IRREDUCIBLE REPRESENTATIONS OF THE SYMMETRIC GROUP

REDMOND MCNAMARA

ABSTRACT. We construct the Specht modules and prove that they completely characterize the irreducible representations of the symmetric group. We will prove certain properties of these representations using combinatorial tools (such as calculating the dimension using Hook's length formula). Only an introductory knowledge of group theory and linear algebra will be assumed and representation theory concepts will be introduced as necessary.

CONTENTS

1. INTRODUCTION

Representation theory studies homomorphisms of a group G into the group of linear transformations. Generally, it is very difficult to find all possible homomorphisms of a group. These problems occupy professional representation theorists, day in and day out. In the business of classifying every possible representation of a group, perhaps the simplest case is the cyclic group. This paper will explore what may be the second most accessible example, the symmetric group.

We will begin by establishing the basics of representation theory by addressing questions such as: what is a representation? what is a module? how many irreducible representations are there? etc. We then apply these basic components to the symmetric group. We define our terms, building the basics of representation theory from introductory knowledge of linear algebra, group theory and the symmetric group. We find an upper bound on the number of inequivalent, irreducible representations. In order to prove that upper bound exists, and it is just the number of conjugacy classes, we will introduce the machinery of group characters and character inner products; similarly, in order to define the Specht modules we will introduce Young diagrams, Young tableaux, tabloids and polytabloids; in order to

Date: August, 2013.

prove that the polytabloids associated with the standard Young tableaux form a basis for the Specht modules, we introduce both an ordering of tabloids and Garnir elements. We will work through several examples to build intuition and illuminate concepts.

The strategy is as follows: in the second section, we define the basic components of representation theory. In the third section, we prove that there is at most one irreducible representation for each conjugacy class. In the fourth section, we will start constructing modules called Specht modules. In the fifth section, we will prove that the Specht moduels are irreducible, which allows us to calculate the number of irreducible representations. In the sixth section, we will find a basis for the Specht modules and use it to calculate the dimensions of the irreducible representations.

2. Basics of Representation Theory

We want to study nice homomorphisms from one group G to another G'. A case when these homomorphisms are useful and workable is when we restrict G' to be the general linear group of size n–the set of all linear transformations from an n dimensional space to itself, or equivalently the set of all n-by-n invertible matrices.

Definition 2.1. A **representation** φ of a group G is a homomorphism $\varphi : G \to GL(n, \mathbb{C})$, where $GL(n, \mathbb{C})$ is the complex general linear group.

Next, we define a module. Modules allow us to take problems about linear maps and turn them into problems about vector spaces. Working with vector spaces can be extremely convenient, making some results immediately clear.

Definition 2.2. A **G − module**, call it V, is a vector space for which there exists a homomorphism $\varphi : G \to GL(V)$, where $GL(V)$ is the set of linear maps $V \to V$. If the group is clear, we will usually just refer to V as a module.

The representation φ induces a multiplication of G in V. Given $g \in G$ and $v \in V$, we define the operation $g \star v$ as $\varphi(g)v$, which is a vector.[1] This operation is linear. If $g, h, e \in G$, where e is the identity element, $\alpha \in \mathbb{C}$ and $v, w \in V$, then,

(1) $g \star (v + w) = \varphi(g)(v + w) = \varphi(g)v + \varphi(g)w = g \star v + g \star w,$

(2) $\alpha(g \star v) = \alpha(\varphi(g)v) = \varphi(g)(\alpha v) = g \star (\alpha v),$

(3) $(gh) \star v = \varphi(gh)v = \varphi(g)\varphi(h)v = \varphi(g)(h \star v) = g \star h \star v,$

(4) $e \star v = \varphi(e)v = Iv = v.$

A natural question arises: can we define a vector space built out of elements of G under some representation φ? It turns out, we can.

Definition 2.3. The **group algebra**, denoted $\mathbb{C}[G]$, is the set of all linear combinations of elements of G. If G has n elements, then $\mathbb{C}[G] = \{c_1g_1 + \cdots + c_ng_n \mid c_1, \ldots, c_n \in \mathbb{C}\}$ and $h \star (c_1g_1 + \cdots + c_ng_n) = c_1(hg_1) + \cdots + c_n(hg_n)$ for all $h \in G$.

A few examples will build our intuition for these objects.

Example 2.4. Consider the group $\mathbb{Z}/2\mathbb{Z}$ and the following three representations:

[1] The notation $\star$ is specific to this paper. It is introduced simply to distinguish between the different types of multiplication. We use $\cdot$ for scalar, vector and matrix multiplication, and $\circ$ for composition.

(1) The *trivial representation*, denoted $\varphi^{(1)}$, which sends both 0 and 1 to $\begin{bmatrix}1\end{bmatrix}$,

(2) The representation $\varphi^{(2)}$ which similarly sends $0 \mapsto \begin{bmatrix}1\end{bmatrix}$ but sends $1 \mapsto \begin{bmatrix}-1\end{bmatrix}$

(3) The representation φ' which sends elements 0 and 1 to the set of two-by-two matrices: $0 \mapsto \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ and $1 \mapsto \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$.

Next, we compute the corresponding modules.

(1) The module $V^{(1)}$ is the one-dimensional vector space $\mathbb{C}$ where multiplication by elements of $\mathbb{Z}/2\mathbb{Z}$ is trivial; for any $g \in \mathbb{Z}/2\mathbb{Z}$ and any $c \in \mathbb{C}$, $g \star c = c$.

(2) The module $V^{(2)}$ is the one-dimensional vector space $\mathbb{C}$ where multiplication by 1 simply switches the sign of a complex number; for any complex number c, $0 \star c = c$ and $1 \star c = -c$.

(3) The module V' is the two-dimensional vector space $\mathbb{C}^2$ where multiplication by 1 just switches the x and y components; for any $\begin{bmatrix}x\\ y\end{bmatrix}$ in $\mathbb{C}^2$, $1 \star \begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}y\\ x\end{bmatrix}$; of course, 0 still leaves any vector unchanged.[2]

For the third example, we ask if there is a subspace of V' that is invariant under any action by G. A trivial example is 0. Another example is $W = \text{span}\left(\begin{bmatrix}1\\ 1\end{bmatrix}\right)$, which has vectors of the form $w = \begin{bmatrix}x\\ x\end{bmatrix}$ for some x in C. Therefore $1 \star w = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}x\\ x\end{bmatrix} = \begin{bmatrix}x\\ x\end{bmatrix} = w$, so $1 \star w = w$, which is in the subspace W.

If a module has a non-trivial, invariant proper subspace, then it is said to be *reducible*. A module that is not reducible is said to be *irreducible*.

We normalize the basis of the invariant subspace so that it is $\left\{\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}\right\}$. We then complete the basis of V', $B' = \left\{\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}, \begin{bmatrix}\frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}\right\}$.

We want to look at φ' in this new basis. Under φ' each element of $\mathbb{Z}/2\mathbb{Z}$ can be treated like a linear transformation. Thus, using the change in basis formula, we note that there exists a change of coordinates matrix such that $T_{B'} = PT_BP^{-1}$. In this case, $P = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$. Similarly, $P^{-1} = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$. The reader can check that $0 \mapsto I_2$, but $1 \mapsto \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$. Therefore, in our new basis, any element $g \mapsto \begin{bmatrix}\varphi^{(2)}(g) & 0\\ 0 & \varphi^{(1)}(g)\end{bmatrix}$. In this form, the representation is clearly reducible.

[2]This example, 2.4.3, and subsequent work on it come from [4].

Proposition 2.5. [3] *Suppose φ' is a reducible representation. There exists an invertible matrix P for which $P\varphi'(g)P^{-1} = \begin{bmatrix} A(g) & B \\ 0 & C(g) \end{bmatrix}$ where A and C are representations themselves.*

Exercise 2.6. [4] Let φ be a representation and P be an invertible matrix. Then $\varphi'(g) = P\varphi(g)P^{-1}$ is itself a representation.

In fact, every representation of a group can be decomposed into a direct sum of irreducible ones; thus, knowing the irreducible representations of a group is of paramount importance. Let us work out another example for the symmetric group.

Example 2.7. Consider the following representations of S_3:

(1) The trivial representation $\varphi^{(1)}$ that sends every element to $\begin{bmatrix}1\end{bmatrix}$,

(2) The *sign representation* $\varphi^{(2)}$ that sends every element to its sign. For example, $(12) \mapsto \begin{bmatrix}-1\end{bmatrix}$ and $(132) \mapsto \begin{bmatrix}1\end{bmatrix}$ (later we use $\operatorname{sgn}\colon S_n \to \mathbb{C}$),

(3) The *defining representation*, φ', which just permutes the columns of the 3-by-3 identity matrix: $(12) \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $(132) \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, etc.

The module corresponding to $\varphi^{(1)}$ is just the one-dimensional vector space $\mathbb{C}$. For all $c \in \mathbb{C}$, $g \star c = c$. The module corresponding to $\varphi^{(2)}$ is also $\mathbb{C}$. Here, for all $c \in C$, $g \star c = c$ if g is an even permutation and $g \star c = -c$ if g is an odd permutation. Both of these modules are irreducible since they are one-dimensional.

The module V' corresponding to φ' is the three-dimensional vector space $\mathbb{C}^3$. An action by a group element just permutes the coordinates of the vector. For example, $(12) \star \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. As above, $W = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$ is an invariant subspace. Therefore φ' is reducible.

There is one more irreducible representation. We find it by looking at the defining representation of S_3 with respect to the basis $B' = \left\{ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{bmatrix} \right\}$. We will simply list what this representation does to the elements of S_3. In particular, $(1) \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $(12) \mapsto \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, $(13) \mapsto \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $(23) \mapsto \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$, $(123) \mapsto \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ and finally $(132) \mapsto \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$. We denote this representation, called the *standard representation*, by $\varphi^{(3)}$.

We will now state a standard result in representation theory. We will not prove this theorem, but it is the bread-and-butter of introductory representation theory. The proof is accessible to the enthusiastic reader.

---

[3] For the complete proof see [8], in particular Section 1.4 on reducibility. The result is closely related to Maschke's theorem, which one can find in [8] Section 1.5.

[4] For the proof, see [4]

Theorem 2.8 (Maschke's Theorem)**.** [5] *Given a group G and a non-zero module V,*

$$V = W^{(1)} \oplus \cdots \oplus W^{(k)},$$

where $W^{(i)}$ are irreducible representations.

Another result at the core of representation theory that we will make use of is the following:

Lemma 2.9 (Schur's Lemma)**.** *Let V and W be two irreducible G-modules and let Φ be a homomorphism that preserves g action, $\Phi\colon V \to W$ and $\Phi(g \star v) = g \star \Phi(v)$. Then either Φ is an isomorphism or Φ is the trivial map.*

Proof. Consider the kernel of Φ, which is a vector space. Then, by definition of Φ, for all $v \in \ker \Phi$, the vector $\Phi(g\star v) = g\star\Phi(v)$. But, since $v \in \ker \Phi$, $\Phi(g\star v) = g\star 0$. Moreover, $g \star 0 = 0$. Thus, $g \star v \in \ker \Phi$, for all $v \in V$. Therefore, the kernel is a vector space invariant under action by g. But, irreducible modules do not have nontrivial subspaces, so either $\ker \Phi = \{0\}$, in which case Φ is an isomorphism, or $\ker \Phi = V$, in which case Φ is the trivial map. □

3. Characters and Conjugacy Classes

Given a group, a natural question to ask is how many irreducible representations it has. For a finite group G, when working over the field $\mathbb{C}$, it is known that the number of conjugacy classes of G equals the number of inequivalent irreducible representations. In this paper, since we are constructing the irreducible representations, we just need to know when we are done. Thus, we will show that the number of conjugacy classes is an upper bound. To understand equivalence of representations, we define a new piece of machinery, the character.

Definition 3.1. The **character** of a group element g with respect to some representation φ, denoted $\chi(g)$ and later $\theta(g)$, is just the trace of the matrix for g: $\chi(g) = \operatorname{tr}(\varphi(g))$.

Proposition 3.2. [6] *For any character χ and group element g, $\chi(g)^* = \chi(g^{-1})$ where $*$ denotes complex conjugation.*

Proposition 3.3. [7] *If g and h are group elements in the same conjugacy class K, then g and h have the same character: $\chi(g) = \chi(h)$.*

We will not prove either of these propositions. But, to work out an example, take the defining representation of S_3 in Example 2.9.3. Here, the identity element has trace 3 and is in a conjugacy class all to itself. The two-cycles $(12), (13)$, and (23) all have character 1 and are in the same conjugacy class; and the three cycles (123) and (132) both have character 0 and are also in the same conjugacy class. Denote these conjugacy classes with $K^{(1)}, K^{(2)}$, and $K^{(3)}$, respectively. In this case, notice that the character is just equal to the number of points fixed by the permutation. We can now compute the character of each conjugacy class for every irreducible representation of S_3.

[5] The proof is related to proposition 2.6. The full proof is given in [8] Section 1.5.

[6] Simply use an orthonormal basis for the corresponding module V.

[7] For proof, see [8], Proposition 1.8.5. The proposition follows immediately from the linear algebra fact that conjugate matrices have the same trace.

(3.4)

	$K^{(1)}$	$K^{(2)}$	$K^{(3)}$
$\chi^{(1)}$	1	1	1
$\chi^{(2)}$	1	−1	1
$\chi^{(3)}$	2	0	−1

For the remainder of the section, we will define a product operation that ensures that such tables have orthogonal rows. We define our inner product below. Note that both χ and θ are group characters, and that $*$ denotes complex conjugation.

$$\langle \chi, \theta \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \cdot \theta(g^{-1}) \tag{3.5}$$

$$= \frac{1}{|G|} \sum_{g \in G} \chi(g) \cdot \theta(g)^* \tag{3.6}$$

$$= \frac{1}{|G|} \sum_{K} |K| \cdot \chi(K) \cdot \theta(K)^* \tag{3.7}$$

The orthogonality holds even if we take out $\frac{1}{|G|}$, but the operation will not be orthonormal. We will not prove this fact, but it is worth leaving a factor of $\frac{1}{|G|}$ in the equation, if only for consistency with outside texts.

Let φ and ψ be inequivalent, irreducible representations of a group G. Call their corresponding characters χ and θ respectively. Later, we prove $\langle \chi, \theta \rangle = 0$. Suppose φ has dimension m and ψ has dimension n. In order to define a homomorphism $\Phi\colon \mathbb{C}^n \to \mathbb{C}^m$ for every pair i, j where $i \leq m$ and $j \leq n$, we first define a matrix $E_{i,j}$, an m-by-n matrix where the i, j^{th} entry is 1 and all other entries are 0. For example, if φ has dimension 2 and ψ has dimension 3,

$$E_{1,3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now Φ has 5 parameters: two representations φ and ψ; two numbers i and j; and a vector v in n-space. Instead of writing $\Phi(\varphi, \psi, i, j, v)$, we prefer $\Phi(v)$ if the other parameters are clear. We define Φ as left multiplication by a matrix $F_{i,j}$, which implicitly depends on φ and ψ.

$$F_{i,j} = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \cdot E_{i,j} \cdot \psi(g^{-1}) \tag{3.8}$$

$$\Phi(v) = F_{i,j} \cdot v. \tag{3.9}$$

Next, we prove that Φ is a homomorphism. By definition, Φ is a homomorphism if, for all $h \in G$,

$$\begin{aligned} \Phi(h \star v) &= h \star \Phi(v) \\ \Phi(\psi(h) \cdot v) &= \varphi(h) \cdot \Phi(v) \\ F_{i,j} \cdot \psi(h) \cdot v &= \varphi(h) \cdot F_{i,j} \cdot v \\ F_{i,j} \cdot \psi(h) &= \varphi(h) \cdot F_{i,j} \\ F_{i,j} &= \varphi(h) \cdot F_{i,j} \cdot \psi(h^{-1}). \end{aligned}$$

Now, we use the definition of $F_{i,j}$ to prove this equality. In particular, we notice that $G = hG$–our summation from Equation 3.8 can range over all $g \in G$ or over

hg for all $g \in G$.

$$\begin{aligned}F_{i,j} &= \frac{1}{|G|}\sum_{g\in G}\varphi(g)\cdot E_{i,j}\cdot\psi(g^{-1})\\ &= \frac{1}{|G|}\sum_{g\in G}\varphi(hg)\cdot E_{i,j}\cdot\psi((hg)^{-1})\\ &= \frac{1}{|G|}\sum_{g\in G}\varphi(h)\varphi(g)\cdot E_{i,j}\cdot\psi(g^{-1})\psi(h^{-1})\\ &= \varphi(h)\cdot F_{i,j}\cdot\psi(h^{-1}).\end{aligned}$$

So Φ is a homomorphism of irreducible representations. By Schur's lemma, Φ is either an isomorphism or the zero map. Since the representations are inequivalent, Φ is the zero map and $F_{i,j}$ is the zero matrix, for all i, j.

Since Φ is the zero map, $F_{i,j}$ is the zero matrix. In particular, the i, j^{th} entry of $F_{i,j}$ is zero for all $i \leq m$, $j \leq n$. Matrix multiplication dictates the following equality, where $\varphi(g)_{i,i}$ denotes the i, i^{th} entry of $\varphi(g)$ and $\psi(g^{-1})_{j,j}$ denotes the j, j^{th} entry of $\psi(g^{-1})$:

$$0 = i, j^{th} \text{ entry of } F_{i,j} \tag{3.10}$$

$$= \frac{1}{|G|}\sum_{g\in G}\varphi(g)_{i,i}\cdot\psi(g^{-1})_{j,j}. \tag{3.11}$$

We are now ready to prove the most important lemma of this section.

Lemma 3.12. *The inner product of any two characters associated with two inequivalent, irreducible representations defined in Equation 3.5 is* 0.

Proof. The character χ of a group element g with respect to a representation φ is given by,

$$\chi(g) = \varphi(g)_{1,1} + \varphi(g)_{2,2} + \cdots + \varphi(g)_{n,n},$$

where $\varphi(g)_{i,i}$ is the i, i^{th} entry of $\varphi(g)$ and φ is n dimensional. Therefore, $\chi(g)\theta(g^{-1})$ is given by the expression,

$$\begin{aligned}\chi(g)\theta(g^{-1}) &= \sum_{i\leq m}\varphi(g)_{i,i}\cdot\sum_{j\leq n}\psi(g^{-1})_{j,j}\\ &= \sum_{i,j}\varphi(g)_{i,i}\cdot\psi(g^{-1})_{j,j},\end{aligned}$$

where ψ is the representation corresponding to θ, and, once again, $\varphi(g)_{i,i}$ is the i, i^{th} entry of $\varphi(g)$ and similarly for $\psi(g^{-1})_{j,j}$. By combining Equation 3.5 and Equation 3.11, we find,

$$\begin{aligned}\langle\chi,\theta\rangle &= \frac{1}{|G|}\sum_{g\in G}\chi(g)\cdot\theta(g^{-1})\\ &= \frac{1}{|G|}\sum_{i,j}\sum_{g\in G}\varphi(g)_{i,i}\cdot\psi(g)_{j,j}\\ &= 0.\end{aligned}$$

□

Lemma 3.13. *Let G be a group with k conjugacy classes denoted $K^{(1)}, \dots, K^{(k)}$ and let φ and ψ be inequivalent, irreducible representations with associated characters χ and θ. Suppose v is a vector where the i^{th} entry of v is given by $\chi(K^{(i)})$ and similarly the $\theta(K^{(j)})^*$ is the j^{th} entry of some vector w. Then v and w are linearly independent.*

Proof. To prove v and w are linearly independent, we temporarily define a new inner product operation on two vectors x and y of length k where x_i denotes the i^{th} entry of x.

$$\langle x, y\rangle = \frac{1}{|G|} \cdot \left(|K^{(1)}| \cdot x_1 \cdot y_1 + \cdots + |K^{(k)}| \cdot x_k \cdot y_k\right).$$

The reader can check that this, in fact, defines an inner product operation. Notice that in our case,

$$\langle w, v\rangle = \langle \chi, \theta\rangle = 0.$$

Consider a linear combination of v and w that gives 0,

$$c_1 v + c_2 w = 0.$$

To show that $c_1 = 0$, we simply take the inner product of both sides and simplify.

$$\begin{aligned} \langle c_1 v + c_2 w, v\rangle &= \langle 0, v\rangle \\ c_1\langle v, v\rangle + c_2\langle w, v\rangle &= \langle 0, v\rangle \\ c_1\langle v, v\rangle + 0 &= 0 \\ \langle v, v\rangle &\neq 0 \\ c_1 &= 0. \end{aligned}$$

A similar argument will show that $c_2 = 0$. □

Clearly, the argument of Lemma 3.13 can be extend to any number of vectors corresponding to any number of irreducible, inequivalent representations. Theorem 2.8 and Lemma 3.13 prove that group character uniquely determines any representation up to equivalence.

Theorem 3.14. *The number of irreducible representations is at most the number of conjugacy classes.*

Proof. Suppose a group G has k conjugacy classes. Since the set of inequivalent, irreducible representation correspond to linearly independent vectors of length k, there can be at most k of them. Alternately, if we arrange the corresponding characters in a table like Table 3.4, then,

$$\begin{aligned} &\text{\# of irreducible representations} = \text{\# of rows} = \text{\# of linearly independent rows} \\ &= \text{\# of linearly independent columns} \leq \text{\# of columns} = \text{\# of conjugacy classes.} \end{aligned}$$

Therefore, the number of conjugacy classes is an upper bound on the number of inequivalent, irreducible representations. □

4. Specht Modules

In this section, we will build a set of modules, known as Specht modules, for each conjugacy class of the symmetric group. We note that the number of conjugacy classes gives an upper bound of the number of irreducible representations and in the following sections, we will in fact see that these Specht modules give us the irreducible representations.

Definition 4.1. A **partition** of a positive integer n is a sequence of positive integers, $\lambda = (\lambda_1, \lambda_2, \cdots)$ in non-increasing order that sum to n.

The partitions of n are canonically associated with the cycle shapes of S_n. For example, the partition $(3, 2, 2, 1)$ of 8 is associated with the permutations on 8 letters with one three-cycle, two two-cycles and one one-cycle.

Lemma 4.2. *The conjugacy classes of S_n are determined entirely by cycle shape.*

Proof. Let p and q be permutations. Let us conjugate q by p. We will first show that to find pqp^{-1}, we just have to apply p point-wise to the cycles of q. Let q send some n to $q(n)$. Then, $(pqp^{-1})(p(n)) = (pq)(p^{-1}p)(n) = p(q(n))$. Since p is applied point-wise to q, the cycle shape is retained. We can also conjugate two permutations of the same cycle shape into one another. If we want to conjugate q into q', we just find some p that maps cycle to cycle. Thus, q and q' are conjugate by this p. $\square$

Example 4.3. Consider two elements of S_3, (12) and (13). Now if the proposition is true, then these two elements are conjugate. For example, if we conjugate by (23), we get $(23) \circ (12) \circ (23) = (13)$. Similarly, the proposition predicts that any conjugation will preserve cycle shape. For example, $(123) \circ (12) \circ (132) = (23)$.

We now introduce a combinatorial object that encodes the cycle shapes and will be used to compute the irreducible representations of the symmetric group.

We want to define some notation to help us represent different permutations, with an emphasis on their conjugacy classes. That notation is of a *Young diagram.* A Young diagram is just an array of boxes, with nonincreasing row length. The length of each row represents the size of one cycle. Let us see some examples.

Example 4.4.

$$\text{one two-cycle, one one-cycle} = (2,1) = \ydiagram{2,1}$$

$$\text{one five-cycle, one four-cycle} = (5,4) = \ydiagram{5,4}$$

We get a *filling* by filling in each box with a number. A *standard filling* satisfies two conditions: the entries are ordered in decreasing value along the rows and down the columns, and it represents a permutation i.e. the numbers 1 through n are used precisely once. The resulting array is a *standard Young tableau.* Let us see some standard Young tableaux.

Example 4.5.

$$(12)(3) = \begin{ytableau} 1 & 2 \\ 3 \end{ytableau}$$

$$(13578)(2469) = \begin{ytableau} 1 & 3 & 5 & 7 & 8 \\ 2 & 4 & 6 & 9 \end{ytableau}$$

We define the group action pointwise: for any permutation g, we just apply g individually to the entries of the tableau T. Let us do some more examples. Notice that we defined action by g to have the same effect as conjugation by g.

Example 4.6.

$$(13) \star \begin{ytableau} 1 & 2 \\ 3 \end{ytableau} = \begin{ytableau} 3 & 2 \\ 1 \end{ytableau} = (13) \circ (12) \circ (13)$$

$$(1589) \star \begin{ytableau} 1 & 3 & 5 & 7 & 8 \\ 2 & 4 & 6 & 9 \end{ytableau} = \begin{ytableau} 5 & 3 & 8 & 7 & 9 \\ 2 & 4 & 6 & 1 \end{ytableau} = (1589) \circ (13578)(2469) \circ (1985)$$

It turns out, in constructing our modules, we do not distinguish between entries in the same row. A *tabloid* is such an equivalence class of Young tableaux. A tabloid corresponding to T is denoted by $\{T\}$.

Example 4.7.

$$\young(123,45) \sim \young(312,54) \sim \young(132,45) \sim \young(321,54)$$

Let us say we have a Young tableau $\young(1234,567)$ and we want to make up an equivalent Young tableau. We can pick any way of rearranging $1,2,3$ and 4. We can also pick any way of rearranging $5,6$ and 7. If we pick the identity, we get the same tableau back. If we choose to swap 5 and 6, and if we swap 3 and 2, we are left with $\young(1324,657)$. If we choose the permutations (123) and (567), we get $\young(2314,675)$. All these different tableaux are equivalent. Conversely, if we have two equivalent tableaux like $\young(1234,567) \sim \young(1243,765)$ we get back a specific permutation: in this case, (34)(57) takes the first diagram into the second. So, the *row stabilizer* for a tableau T, denoted $R(T)$, is the same as picking a permutation on the first set of letters, then one on the second set of letters and so on. In our example, $R(T) \cong S_4 \times S_3$.

It turns out, the *column stabilizer*, denoted $C(T)$, is surprisingly important for our purposes. We found elements of the row stabilizer by just rearranging the entries of the rows. We find the column stabilizer the same way. Now, consider the tableau $T = \young(123,456)$. We know that the column stabilizer is $C(T) = \{(1),(14),(25),(36),(14)(25),(14)(36),(25)(36),(14)(25)(36)\}$. We want to find the column stabilizer after we apply a permutation g to T. We want to find $C(g \star T)$. Suppose we know $g = (145)$. We note that $g \star T = \young(423,516)$. Our new column stabilizer is $C(g \star T) = \{(1),(45),(12),(36),(12)(45),(45)(36),(12)(36),(12)(45)(36)\}$. Note that $C(g \star T)$ can be obtained by applying g point-wise to $C(T)$. Therefore, $C(g \star T) = g \circ C(T) \circ g^{-1}$. This holds for the row stabilizer as well. i.e., $R(g \star T) = g \circ R(T) \circ g^{-1}$.

Now, let T be a Young tableau. We define the associated *polytabloid*, denoted by e_T as follows:

$$e_T = \sum_{\pi \in C(T)} \operatorname{sgn}(\pi)\pi \star \{T\} \tag{4.8}$$

Example 4.9.

$$\begin{aligned}
e_{\young(123)} &= \{\young(123)\} \\
e_{\young(1234,56)} &= \operatorname{sgn}((1))(1) \star \left\{\young(1234,56)\right\} + \operatorname{sgn}((15))(15) \star \left\{\young(1234,56)\right\} \\
&\quad + \operatorname{sgn}((15)(26))(15)(26) \star \left\{\young(1234,56)\right\} + \operatorname{sgn}((26))(26) \star \left\{\young(1234,56)\right\} \\
&= \left\{\young(1234,56)\right\} - \left\{\young(5234,16)\right\} + \left\{\young(5634,12)\right\} - \left\{\young(1634,52)\right\}
\end{aligned}$$

Proposition 4.10. [8] *For all $g \in G$, $g \star e_T = e_{g \star T}$.*

Next, we define the Specht module, usually denoted $S^{(\lambda)}$, where λ just specifies a cycle shape.

[8]For proof, see [8] Lemma 2.3.3.

Definition 4.11. A **Specht module** is a module spanned by polytabloids e_T, where T is taken over all tableaux of shape λ i.e. $S^{(\lambda)} = \{c_1 e_{T_1} + c_2 e_{T_2} + c_3 e_{T_3} + \cdots \mid c_1, c_2 \ldots \in \mathbb{C}, T_1, T_2 \ldots$ are tableaux of shape $\lambda\}$.

Let us work out some examples. When $\lambda = \square\square\square$, there is only one tabloid which we saw in Example 4.9. For any group element g, $g \star \{\boxed{1}\boxed{2}\boxed{3}\} = \{\boxed{1}\boxed{2}\boxed{3}\}$ so the corresponding representation is just the trivial action.

When $\lambda = \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$, working out the example is a bit harder: there are 3! tableaux and each associated polytabloid has 3! terms because $C(T) \cong S_3$ for all T. The trick to figuring out what $S^{(\lambda)}$ looks like is to look at g action on a basis element e_T. For any group element g, we have the following equalities:

$$\begin{aligned}
g \star e_T &= g \star \sum_{\pi \in C(T)} \operatorname{sgn}(\pi) \pi \star \{T\} \\
&= \sum_{\pi \in S(3)} \operatorname{sgn}(\pi) \cdot g \circ \pi \star \{T\} \\
&= \sum_{\pi \in S(3)} \operatorname{sgn}(g^{-1} g \pi) \cdot g \circ \pi \star \{T\} \\
&= \sum_{\pi \in S(3)} \operatorname{sgn}(g^{-1}) \operatorname{sgn}(g\pi) \cdot g \circ \pi \star \{T\} \\
&= \sum_{g\pi \in S(3)} \operatorname{sgn}(g^{-1}) \operatorname{sgn}(g\pi) \cdot g \circ \pi \star \{T\} \\
&= \operatorname{sgn}(g^{-1}) e_T \\
&= \operatorname{sgn}(g) e_T
\end{aligned}$$

So $S^{(\lambda)}$ is the sign representation. If we were to write out all the e_T, we would realize that for all T, $e_T = \pm e_{\begin{smallmatrix}\boxed{1}\\\boxed{2}\\\boxed{3}\end{smallmatrix}}$. Notice that, even though there were 3! different tableaux, the resulting Specht module was still only one-dimensional.

5. Irreducibility of the Specht Modules

We have already constructed a Specht module for each conjugacy class of S_n and have shown that the number of conjugacy classes gives an upper bound for the number of irreducible, inequivalent modules. In this section, we will show that these Specht modules have no proper, nontrivial submodules invariant under the action of g. Before we prove the big result of this section, we need some tools.

Lemma 5.1. *Let T and T' be two λ tableaux. Then* $\sum_{\pi \in C(T')} sgn(\pi) \pi \star \{T\} = \pm e_{T'}$.

Proof. The argument is similar to the argument we used to prove $S^{(\lambda)}$ gives the sign representation when $\lambda = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}$.

$$\begin{aligned} T &= g \star T' \text{ for some } g \in G \\ \sum_{\pi \in C(T')} \mathrm{sgn}(\pi)\pi \star \{T\} &= \sum_{\pi \in C(T')} \mathrm{sgn}(\pi)\pi \star \{g \star T'\} \\ &= \sum_{\pi \in C(T')} \mathrm{sgn}(\pi)\pi \circ g \star \{T'\} \\ &= \sum_{\pi \in C(T')} \mathrm{sgn}(\pi \circ g)\mathrm{sgn}(g^{-1})\pi \circ g \star \{T'\} \\ &= \mathrm{sgn}(g^{-1}) \sum_{\pi \in C(T')} \mathrm{sgn}(\pi \circ g)\pi \circ g \star \{T'\} \\ &= \mathrm{sgn}(g^{-1}) e_{T'} = \pm e_{T'} \end{aligned}$$

□

Lemma 5.2. *Let W be a nontrivial subspace of $S^{(\lambda)}$ for some λ and let $w \in W$ such that $w \neq 0$. Choose some tableau T of shape λ. Then,*

$$\sum_{\pi \in C(T)} sgn(\pi)\pi \star w = c \cdot e_T,$$

for some $c \in \mathbb{C}$. Since $w \neq 0$, $c \neq 0$.

Proof. Since $w \in S^{(\lambda)}$, it must be the sum of tabloids of shape λ indexed by i,

$$w = \sum_i c_i \{T_i\}.$$

By the previous lemma,

$$\sum_{\pi \in C(T)} \mathrm{sgn}(\pi)\pi \star \{T_i\} = \pm e_T,$$

for all i. Then, we have the following equation:

$$\begin{aligned} \sum_{\pi \in C(T)} \mathrm{sgn}(\pi)\pi \star w &= \pm c_1 e_T \pm c_2 e_T \pm \cdots \\ &= c \cdot e_T, c \in \mathbb{C}. \end{aligned}$$

□

Theorem 5.3. [9] *The Specht modules are irreducible.*

Proof. Since W is a nontrivial subspace, it contains $w \neq 0$. By the previous lemma,

$$\sum_{\pi \in C(T)} \mathrm{sgn}(\pi)\pi \star w = c \cdot e_T,$$

---

[9] This is essentially the Submodule Theorem, which is presented in Section 2.4 of [8].

for some $c \in \mathbb{C}$. Since W is invariant under group action, $\pi \star w \in W$, for all $\pi \in C(T) \subset G$. Therefore, the linear combination,

$$\sum_{\pi \in C(T)} \operatorname{sgn}(\pi)\pi \star w \in W.$$

But, this implies that $c \cdot e_T \in W$ for some $c \neq 0$, and thus $e_T \in W$. However, since $g \star e_T = e_{g\star T}$, we can obtain any tableau T' from T by this g action. Since W is invariant under this g action, $e_{T'} \in W$ for all T' of shape λ. Therefore, $W = S^{(\lambda)}$. □

6. Dimensions of the Irreducible Representations

In this section, we prove that the polytabloids associated with the standard Young tableaux of shape λ form a basis for the Specht module $S^{(\lambda)}$. We also exhibit a formula for finding the dimension of a Specht module of shape λ. In order to prove linear independence, we introduce an order on tabloids of shape λ. In order to prove that the polytabloids associated with the standard Young tableaux span, we introduce Garnir elements. In order to exhibit the hook-length formula, we introduce the concept of hook-length. These tools give us a combinatorial way of understanding the Specht modules.

Theorem 6.1. *The polytabloids associated with the standard Young tableaux form a basis for the corresponding Specht module:* $S^{(\lambda)} = \{c_1 e_{T_1} + \cdots + c_k e_{T_k} \mid c_1, \ldots, c_k \in \mathbb{C}, T_1, \ldots, T_k$ *are standard tableaux of shape* $\lambda\}$.

We will prove this in parts; we will first show that the standard polytabloids are linearly independent and then show that they span the Specht modules. We begin with some machinery.

Definition 6.2. We say $\{T\} < \{T'\}$ if there exists some i such that,

(1) for all $j > i$, j is in the same row of both $\{T\}$ and $\{T'\}$,

(2) i is in a higher row of $\{T\}$ than $\{T'\}$.

The ordering extends to tableaux just as one would expect: $T < T'$ if $\{T\} < \{T'\}$, where $T \leq T'$ and $T' \leq T$ implies $\{T\} = \{T'\}$.

Proposition 6.3. [10] *The relation in Definition 6.2 defines an ordering.*

Lemma 6.4. *Suppose* $\pi \in C(T)$ *for some standard Young tableau* T. *Then,* $\{\pi \star T\} \leq \{T\}$.

Proof. Take the largest entry permuted by π and label it i. All $j > i$ are not permuted. Therefore, they are in the same row of T as $\pi \star T$. The columns are already in descending order. Therefore, any permutation must take the largest entry permuted to a higher row. Thus, $\{\pi \star T\} \leq \{T\}$. □

Theorem 6.5. *The polytabloids associated with the standard Young tableaux are linearly independent.*

[10]This proof is fairly routine. Although [6] does not explicitly prove the proposition, [6] is the source to which concerns should be directed. It is to [6] that we owe the basic strategy for linear independence, while the presentation and especially Lemma 6.4 are somewhat original.

Proof. Suppose we apply this ordering to the standard tableaux, $T_1 < T_2 < \cdots < T_k$ and that there is a linear combination of the associated polytabloids that gives 0,

$$c_1 e_{T_1} + c_2 e_{T_2} + \cdots + c_k e_{T_k} = 0.$$

We will begin by showing that $c_k = 0$. When we expand e_{T_k}, we get $\{T_k\} \pm \cdots$. In order to cancel $\{T_k\}$, it must show up again later down in the line. But, the terms in any other polytabloid are of the form $\pi \star \{T_\ell\}$ for some $\ell < k$. Applying the previous lemma, $\pi \star \{T_\ell\} \leq \{T_\ell\} < \{T_k\}$, so we cannot possibly cancel $\{T_k\}$ with any other polytabloid. Thus, $c_k = 0$. We can apply an inductive argument to show that $c_i = 0$, for all $i \leq k$. ☐

We will now demonstrate a combinatorial process known as the *straightening algorithm*, which takes any standard tableau T and writes the associated polytabloid e_T in terms of other polytabloids closer to a linear combination of polytabloids associated with standard Young tableaux. We use induction on this process to arrive at a linear combination of standard polytabloids for any element of $S^{(\lambda)}$ proving the standard polytabloids span the module. The procedure goes as follows:

(1) Take a tableau T. Order the columns in decreasing order (i.e. the value of the entries decreases down the columns), which only changes the sign of the final linear combination. For instance if you have the tableau $T = \begin{array}{|c|c|c|c|}\hline 1 & 9 & 3 & 6 \\ \hline 4 & 2 & 5 \\ \cline{1-3} 8 & 7 \\ \cline{1-2}\end{array}$, then you would end up with $T' = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 6 \\ \hline 4 & 7 & 5 \\ \cline{1-3} 8 & 9 \\ \cline{1-2}\end{array}$.

(2) If the tableau is not yet standard, then there must be two adjacent entries in the same row where the left is greater than the right. If there are more than two such entries, we can just apply the algorithm to the top-most and then left-most pair for consistency. If, for instance, $T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 6 \\ \hline 4 & 7 & 5 \\ \cline{1-3} 8 & 9 \\ \cline{1-2}\end{array}$ we focus on the pair 7 and 5. We isolate all the entries below the left out-of-order entry and above the right out-of-order entry. For example, we isolate 3, 5, 7 and 9. We call the entries below the left-out-of-order entry A and the entries above the right out-of-order entry B.

(3) Calculate the *Garnir element*, denoted by $g_{A,B}$, which is just the signed sum of all permutations of the isolated entries that keep both subsets A and B without column ascent, $g_{A,B} = \sum_{\pi} \text{sgn}(\pi)\pi$. For instance, in the above example $g_{A,B} = (1) - (57) + (579) - (375) + (37)(59) - (3795)$.

Lemma 6.6. *Let T be a tableau and $g_{A,B}$ be a corresponding Garnir element. Then, $g_{A,B} \star e_T = \sum_{\pi} (sgn(\pi)\pi \star e_T) = 0$.*

Proof. Consider $\sum_{\sigma \in S_{A \cup B}} \text{sgn}(\sigma)\sigma \star \{T\}$ where $S_{A\cup B}$ is the permutations of A and B. For all σ, there exist two adjacent elements a and b, and a transposition of the two elements denoted (ab). Then, $(ab) \star (\sigma \star \{T\}) = \sigma \star \{T\}$ and since $\text{sgn}((ab) \circ \sigma)(ab) \circ \sigma \star \{T\} = -\text{sgn}(\sigma)(ab) \circ \sigma \star \{T\}$ and $\text{sgn}(\sigma)\sigma \star \{T\} = \text{sgn}(\sigma)(ab) \circ \sigma \star \{T\}$ are both terms in the summand, the whole expression cancels to 0.

Now factor out the elements of the column stabilizer,

$$\sum_{\sigma \in S_{A\cup B}} \operatorname{sgn}(\sigma)\sigma \star \{T\} = \sum_{\pi} \sum_{\sigma \in C(T)} \operatorname{sgn}(\pi \circ \sigma)\pi \circ \sigma \star \{T\}.$$

For the remaining sum, we only need to choose one representative for each possible composition of the columns. If we choose each π so that the columns are in descending order, then we get back $g_{A,B} \star e_T$,

$$\begin{aligned}\sum_{\pi} \sum_{\sigma \in C(T)} \operatorname{sgn}(\pi \circ \sigma)\pi \circ \sigma \star \{T\} &= \sum_{\pi} \operatorname{sgn}(\pi)\pi \star \left(\sum_{\sigma \in C(T)} \operatorname{sgn}(\sigma)\sigma \star \{T\} \right) \\ &= g_{A,B} \star e_T \\ &= 0.\end{aligned}$$

□

Theorem 6.7. *The standard polytabloids span the corresponding Specht module.*

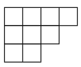
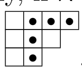
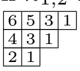
Proof. If $g_{A,B} = (1) \pm \pi_1 \pm \pi_2 \pm \ldots \pm \pi_k$ then we can multiply by e_T on the right to get $g_{A,B} \star e_T = e_T \pm \pi_1 \star e_T \pm \pi_2 \star e_T \pm \ldots \pm \pi_k \star e_T$. Therefore, by the previous proposition, $e_T = \pm\pi_1 \star e_T \pm \pi_2 \star e_T \pm \ldots \pm \pi_k \star e_T$ and we have e_T written in terms of other polytabloids to which we can reapply the algorithm. These other polytabloids are somehow "closer" to the polytabloids associated with the standard tableaux, which we could formalize by defining a partial ordering on the rows.[11] By induction, e_T is spanned by the standard polytabloids. □

Theorem 6.1. *The polytabloids associated with the standard Young tableaux form a basis for the corresponding Specht module:* $S^{(\lambda)} = \{c_1 e_{T_1} + \cdots + c_k e_{T_k} \mid c_1, \ldots, c_k \in \mathbb{C}, T_1, \ldots, T_k$ *are standard tableaux of shape* $\lambda\}$.

Proof. By Theorem 6.5 and Theorem 6.7, the polytabloids associated with the standard Young tableaux a linearly independent and span, so they form a basis. □

We have shown that the Specht modules give us all the irreducible representations of S_n. We conclude by calculating the dimensions of these representations.

Definition 6.8. The **hook − length** of a given entry indexed i, j in a Young tableau T of shape λ, denoted $h_{i,j}$, is the number of entries to the right of i, j in row i plus the number of entries underneath i, j in column j plus 1.

Visually, if $\lambda =$,  then $h_{1,2}$ is computed by counting dots in the following diagram:  . The tableau 

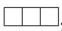
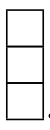
has $h_{i,j}$ as its i, j^{th} entry. Using this definition of the hook-length, we have the following formula, which counts the number of standard Young tableaux for a given shape λ of size n:[12]

$$\dim(S^{(\lambda)}) = \frac{n!}{\prod_{i,j} h_{i,j}} \tag{6.9}$$

[11] A more systematic treatment of Garnir elements is found in [8] section 2.6. We followed [5] in simply describing the algorithm used and providing the necessary picture.

[12] The proof of this formula is found in [8]. We simply exhibit it here.

The dimension of a representation corresponding to a partition λ is given by the standard Young tableaux of that shape, which we calculate using Equation 6.9.

Example 6.10. Take, for instance, S_3. We have corresponding tableaux shapes ,  and . Then, by the hook-length formula, we have that the dimension of the irreducible representations corresponding to these shapes are $1, 2$, and 1, respectively, which agrees with the discussion so far.

In conclusion, in this paper, we have been able to state how many irreducible representations of S_n there are (the number of partitions of n), explicitly construct these irreducible representations (via the Specht modules), and calculate their dimensions (using the hook-length formula). The symmetric group is one of the exceptional cases in allowing such explicit constructions and is a good starting point in studying representation theory. Moreover, these constructions motivate further topics in combinatorics as Young diagrams and Young tableaux are ubiquitous in applications of algebraic combinatorics to representation theory, enumerative geometry and even statistical mechanics! The topic of this paper is intimately connected with the representation theory of the complex general linear group, which might be a natural continuation of the work done here.

Acknowledgments. I want to thank Rediet Abebe for her help. Without her, there would be no topic, no primary reading material, no guidance, no editing. It is not an exaggeration to say that without her, this paper would not have happened. Of course, I am also grateful for the University of Chicago REU and especially Peter May, Jessica Lin and Ben Fehrman for enabling me to study math this summer.

References

[1] Jean-Franois Arbour. Young symmetrizer, Specht modules and examples. http://arbourj.wordpress.com/2013/01/22/young-symmetrizer-specht-modules-and-examples/
[2] Volkers Böchers, Stephen Gieseke. The youngtab package. http://www.ctex.org/documents/packages/math/youngtab.pdf
[3] Character Orthogonality Theorem. http://groupprops.subwiki.org/wiki/Character_orthogonality_theorem
[4] Robert de Mello Koch. Group Theory. http://www.youtube.com/watch?v=3wNPrSwbtQ8
[5] Garnir Relations. http://en.wikipedia.org/wiki/Garnir_relations
[6] G. D. James. The Representation Theory of the Symmetric Group. Springer. 1978.
[7] Evan Jenkins. Representation Theory of S_n. http://www.math.uchicago.edu/ ejenkins/notes/symmetric.pdf
[8] Bruce Sagan. The Symmetric Group. Springer-Verlag New York, Inc. 2001.