the holder the right but not the obligation to purchase a specified quantity of a financial instrument at a *strike price* K from the writer at some specified time on or prior to the options contract's expiration. An analogous definition of a *put* may be given, but as a right to sell. The time of purchase in a call – or, analogously, the time of sale in a put – is known as the *exercise time.* When the underlying financial instrument costs more than the strike price (in a call) or less (in a put), then the options contract is said to be *in-the-money.* Otherwise, we say it is *out-of-the-money* or *at-the-money* if the price is very close to the strike.

Definition 2.4. A *European* options contract is one that must be exercised at a specific date. In contrast, an *American* options contract is one that does not have a specified exercise time; it may be exercised at any time prior to its expiration.

Definition 2.5. The *intrinsic value* of an options contract for the option holder is a function $g\colon \mathbb{R} \to \mathbb{R}$ that reflects the difference in the strike price K and the underlying stock price s of the option. For a call, $g(s) = s - K$. For a put, $g(s) = K - s$.

Definition 2.6. The *interest rate* r represents a riskless return that can be made that yields $1 + r$ dollars at time one for a dollar invested in the money market at time zero. Similarly, a dollar borrowed at time zero from the money market results in a debt of $1 + r$ dollars.

Definition 2.7. *Arbitrage* is a trading strategy that arises in inefficient markets. An investor taking advantage of arbitrage can start with no money, have zero probability of losing money, and have positive probability of making money.

Definition 2.8. *Hedging* is the process whereby an investor minimizes risk by investing in a way so as to control for market fluctuations.

Definition 2.9. *Volatility* of a financial instrument quantifies how drastically the instrument may change in value.

Remark 2.10. The definition for volatility may seem a bit loose, and it really is. It is difficult to quantify volatility, and there are many approaches to do it, none of which is 100% correct. For our purposes, the volatility calculated will be the standard deviation of returns.

3. The Black-Scholes-Merton Model

The Seven Assumptions. We are now ready to derive the Black-Scholes-Merton model for pricing European calls and puts that do not pay dividends. In order to do so, we shall make the following assumptions:

1. There are no riskless arbitrage opportunities. After all, if such arbitrage opportunities existed, they would probably be taken advantage of very quickly.
2. Security trading is continuous, meaning that there are no "jumps" in price quotes.
3. Securities are perfectly divisible and there are no transactions costs of any sort.
4. There is a constant risk-free rate of interest r that applies to all securities and maturities.
5. The securities in question do not pay dividends.
6. Individuals can short securities and reinvest all proceeds immediately.
7. The price of the underlying stock follows the process

$$dS = \mu S dt + \sigma S dz$$

Derivation of the Differential Equation. With these assumptions made, we may derive the key pricing equations for a European options contract, beginning with a call [2]. Analogous logic provides the price for the put. First, we shall add a lemma:

Lemma 3.1. *Our stock price may be written as*

$$\ln(S_T) \sim \phi\left(\ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right)$$

Proof. Recall Ito's Lemma

$$dG = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial G}{\partial S}\sigma S dz$$

Let $G = \ln S$. So

$$\frac{\partial G}{\partial S} = \frac{1}{S} \qquad \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2} \qquad \frac{\partial G}{\partial t} = 0$$

If we substitute this into the formula from Ito's Lemma, we get

$$\begin{aligned} dG &= \left(\frac{1}{S}\mu S + 0 + \frac{1}{2}\frac{-1}{S^2}\sigma^2 S^2\right)dt + \frac{1}{S}\sigma S dz \\ &= \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dz \end{aligned}$$

Therefore, G follows a Wiener process with mean $\mu - \dfrac{\sigma^2}{2}$ and variance σ^2. So over any period of time T, the stock price also follows a process with a mean of $\left(\mu - \dfrac{\sigma^2}{2}\right)T$ and variance $\sigma^2 T$, meaning its standard deviation is $\sigma\sqrt{T}$. We may write this as

$$\ln(S_T) - \ln(S_0) \sim \phi\left(\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right)$$

Because the normal distribution is additive under transformations, we can just add $\ln(S_0)$ to each side and complete the proof. ∎

Before proceeding, we must introduce a seemingly random lemma. Its significance will become apparent immediately after it is proved.

Lemma 3.2. *A random variable V with lognormal distribution and with stdev $(\ln(V)) = w$ has mean*

$$m = \ln[\mathbb{E}(V)] - \frac{w^2}{2}$$

Proof. Let $X = \ln(V)$. Then X is normally distributed with $N(m, w^2)$. The Gaussian distribution for X is given by

$$f(X) = \frac{1}{\sqrt{2\pi}w}\exp\left(-\frac{(X-m)^2}{2w^2}\right)$$

Substituting in $\ln(V)$ for X, we can find the probability density function of V [3]:

$$\begin{aligned}\int_{-\infty}^{\infty} f(X)dX &= \int_{-\infty}^{\infty} f(\ln V)d\ln V \\ &= -\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}wV}\exp\left(-\frac{(\ln V - m)^2}{2w^2}\right)dV\end{aligned}$$

Realizing that this is a probability density function of V, we can write

$$h(V) = \frac{1}{\sqrt{2\pi}wV}\exp\left(-\frac{(\ln(V)-m)^2}{2w^2}\right)$$

Consider the following integral:

$$\begin{aligned}\int_0^{\infty} V^n h(V)dV &= \int_{-\infty}^{\infty} \frac{\exp(nX)}{\sqrt{2\pi}w\exp(x)}\exp\left(-\frac{(\ln(V)-m)^2}{2w^2}\right)\exp(x)dX \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}w}\exp\left(-\frac{(X-m-nw^2)^2}{2w^2}\right)\exp\left(\frac{2mnw^2+n^2w^4}{2w^2}\right)dX \\ &= \exp\left(nm + n^2w^2/2\right)\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}w\exp\left(-\frac{(X-m-nw^2)^2}{2w^2}\right)dX\end{aligned}$$

The integral on the right-hand side is a normal probability distribution of a function with mean $m + nw^2$ and standard deviation w, so since we are integrating over this whole normal distribution, it is equal to one. We have so far shown that

$$\int_0^{\infty} V^n h(V)dV = \exp\left(nm + n^2w^2/2\right)$$

Letting $n = 1$, we get the expected value:

$$\begin{aligned}\int_0^{\infty} Vh(V)dV = \mathbb{E}(V) &= \exp\left(m + w^2/2\right) \\ \ln[\mathbb{E}(V)] &= m + \frac{w^2}{2} \\ m &= \ln[\mathbb{E}(V)] - \frac{w^2}{2}\end{aligned}$$

□

If we now consider our stock process, we realize that we have $m = \ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right) T$ and $w = \sigma\sqrt{T}$. Plugging this into the penultimate equation in our proof of Lemma 3.2, we get

$$\begin{aligned}
\ln[\mathbb{E}(S_T)] &= \ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right) T + \frac{(\sigma\sqrt{T})^2}{2} \\
&= \ln(S_0) + \mu T \\
\mathbb{E}(S_T) &= S_0 \exp(\mu T)
\end{aligned}$$

We can also calculate the variance of S_T. Let's use our approach from Lemma 3.2, instead plugging in $n = 2$, to get

$$\int_0^\infty V^2 h(V) dV = \mathbb{E}(V^2) = \exp(2m + 2w^2)$$

Using our definition of variance, we can calculate

$$\begin{aligned}
\text{var } (V) &= \mathbb{E}(V^2) - [\mathbb{E}(V)]^2 \\
&= \exp(2m + 2w^2) - [\exp(m + w^2/2)]^2 \\
&= \exp(2m + w^2)[\exp(w^2) - 1] \\
\text{var } (S_T) &= \exp\left(2\left(\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right) T\right) + (\sigma\sqrt{T})^2\right)[\exp\left((\sigma\sqrt{T})^2\right) - 1] \\
&= S_0^2 \exp(2\mu T)[\exp(\sigma^2 T) - 1]
\end{aligned}$$

This yields the mean and variance of the stock process, from which we can easily find the standard deviation by taking the square root of each side.

Up until this point, we have considered the stock process from a continuous standpoint. For ease of proceeding in the next part of the proof, we will consider the discrete case. A more formal reference is given at the end of the section for curious readers.

$$\begin{aligned}
dS &= \mu S dt + \sigma S dz \\
df &= \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right) dt + \frac{\partial f}{\partial S}\sigma S dz
\end{aligned}$$

Discretizing yields

$$\begin{aligned}
\Delta S &= \mu S \Delta t + \sigma S \Delta z \\
\Delta f &= \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right) \Delta t + \frac{\partial f}{\partial S}\sigma S \Delta z
\end{aligned}$$

We don't like that pesky risk component (the Δz) that is random (it depends on a draw from a normal distribution) rather than deterministic (based on time). So we can create a portfolio Π that involves going short one derivative and going long $\frac{\partial f}{\partial S}$ shares of the underlying stock to yield

$$\begin{aligned}
\Pi &= -f + \frac{\partial f}{\partial S} S \\
\Delta\Pi &= -\Delta f + \frac{\partial f}{\partial S} \Delta S \\
&= -\left[\left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \right) \Delta t + \frac{\partial f}{\partial S}\sigma S \Delta z \right] + \frac{\partial f}{\partial S}\left(\mu S \Delta t + \sigma S \Delta z \right) \\
&= \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \right) \Delta t
\end{aligned}$$

Luckily, using our assumptions, we were able to remove the risk component, but we still have a discrete time component that we would like to get rid of. So intuitively, let's realize how we expect our portfolio to grow. As per the no-arbitrage assumption, during a small change in time, our portfolio will change a small amount equal to the prevailing risk-free interest rate r. We may write this and substitute as

$$\begin{gathered}
\Delta\Pi = r\Pi\Delta t \\
\left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \right) \Delta t = r\left(f - \frac{\partial f}{\partial S} S \right) \Delta t
\end{gathered}$$

$$\boxed{\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf}$$

The final equation above is the Black-Scholes differential equation. It contains no random components, and no discrete time components: only partial derivatives and observable variables. If we impose the following boundary conditions for European calls and puts (respectively) at time $t = T$ and depending on their strike price K and the stock price S at expiration, we will see we have a reasonable closed-form solution:

$$\begin{gathered}
f = \max(S - K, 0) \\
f = \max(K - S, 0)
\end{gathered}$$

As long as we have the right amounts of derivative and stock in our portfolio, it will be riskless...for an infinitesimally short period of time. Thus, it requires frequent adjustments.

Derivation of the Pricing formulæ. Now we will look at the two other important formulæ in the Black-Scholes-Merton model. These are the pricing formulæ for the European call or put whose underlying stock does not pay dividends. The minor adjustments will not be covered in this paper, because an analysis of only these formulæ will be suitable for our purposes of analyzing the efficiency of this model.

After realizing we must calculate the expectation of our boundary condition, we see that for a call, we may write

$$f = \mathbb{E}[\max(V-K,0)]$$

This convention shall be followed for the rest of the proof, and it may be shown for completeness in a more advanced proof that this definition of f not only satisfies the above Black-Scholes differential equation, but is also the unique solution and provides the later pricing formulæ. We shall prove a quick lemma before proceeding so that the rest of the proof becomes simple:

Lemma 3.3. *For a random variable V with a lognormal distribution and with stdev $(\ln(V)) = w$, we may write*

$$\mathbb{E}[\max(V-K,0)] = \mathbb{E}(V)N\left(\frac{\ln[\mathbb{E}(V)/K]+w^2/2}{w}\right) - KN\left(\frac{\ln[\mathbb{E}(V)/K]-w^2/2}{w}\right)$$

Proof. Define $g(V)$ as the probability density function of V.

$$\mathbb{E}[\max(V-K,0)] = \int_K^\infty (V-K)g(V)dV$$

As shown in Lemma 3.2, the mean of $\ln(V)$ in this case is

$$m = \ln[\mathbb{E}(V)] - \frac{w^2}{2}$$

Define a new variable

$$Q = \frac{\ln(V)-m}{w}$$

Based on our transformation of $\ln(V)$, the variable Q is normally distributed with mean zero and standard deviation one. This means we may write it as our standard Gaussian distribution given by

$$h(Q) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{Q^2}{2}\right)$$

Using the Gaussian distribution as our probability density function for Q, we may transform the given integral as

$$\begin{aligned}\mathbb{E}[\max(V-K,0)] &= \int_{\frac{\ln K-m}{w}}^\infty (\exp(Qw+m)-K)h(Q)dQ \\ &= \int_{\frac{\ln K-m}{w}}^\infty \exp(Qw+m)h(Q)dQ - K\int_{\frac{\ln K-m}{w}}^\infty h(Q)dQ\end{aligned}$$

To proceed, we note that

$$\begin{aligned}\exp(Qw+m)h(Q) &= \frac{1}{\sqrt{2\pi}}\exp\left((-Q^2+2Qw+2m)/2\right) \\ &= \frac{1}{\sqrt{2\pi}}\exp\left([-(Q-w)^2+2m+w^2]/2\right) \\ &= \frac{\exp\left(m+w^2/2\right)}{\sqrt{2\pi}}\exp\left([-(Q-w)^2]/2\right) \\ &= \exp\left(m+w^2/2\right)h(Q-w)\end{aligned}$$

We substitute this back in to our first integral and rewrite this as

$$\mathbb{E}[\max(V-K,0)] = \exp\left(m+w^2/2\right)\int_{\frac{\ln K-m}{w}}^{\infty} h(Q-w)dQ - K\int_{\frac{\ln K-m}{w}}^{\infty} h(Q)dQ$$

We may readily evaluate the first integral in this form and write this as

$$\begin{aligned}\int_{\frac{\ln K-m}{w}}^{\infty} h(Q-w)dQ &= 1-N[(\ln K-m)/w-w]\\ &= N[-(\ln K+m)/w+w]\end{aligned}$$

where from now on

$$N(x)=\int_{-\infty}^{x}\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right)dt$$

If we substitute in for m, we get

$$\begin{aligned}&= N\left(\frac{\ln[\mathbb{E}(V)/K]+w^2/2}{w}\right)\\ &= N(d_1)\end{aligned}$$

This completes the first half of the right side, and analogously we may show that

$$\int_{\frac{\ln K-m}{w}}^{\infty} h(Q)dQ = N\left(\frac{\ln[\mathbb{E}(V)/K]-w^2/2}{w}\right) = N(d_2)$$

Therefore, we may quickly substitute our results in and prove the lemma

$$\begin{aligned}\mathbb{E}[\max(V-K,0)] &= \exp\left(m+w^2/2\right)N(d_1)-KN(d_2)\\ &= \exp\left(\ln[\mathbb{E}(V)]-w^2/2+w^2/2\right)N(d_1)-KN(d_2)\\ &= \mathbb{E}(V)N(d_1)-KN(d_2)\end{aligned}$$

□

With this lemma proved, we may now easily find the pricing formulæ. We assumed that the underlying stock price follows the process

$$dS=\mu Sdt+\sigma Sdz$$

We also showed earlier that it was lognormally distributed with a standard deviation of $\sigma\sqrt{T}$. We also know that because we assume the stock follows a Markov Process, the stock price today should be equal to the discounted value of the future stock price at all periods, so

$$\mathbb{E}(S_t)=S_0\exp(rT)$$

We can similarly say that, for a call, the price today will be discounted from the expected price at the expiration date

$$c=\exp(-rT)\mathbb{E}[\max(S_T-K,0)]$$

If we now substitute in using our lemma's results, we get

$$\begin{aligned}c &= \exp(-rT)[S_0\exp(rT)N(d_1)-KN(d_2)]\\ &= S_0N(d_1)-K\exp(-rT)N(d_2)\end{aligned}$$

where

$$d_1=\frac{\ln[\mathbb{E}(S_T)/K]+\sigma^2T/2}{\sigma\sqrt{T}}=\frac{\ln(S_0/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}$$

and similarly

$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}$$

This is the pricing formula for a European call that does not pay dividends. We may quickly show, using a similar process, that the pricing formula for a non-dividend paying European put is given by

$$p = K\exp(-rT)N(-d_2) - S_0N(-d_1)$$

This concludes the formal proof of the pricing formulæ of the model. While some intuition was used to shortcut some highly advanced mathematics, these results may still be proved using a more rigorous approach. For such an approach, I recommend that the interested reader searches for the Feynman-Kac Theorem or the Kolmogorov backward equation.

4. Discussion of Results in Practice

Mathematical Limits. Before examining whether or not this model as a whole makes sense, let's first consider how realistic the resulting formulæ are. This will be accomplished through considering the mathematical limits of different variables in the call pricing formula for extreme cases: near zero, and near infinity. Analogous logic may be supplied for proving the put holds true. This section will prove that the model appears quite attractive in theory.

As $S_0 \to 0$, we have $\ln(S_0/K) \to -\infty$, so $N(d_2) \to 0$. Also, since $N(d) \leq 1$ for all d, we quickly see $c \to 0$. As $S_0 \to \infty$, we have $c \to \infty$ because $N(d_1), N(d_2) \to 1$. More practically, when S_0 gets very large, we see that c gets very large too, practically as large as S_0. Thus $c \to S_0 - Ke^{-rT}$. These cases make intuitive sense because if the underlying stock is almost worthless, it is unlikely for the call to ever pass the strike price, so it must also be almost worthless. Similarly, if the underlying stock is extremely expensive, the strike price is inconsequential. Therefore, the call becomes almost as expensive as the stock.

As $K \to 0$, we have $\ln(S_0/K) \to \infty$ so $N(d_1), N(d_2) \to 1$, and therefore $c \to S_0$. As $K \to \infty$, we have $\ln(S_0/K) \to 0$, so $N(d_1)$ and $N(d_2)$ approach finite, nonzero values, so as a whole $c \to 0$ because we subtract by a number far larger than the finite $S_0N(d_1)$, and c cannot be worth a negative amount. These cases make intuitive sense because if the strike is very low, the call approaches the underlying stock price because of the ability to exercise it for a very small price. Similarly, if the strike is very high, the call becomes worthless because it is unlikely that the underlying stock will ever reach that value.

As $\sigma \to 0$, it can be shown that $N(d_1), N(d_2) \to 1$ by splitting up the fractions associated with d_1 and d_2. As $\sigma \to \infty$, we have $d_1 \to \infty$ so $N(d_1) \to 1$, and $d_2 \to -\infty$ so $N(d_2) \to 0$, meaning $c \to S_0$. These cases make intuitive sense because if the volatility is very low, the underlying stock will move at an almost imperceptible rate, meaning that if it is lower than the strike, the call will expire worthless, and if it is higher than the strike, the call will be worth the stock minus the present value of the strike. Similarly, if volatility for the stock is very high, the call will become worth approximately the same as the stock. After all, with such wild swings, the call will rapidly switch between being worthless (if the stock price is very low) and being as valuable as the stock itself (if the stock price is very high).

As $T \to 0$, it can be shown that $N(d_1), N(d_2) \to 1$, and since $\exp(-r \cdot 0)) = 1$, we get $c \to S_0 - K$. As $T \to \infty$, it can be shown that $N(d_1), N(d_2) \to 1$, but $\exp(-r \cdot T) \to 0$, so $c \to S_0$. These cases make intuitive sense because if the time to expiration is nigh, then there is no discounting of present value to factor into the strike price, and the call becomes worth its intrinsic value. Perhaps more importantly, the volatility component becomes irrelevant because there is no time for the volatility swings to express themselves in the call price. Similarly, if the time to expiration is quite distant, the call approaches the underlying stock itself. This is because the stock has huge possibilities for movement, and the discounting of present value consideration is of little importance based on the extended timeframe.

The limits of the interest rate r become far more interesting. As $r \to 0$, we don't have $N(d_1)$ or $N(d_2)$ tending towards anything simple. In fact, we get that

$$c \to SN\left(\frac{\ln(S_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}}\right) - KN\left(\frac{\ln(S_0/K) - \sigma^2 T/2}{\sigma\sqrt{T}}\right)$$

This is not a simple expression at all compared to the previous example, but this answer still makes sense. This is because as $r \to 0$, the trader does not have any other risk-free investments to enjoy. Therefore, he does not forgo any simple profits, so the call could provide an attractive investment provided it is priced correctly. Such an investment depends on all of the other variables when risk-free interest rates are low or nonexistent. As $r \to \infty$, we simply have that $N(d_1), N(d_2) \to 1$ and simply that $c \to S_0$. This reflects the fact that if a trader is missing out on very large risk-free interest rates, he will only buy the call if it is virtually the same price as the stock; the call is worthless to be held, and so the optionality no longer should be priced in. Obviously a very large risk-free interest rate would have implications on the underlying stock price itself, too...but this logic should suffice to describe why $c \to S_0$.

Practical Limits. From our theoretical point of view, the Black-Scholes-Merton model appears great! Armed with this toolkit of formulæ and enough capital and speed, a trader could always know the "true" value of any option or its underlying stock by comparing the two's prices and taking advantage of any mispriced opportunities.

All well and good in theory...but what about in practice?

In practice, there are problems. While this model enjoyed considerable success during its first years used in practice, some spectacular losses have resulted from when its assumptions have failed. In this section, we will more closely investigate the assumptions we made in our derivation of the formulæ. Then we shall consider possible adjustments to the assumptions and suggest other models that use more realistic assumptions:

Assumption 1. There are no riskless arbitrage opportunities.

Reality 1. This assumption is generally true. Many miniscule mispricings are taken advantage of within milliseconds by high-frequency, automated trading systems, far more quickly than even an instant reaction time for a human being. But higher-level arbitrage can exist. One example that existed for many years includes a mispricing in customer-provider relationships amongst public firms. If Firm X is a publically traded company, and Firm Y is another public company that is one of Firm X's largest customers by far, then if Firm Y misses its projected earnings substantially, Firm X most likely will miss *its* earnings. The analogous logic applies for exceeding expectations. While this is not purely riskless arbitrage in the sense that it is not totally riskless, it is a fundamental strategy that has continued to provide incredible returns with *very* minimal risk, with returns above and beyond standard benchmark performances. For more detail on this strange behavior known as the "limited attention hypothesis," please refer to the references page.

When such an event exists, the underlying stock is fundamentally under/overpriced. Therefore, the calls or puts on the stock would all be fundamentally mispriced, if we just plugged in numbers to the formulæ. In conclusion, while riskless arbitrage opportunities are truly rare and virtually impossible to take advantage of without automated trading, they do exist. And this may have drastic results, positive or negative.

Assumption 2. Security trading is continuous.

Reality 2. This assumption is highly questionable. Jumps in prices are common whenever a large order is placed and the market rushes to react.

Assumption 3. Securities are perfectly divisible and there are no transaction costs of any sort.

Reality 3. The assumption that securities are perfectly divisible is false, however this failure is relatively benign. Unless the security is traded at an ultra-high-frequency, the subdivisibility argument is unlikely to be noticed.

The transaction costs are an entirely different matter. Transaction costs can be massive, especially for high-frequency traders. In fact, one of the reasons some arbitrage opportunities may exist is that the arbitrage may be unobtainable because the potential profit is more than offset by the transaction cost. While ignoring transaction costs can simplify calculations, one might decide upon a course of action that is hindered by these small but numerous costs.

Assumption 4. There is a constant risk-free rate of interest r that applies to all securities and maturities.

Reality 4. This assumption is false. The closest concept to a risk-free rate of interest is embodied by a U.S. Treasury Bill, which pays a guaranteed amount over a certain period of time. The only inherent risk is that the U.S. defaults on its debt. While the probability of this event is nonzero, it is very small. And besides, in this case, traders would have more worries than a slight fudge in a pricing formula...

Normally, an interest rate r is determined by these prevailing rates, but securities do not all have one common rate, or even their own unique ones. Even different traders could have different rates.

Assumption 5. The securities in question do not pay dividends.

Reality 5. This assumption may be true. For securities that do pay dividends, simple adjustments may be made to the pricing formulæ. Otherwise, there are multiple securities that do not pay dividends.

Assumption 6. Individuals can short securities and reinvest all proceeds immediately.

Reality 6. Not all securities may be shorted, but it's safe to assume that we would not apply our models to a security we would not short. Also, not all proceeds may be immediately reinvested. For particularly large institutions, a huge deployment of capital might be impossible. The reasons are numerous, including the fear of moving the market, or special restrictions imposed by investors or a fund itself.

Assumption 7. The price of the underlying stock follows the process

$$dS = \mu S dt + \sigma S dz$$

Reality 7. This assumption is the most questionable one in the model by far. The notion that stocks follow a lognormal distribution appears acceptable from a theoretical standpoint. Especially to the untrained eye, stock prices may appear entirely random. To many academics, this assumption seems perfectly valid.

But this assumption is wrong. Distributions are notoriously "fat-tailed," especially on the downside. This means that there is substantially more risk for a large fall than many theoreticians assume. An analysis of this assumption would merit an entire paper for itself, but here are some of the most startling counterexamples to this assumption:

- In August 1998, the Dow Jones Industrial Average fell 3.5% on August 4th, 4.4% during the next three weeks, and 6.8% on August 31st. The August 31st fall alone would have been estimated by most standard models such as Black-Scholes-Merton to have occurred at odds of one in twenty million. Said another way, an individual who trades every day for 100,000 years should *never* have had this occur to them. The fact that this occurred three times during one month is approximately one in five hundred billion.
- In 1997, the Dow Jones Industrial Average fell 7.7% in a single day, at a probability of one in fifty billion, assuming total randomness of the markets.
- In July 2002, the DJIA fell three times in seven trading days, at a probability of one in four trillion.
- On October 19, 1987 (Black Monday), the DJIA fell 29.2%. This was the worst day of trading in at least a century, and under this single assumption in Black-Scholes-Merton, this should have occurred with a probability of one in 10^{50}. The scale of this number is almost unfathomable. As Benoit Mandelbrot – the father of fractal geometry – describes it, the "odds are so small they have no meaning. It is a number outside the scale of nature. You could span the powers of ten from the smallest subatomic particle to the breadth of the measurable universe – and still never meet such a number." [4]

5. Concluding Remarks

The Black-Scholes-Merton model is a perfect example of a financial mathematics model that appears wonderful in theory but disastrous in practice. Taking mathematical limits of the resulting equations yielded sensible results. The proof presented was reasonably rigorous, and many entirely rigorous proofs exist, notably involving the Feynman-Kac Theorem or Kolmogorov backward equation. And the assumptions themselves seemed benign. In fact, I would suspect most readers of this paper – myself included, when I first learned of the model – accepted the entire presentation of the model, up until the assumptions themselves were explicitly questioned.

And so did many financial firms. The Black-Scholes-Merton model, was once met with great enthusiasm. Yet in the present day, few firms actually believe the model, and even fewer use it.

I hope that this paper has accomplished both of the goals I have set out to achieve. The first goal was stated pretty clearly, and that was to introduce the reader to a historically important financial mathematics model, with its accompanying mathematical and financial jargon.

The second goal was unstated, until now. This paper primarily sought to demonstrate the dangers of blindly following assumptions and too quickly accepting a conclusion. It is indeed a valid conclusion in the theoretical sense. But in a practical sense, the model fails, as have many other financial models before and after it.

I hope that the reader will come away from this paper with a deeper appreciation for the discipline of financial mathematics. And especially for the practically minded individual, I hope this paper has also resulted in a healthier dose of skepticism.

Acknowledgments. I am deeply indebted to Professor John C. Hull from the Rotman School of Management at the University of Toronto for the excellent description of the Black-Scholes-Merton Model he provided in his famous work, *Options, Futures, and Other Derivatives.* I enhanced my knowledge of financial mathematics greatly through reading this work, and numerous proofs presented in the book were woven together and adjusted to make a readable Section Three for this paper.

The book *Introduction to Probability Models* by Professor Sheldon M. Ross at the University of California at Berkeley proved particularly helpful in most of the definitions in Section One for this paper. I found it to be an excellent primer in probability models and recommend it in general for the interested reader.

The (mis)Behavior of Markets by the famous Benoit Mandelbrot changed almost everything I believed about financial mathematics. I had been a believer in Black-Scholes-Merton before reading this wake-up call. Interestingly enough, this work – at odds with conventional models such as Black-Scholes-Merton – proved to be the primary inspiration for writing this paper in the first place. There is very little published work on fractal finance out there, but I believe that these models will become invaluable in the near future...if they are not already being used by secretive firms. I deeply recommend this book for anyone interested in finance whatsoever, especially financial mathematics.

Finally, I would like to thank my two mentors, Hyomin Choi and Janet Jenq. The three of us spent many hours in meetings covering a wide array of topics. And while most of the time was dedicated to creating this paper, Hyomin and Janet volunteered extra time to instruct me in other miscellaneous concepts. These concepts ranged from extra probability exercises, to extra options pricing terminology, and even to a book of brainteasers asked in financial interviews. Hyomin and Janet each contributed their strengths to make sure that I would develop a deeper understanding of the material. In this way, I developed a better paper. But more importantly, I experienced a better preparation for a future career in the financial world. Hyomin's and Janet's collective patience and dedication truly went above and beyond in making this Summer REU experience an enjoyable one. Whatever future I have in finance will partially be owed to the efforts they made this summer. I am thankful for this and will not forget their help.

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