THE RIEMANN MAPPING THEOREM

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Abstract. We will develop some of the basic concepts of complex function theory and prove a number of useful results concerning holomorphic functions. We will focus on derivatives, zeros, and sequences of holomorphic functions. This will lead to a brief discussion of the significance of biholomorphic mappings and allow us to prove the Riemann mapping theorem.

Contents

1. Introduction

Holomorphic functions are central to complex function theory, much as real differentiable functions are one of the main subjects of calculus. We call a function biholomorphic if it is a holomorphic bijection. Spaces between which biholomorphisms exist are called biholomorphically equivalent. The idea of biholomorphic equivalence is significant in complex analysis because it allows us to substitute nice spaces for more complicated ones. For example, suppose that U and V are open subsets of $\mathbb{C}$. If $f : U \longrightarrow V$ is a biholomorphic map, then any function $g : V \longrightarrow \mathbb{C}$ is holomorphic if and only if $g \circ f : U \longrightarrow \mathbb{C}$ is holomorphic. Basically, in complex function theory, biholomorphically equivalent spaces are essentially identical.

The Riemann mapping theorem is a major result in the study of complex functions because it states conditions which are sufficient for biholomorphic equivalence. Biholomorphic mappings between spaces are often difficult to construct. The theorem is useful because it guarantees the existence of such a function between certain kinds of spaces, making the construction of biholomorphic functions unnecessary.

Although the reader should be somewhat familiar with holomorphic functions, no more than a basic understanding is necessary. In the first section we will discuss some of the more important characteristics of holomorphic functions. We will derive several formulas concerning the derivatives and zeros of holomorphic functions. We will also prove the open mapping theorem, which states that the images of open sets under nonconstant holomorphic functions are open.

In the next section, we will discuss sequences of holomorphic functions. These will be significant in the proof of the Riemann mapping theorem because they will allow us to demonstrate the existence of a particular function, which we can then prove to be biholomorphic. In the penultimate section, we will formally introduce biholomorphic mappings and discuss their significance in complex function theory. We will consider several special kinds of biholomorphic functions, including Möbius transformations. Möbius transformations will be especially relevant in the proof of the main theorem, which will be completed in the final section of the paper.

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Remark 1.1. We will use the notation "$D(p,r)$" to mean the open disc of radius r with center p. In the last section, we will use "$D = D(0,1)$" to denote the (open) unit disc.

2. Holomorphic Functions

For some open $U \subseteq \mathbb{C}$, a function $f : U \longrightarrow \mathbb{C}$ is *holomorphic* if it is complex differentiable. If $f = u + iv$, then the derivatives with respect to u and v satisfy the Cauchy Riemann equations. That is,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

The word "analytic" is sometimes used instead because holomorphic functions can be given by convergent Taylor series.

We will assume that holomorphic functions on simply connected open sets have antiderivatives (this can be shown through topological arguments and the Cauchy integral theorem for multiply connected domains).

We will begin with a fairly easy but useful fact about holomorphic functions.

Lemma 2.1. *Suppose that $U \subset \mathbb{C}$ is simply connected and open and that $f : U \longrightarrow \mathbb{C}$ is holomorphic and nowhere vanishing. Then there exists a holomorphic function $g : U \longrightarrow \mathbb{C} \setminus \{0\}$ such that*

$$[g(z)]^2 = f(z)$$

for all $z \in U$.

Proof. Note that the function $f'(z)/f(z)$ is holomorphic because f is nonvanishing. Since U is simply connected, there exists some holomorphic function $h : U \longrightarrow \mathbb{C}$ such that $h'(z) = f'(z)/f(z)$. Pick some $z_0 \in U$. We can add a constant to h such that

$$e^{h(z_0)} = f(z_0).$$

Then for $z \in U$, we have

$$\begin{aligned}
\frac{\partial}{\partial z}\left(f(z) \cdot e^{-h(z)}\right) &= \frac{\partial f}{\partial z}(z) \cdot e^{-h(z)} + f(z)\left(-\frac{\partial h}{\partial z}(z) \cdot e^{-h(z)}\right) \\
&= \frac{\partial f}{\partial z}(z) \cdot e^{-h(z)} - f(z) \cdot \frac{\partial f(z)/\partial z}{f(z)} \cdot e^{-h(z)} \\
&= 0.
\end{aligned}$$

Hence $f(z) \cdot e^{-h(z)}$ must be constant. We constructed h such that $f(z_0) \cdot e^{-h(z_0)} = 1$, so we must have

$$f(z) = e^{h(z)}$$

on U. Then set

$$g(z) = e^{h(z)/2},$$

which concludes the proof. □

We will now move on to more interesting aspects of holomorphic functions.

2.1. The Cauchy Integral Theorem and the Cauchy Estimates.

The Cauchy integral theorem is fairly simple and follows from something which looks rather similar to the fundamental theorem of calculus. We will not derive it completely, but the calculations are not terribly complicated. They follow from the definition of a contour integral and the Cauchy Riemann equations.

Theorem 2.2 (Cauchy Integral Theorem). *Suppose that $U \subset \mathbb{C}$ is open and connected and that $f : U \longrightarrow \mathbb{C}$ is holomorphic. Let $\gamma : [a, b] \longrightarrow U$ be a closed C^1 curve. Then*

$$\oint_\gamma f(z)\,dz = 0.$$

Proof. Since U is open and connected, we can find some holomorphic function $g : U \longrightarrow \mathbb{C}$ such that $g' \equiv f$. We have $g(\gamma(a)) = g(\gamma(b))$ because γ is closed. By basic computation, we can show

$$\begin{aligned} 0 &= g(\gamma(a)) - g(\gamma(b)) \\ &= \oint_\gamma g'(z)\,dz \\ &= \oint_\gamma f(z)\,dz. \end{aligned}$$

□

The following result is a useful tool which produces a bound on the derivative of a holomorphic function. It is particularly interesting because there is no corresponding concept in real analysis. The Cauchy estimates have a surprising number of applications in complex function theory.

In fact, bounding the derivative of a holomorphic function will be an important part of the proof of the Riemann mapping theorem. The Cauchy estimates will allow us to find a function which maximizes the derivative at a particular point. Interestingly enough, this function will turn out to be the biholomorphic map which will prove the theorem.

Theorem 2.3. *Let $U \subset \mathbb{C}$ be open, and let $f : U \longrightarrow \mathbb{C}$ be a holomorphic function. Suppose that $p \in U$ and that for some $r > 0$ we have $\overline{D(p, r)} \subset U$. Set $M = \sup_{z \in \overline{D(p,r)}} |f(z)|$. Then for $k \in \mathbb{Z}^+$, we have*

$$\left| \frac{\partial^k f}{\partial z^k}(p) \right| \le \frac{M k!}{r^k}.$$

Proof. For $k \in \mathbb{Z}^+$, we have

$$\frac{\partial^k f}{\partial z^k}(p) = \frac{k!}{2\pi i} \oint_{|\zeta - p| = r} \frac{f(\zeta)}{(\zeta - p)^{k+1}}\,d\zeta.$$

This assertion follows from the definition of complex differentiability and manipulation of limits. For a full proof, see Greene and Krantz ([1], Theorem 3.1.1, p. 69).

Let $\gamma : [0, 1] \longrightarrow \partial D(p, r)$ be the (counterclockwise) path around the boundary of the disc $D(p, r)$. Then we see

$$\begin{aligned} \left| \oint_\gamma \frac{f(\zeta)}{(\zeta - p)^{k+1}}\,d\zeta \right| &= \left| \int_0^1 \frac{f(\gamma(t))}{(\gamma(t) - p)^{k+1}} \cdot \frac{d\gamma}{dt}\,dt \right| \\ &\le \int_0^1 \left| \frac{f(\gamma(t))}{(\gamma(t) - p)^{k+1}} \right| \cdot \left| \frac{d\gamma}{dt} \right| dt \\ &\le \sup_{t \in [0,1]} \frac{|f(\gamma(t))|}{|\gamma(t) - p|^{k+1}} \int_0^1 \left| \frac{d\gamma}{dt} \right| dt. \end{aligned}$$

The integral in this last line is simply the length of the path γ, which is $2\pi r$. Thus we have

$$\begin{aligned} \left| \frac{\partial^k f}{\partial z^k}(p) \right| &\le \frac{k!}{2\pi} \cdot \sup_{|z - p| = r} \frac{|f(z)|}{r^{k+1}} \cdot 2\pi r \\ &\le \frac{M k!}{r^k}. \end{aligned}$$

□

This theorem is a powerful tool in complex analysis, and we will use it later to bound derivatives of holomorphic functions. We will now move on to discuss the zeros of holomorphic functions, which turn out to be surprisingly well-behaved.

2.2. **Zeros of Holomorphic Functions.**

When considering differentiable functions on the real numbers, zeros can be rather nasty. For example, consider

$$x^2 \sin\left(\frac{1}{x}\right).$$

This function is differentiable everywhere. However, it has infinitely many zeros in any neighborhood of 0. This cannot be true of holomorphic functions.

We say that a set of points S is *discrete*, or that these points are *isolated*, if for each $z \in S$ there exists some $r > 0$ such that $S \cap D(z, r) = \{z\}$. We will now demonstrate that the zeros of holomorphic functions which are not identically zero are isolated.

Theorem 2.4. *Let $U \subset \mathbb{C}$ be connected and open, and let $f : U \longrightarrow \mathbb{C}$ be holomorphic. If f is not identically zero, then the zeros of f are isolated.*

Proof. Suppose that f is not identically zero. If f has no zeros, the assertion is clearly true. If f has zeros, let z_0 be a zero of f. Then set

$$k = \min\left\{n \in \mathbb{Z}^+ : \left(\frac{\partial}{\partial z}\right)^n f(z_0) \neq 0\right\}.$$

Since f is holomorphic, it has a Taylor expansion on some disc $D(z_0, r)$, as follows:

$$f(z) = \sum_{j=k}^{\infty} \left(\frac{\partial}{\partial z}\right)^j f(z_0) \frac{(z - z_0)^j}{j!}.$$

Then set

$$g(z) = \sum_{j=k}^{\infty} \left(\frac{\partial}{\partial z}\right)^j f(z_0) \frac{(z - z_0)^{j-k}}{j!}.$$

Clearly g must be holomorphic on $D(z_0, r)$, so it is continuous on the same interval. Since $g(z_0) \neq 0$ (because $(\frac{\partial}{\partial z})^k f(z_0) \neq 0$), there exists some positive $\delta < r$ such that for all $z \in D(z_0, r)$, if $|z - z_0| < \delta$, then $|g(z) - g(z_0)| < |g(z_0)|$. But then we know that g contains no zeros in $D(z_0, \delta)$, and since we have

$$f(z) = g(z) \cdot (z - z_0)^k,$$

we know that for $z \in D(z_0, \delta)$, $f(z) = 0$ only if $z = z_0$. Such a disc can be found for all zeros of f, so they must be isolated. $\square$

Theorem 2.4 has a number of very useful consequences. For example, if two holomorphic functions are equal on any open set, then they are equal everywhere. It also allows us to transfer properties of functions like e^x and trigonometric functions from $\mathbb{R}$ to $\mathbb{C}$. This theorem leads to many interesting results which are certainly worth exploring. However, we will only use it in order to prove the open mapping theorem, a necessary result for the main theorem. We will also need the following method for counting multiplicities of zeros.

We sometimes consider multiplicities of zeros with functions on the reals. For example, a polynomial of which $(x - a)^n$ is a factor has a zero at a of multiplicity n. A similar concept applies to holomorphic functions. Formally speaking, if p is a zero of a holomorphic function f of order n, then the first nonzero term of the power series expansion of f about p is the n^{th} term. That is,

$$f(z) = \sum_{j=n}^{\infty} \frac{\partial^j f}{\partial z^j}(p) \cdot \frac{(z - p)^j}{j!}.$$

Then n is the least positive integer such that $f^{(n)}(p) \neq 0$.

Lemma 2.5. *Let $U \subset \mathbb{C}$ be open, and let $p \in U$ with $\overline{D(p,r)} \subset U$. Suppose that f is a holomorphic function on U which has a zero of order n at p and no other zeros in $\overline{D(p,r)}$. Then*

$$\frac{1}{2\pi i} \oint_{\partial D(p,r)} \frac{f'(z)}{f(z)}\, dz = n.$$

Proof. Consider the power series expansions for f and f'. Set

$$h(z) = \frac{f(z)}{(z-p)^n} = \sum_{j=n}^{\infty} \frac{1}{j!} \cdot \frac{\partial^j f}{\partial z^j}(p)(z-p)^{j-n}.$$

It follows by basic computation that

$$\frac{f'(z)}{f(z)} = \frac{h'(z)}{h(z)} + \frac{n}{z-p}.$$

Then h'/h is holomorphic and nowhere vanishing on some neighborhood of $\overline{D(p,r)}$. Integrating and applying Cauchy's integral theorem proves the lemma. For more details, see [1] (Lemma 5.1.1, p. 158). □

This formula extends naturally to discs with more than one zero, as shown in the following result.

Theorem 2.6 (Argument Principle)**.** *Let $U \subset \mathbb{C}$ be open, and let $p \in U$ with $\overline{D(p,r)} \subset U$. Suppose that $f : U \longrightarrow \mathbb{C}$ is holomorphic and has no zeros in $\partial D(p,r)$. If $z_1, \dots, z_k$ are (distinct) zeros of f in $D(p,r)$ with multiplicities $m_1, \dots, m_k$, then*

$$\frac{1}{2\pi i} \oint_{\partial D(p,r)} \frac{f'(z)}{f(z)}\, dz = \sum_{j=1}^{k} m_j.$$

Proof. Set

$$h(z) = \frac{f(z)}{(z-z_1)^{m_1} \cdot (z-z_2)^{m_2} \cdot \ldots \cdot (z-z_k)^{m_k}}$$

for $z \in U \setminus \{z_1, z_2, \dots, z_k\}$. Then for $j \in \{1, 2, \dots, k\}$, we have

$$h(z) = \frac{f(z)}{(z-z_j)^{m_j}} \cdot \prod_{l \neq j} \frac{1}{(z-z_l)^{m_l}}.$$

Note that the first term is the same holomorphic function from the lemma and that the second term is clearly holomorphic on a neighborhood of z_j. Then h must be holomorphic on a neighborhood of $\overline{D(p,r)}$. As in the lemma, we have

$$\frac{f'(z)}{f(z)} = \frac{h'(z)}{h(z)} + \sum_{l=1}^{k} \frac{m_l}{z-z_l}.$$

We also know that h is nonvanishing in $\overline{D(p,r)}$, so h'/h is holomorphic. Thus integrating and applying Cauchy's integral theorem proves the theorem. □

Note that this is only the portion of the argument principle which applies to holomorphic functions. There is another version for meromorphic functions, which includes multiplicities of poles as well.

The argument principle is especially useful because it is such a simple formula for counting zeros. This principle will be used several times in later proofs, including in the following proof of the open mapping theorem.

Theorem 2.7 (Open Mapping Theorem)**.** *Suppose that $U \subset \mathbb{C}$ is open and connected and that $f : U \longrightarrow \mathbb{C}$ is nonconstant and holomorphic. Then $f(U)$ is open.*

Proof. Pick some arbitrary $q \in f(U)$, and choose $p \in U$ such that $f(p) = q$. Define a function $g_q(\zeta) = f(\zeta) - q$ on U. This function is clearly nonconstant and holomorphic, so its zeros are isolated. Therefore, we can find some $r > 0$ such that $\overline{D(p,r)} \subset U$ and g_q does not vanish on $\overline{D(p,r)} \setminus \{p\}$. Since $\partial D(p,r)$ is compact and contains no zeros of g_q, there exists some $\varepsilon > 0$ such that $|g_q(\zeta)| > \varepsilon$ for all $\zeta \in \partial D(p,r)$. We will show that $D(q,\varepsilon) \subset f(U)$.

For $z \in D(q, \varepsilon)$, define $g_z(\zeta) = f(\zeta) - z$, and define

$$N(z) = \frac{1}{2\pi i} \oint_{\partial D(p,r)} \frac{f'(\zeta)}{f(\zeta) - z} \, d\zeta$$

in order to count the multiplicities of the zeros of g_z on $D(p, r)$. Note that the integrand is defined because for $z \in D(q, \varepsilon)$ and $\zeta \in \partial D(q, \varepsilon)$, we have

$$\begin{aligned} |f(\zeta) - z| &\geq |f(\zeta) - q| - |z - q| \\ &= |g_q(\zeta)| - |z - q| \\ &> \varepsilon - |z - q| \\ &> 0. \end{aligned}$$

Then we also know that g_z is nonvanishing on $\partial D(q, \varepsilon)$, which allows us to apply the argument principle. Thus g_q vanishes to order $N(q)$ at p, so $N(q)$ is a positive integer.

Clearly N is continuous and takes on only integer values. However, this implies that N is constant, so $N(z) > 0$ for all $z \in D(q, \varepsilon)$. Therefore, for all $z \in D(q, \varepsilon)$, g_z vanishes on $D(p, r)$, which implies that $D(q, \varepsilon) \subset f(U)$. Our choice of q was arbitrary, so this means that $f(U)$ must be open. □

The open mapping theorem gives rise to two convenient corollaries.

Corollary 2.8 (Maximum Modulus Principle)**.** *Let $U \subset \mathbb{C}$ be open and connected, and let f be a holomorphic function on U. Suppose that there exists some $p \in U$ such that $|f(z)| \leq |f(p)|$ for all $z \in U$. Then f is constant.*

Proof. Suppose that f is not constant. Then the open mapping theorem applies, so $f(U)$ is open. However, if $f(U)$ is open, there must be some $\varepsilon > 0$ such that $D(f(p), \varepsilon) \subset f(U)$, which is a contradiction. Thus f must be constant. □

Corollary 2.9 (Maximum Modulus Theorem)**.** *Let $U \subset \mathbb{C}$ be bounded, open, and connected. Let f be a function which is continuous on $\overline{U}$ and holomorphic on U. Then the maximum value of $|f|$ must occur on ∂U.*

Proof. First note that the maximum value of $|f|$ must occur because $\overline{U}$ is closed and bounded. If f is constant, the conclusion is obvious. If f is not constant, then the maximum modulus principle states that f cannot have a maximum on U, so it must occur on ∂U. □

Here we will conclude our discussion of the behavior of individual holomorphic functions and move on to sequences of holomorphic functions. We will find that several of the results concerning the zeros of holomorphic functions will be important in proofs concerning sequences. Although our discussion of sequences of holomorphic functions will not be particularly lengthy, we will introduce several powerful concepts which will be pivotal to the Riemann mapping theorem.

3. Sequences of Holomorphic Functions

We want to study families of holomorphic functions, and we will do this through sequences in these families. We will discuss only a few results concerning sequences of holomorphic functions, those which will be used in the proof of the main theorem. In the proof, we will consider a family of functions which have a specific set of properties. We will want to maximize the derivative at a certain point, and Montel's theorem and Theorem 3.2 allow us to find a function which does this. Hurwitz's theorem will allow us to prove that this function is injective. In order to prove these results, we need several definitions.

Consider an open set $U \subset \mathbb{C}$ and a sequence of functions $\{f_j\}$ which are holomorphic on U. We say that $\{f_j\}$ *converges uniformly* to a function f_0 if for every $\varepsilon > 0$, there exists some J such that for all $j > J$ and for all $z \in U$, we have $|f_j(z) - f_0(z)| < \varepsilon$. Note that this J must work for all $z \in U$; it depends only on ε.

We say that $\{f_j\}$ *converges normally* to f_0 if for each compact $K \subset U$ and each $\varepsilon > 0$, there exists some J such that for all $j > J$ and for all $z \in K$, we have $|f_j(z) - f_0(z)| < \varepsilon$. Essentially, this means that the sequence converges uniformly on compact subsets of U.

We also sometimes want to talk about boundedness of functions. We say that a family of functions F is *uniformly bounded* if there is some constant M such that for all $f \in F$ and for all $z \in U$, we have

$$|f(z)| \leq M.$$

When considering holomorphic functions, uniform boundedness is actually enough to guarantee a normally convergent subsequence, as stated in the following theorem.

Theorem 3.1 (Montel's Theorem)**.** *Suppose that $U \subset \mathbb{C}$ is open and that F is a family of uniformly bounded holomorphic functions on U. Then for every sequence $\{f_j\} \subset F$ there is a subsequence $\{f_{j_k}\}$ which converges normally to a (necesarily holomorphic) function f_0.*

Proof. See Stein and Shakarchi ([2], Theorem 3.3, p. 225). □

Montel's theorem is very useful when discussing families of holomorphic functions. It allows us to prove the following theorem, which will play a key role in the proof of the Riemann mapping theorem.

Theorem 3.2. *Let $U \subset \mathbb{C}$ be open, and fix $p \in U$. Let F be a family of holomorphic functions $f : U \longrightarrow D(0,1)$ satisfying $f(p) = 0$. Then there is a sequence $\{f_j\}$ in F which converges normally to a holomorphic function $f_0 : U \longrightarrow D(0,1)$ such that $|f'(p)| \leq |f_0'(p)|$ for all $f \in F$.*

Proof. Note that the Cauchy Estimates give us an upper bound on $|f'(p)|$. This allows us to set

$$w = \sup\{|f'(p)| : f \in F\}.$$

Then there exists a sequence $\{f_j\} \subset F$ such that $|f_j'(p)| \to w$. Since each function maps to the unit disc, the sequence $\{f_j\}$ is bounded uniformly by 1. This means that we can apply Montel's theorem, which states that there is a subsequence $\{f_{j_k}\}$ which converges uniformly on compact subsets to some function f_0. By the Cauchy Estimates, we know $\{|f_{j_k}'(p)|\}$ is uniformly bounded and must converge by Montel's theorem. Then it must converge to $|f_0'(p)|$. Thus $|f_0'(p)| = w$, proving the theorem. □

In the main proof, Montel's theorem and Theorem 3.2 will produce a function $f : U \longrightarrow \mathbb{C}$ which will prove the theorem. However, we will also need to show that it is a bijection. Hurwitz's theorem will help us show that it must be injective. We will use it to show that for any $z_0 \in U$, the function $f(z) - f(z_0)$ is nowhere vanishing on $U \setminus \{z_0\}$, proving injectivity.

Theorem 3.3 (Hurwitz's Theorem)**.** *Let $U \subset \mathbb{C}$ be open and connected. Suppose that $\{f_j\}$ is a sequence of nowhere vanishing functions which are holomorphic on U. If this sequence converges normally to a holomorphic function f_0, then either f_0 is nowhere vanishing or $f_0 \equiv 0$.*

Proof. Suppose that f_0 is not identically zero but that it vanishes at some point $p \in U$ with multiplicity n. Since the zeros of f are isolated, we can pick $r > 0$ such that $\overline{D(p,r)} \subset U$ and f_0 does not vanish on $\overline{D(p,r)} \setminus \{p\}$. By the argument principle, we have

$$\frac{1}{2\pi i} \oint_{|z-p|=r} \frac{f_0'(z)}{f_0(z)}\, dz = n. \tag{3.4}$$

Since each f_j is nowhere vanishing, for each j we have

$$\frac{1}{2\pi i} \oint_{|z-p|=r} \frac{f_j'(z)}{f_j(z)}\, dz = 0 \tag{3.5}$$

by the argument principle. Since the sequences $\{f_j\}$ and $\{f_j'\}$ converge uniformly on $\partial D(p,r)$ to f_0 and f_0', respectively, the integrals in (3.5) must converge to the integral in (3.4). However, this is a contradiction because n is a positive integer. Thus if f_0 is not identically zero, then it is nowhere vanishing. □

This concludes our discussion of general holomorphic functions. In the next section we will talk about special kinds of holomorphic functions.

4. Biholomorphic Mappings

A mapping is called *biholomorphic* if it is a holomorphic bijection. Note that if f is a biholomorphic mapping, then f' is nowhere vanishing (see [1], Theorem 5.2.2, p. 164).

The term "conformal" refers to maps which preserve angles. In this context, it is synonymous with "biholomorphic," as the conditions are equivalent in the complex plane. For more on conformal mappings and their significance in complex analysis, see Lang ([3], **I** §7 and **VII**).

We call two spaces U and V *biholomorphically equivalent* (or conformally equivalent) if there exists a biholomorphic mapping $f : U \longrightarrow V$. This concept is significant because it allows us to treat biholomorphically equivalent spaces as if they are identical.

In this section we will discuss Möbius transformations, which are biholomorphic self-maps (automorphisms) of the unit disc, and the Cayley transform, which is a biholomorphic map of the upper half plane onto the unit disc.

A *linear fractional transformation* is a function of the form

$$z \mapsto \frac{az+b}{cz+d},$$

where $ad - bc \neq 0$.

A *Möbius transformation* is a linear fractional transformation of the form

$$z \mapsto \frac{z-a}{1-\bar{a}z},$$

where $a \in \mathbb{C}$ and $|a| < 1$. It is clear that such a function is holomorphic. Elementary algebra shows that Möbius transformations are biholomorphic self-maps of the unit disc. In fact, all automorphisms of the unit disc are rotations of Möbius transformations (see [1], Theorem 6.2.3, p. 183).

Möbius transformations are convenient because they allow us to manipulate biholomorphic functions which map to the unit disc. For example, suppose that we have some function f which is biholomorphic and maps to the unit disc. Suppose that for $a \in \mathbb{C}$ and $b \in D(0,1) \setminus \{0\}$, we have $f(a) = b$, but we want a biholomorphic function which maps a to zero (this is something which we will need to do in the proof of the main theorem). Consider the Möbius transformation

$$\phi(z) = \frac{z-b}{1-\bar{b}z}.$$

Then $\phi \circ f$ is biholomorphic and still maps to the unit disc, but it maps a to zero. Möbius transformations have many useful applications such as this and are thus very significant when discussing biholomorphic equivalence to the unit disc.

One known biholomorphic map is the Cayley transform, which is the map

$$z \mapsto \frac{z-i}{z+i}.$$

This function is holomorphic on the upper half plane (which is defined to be $\{z \in \mathbb{C} : \text{Im } z > 0\}$). It maps to the unit disc and is bijective, so the upper half plane and the unit disc must be biholomorphically equivalent. Proving this is fairly easy, given the function. However, attempting to create the function from scratch is far more challenging. This is why the Riemann mapping theorem is so useful.

5. The Riemann Mapping Theorem

The Riemann mapping theorem states conditions which are sufficient (and, in fact, necessary) for biholomorphic equivalence with the unit disc. Recall that in this in proof, D will refer to $D(0,1)$, the open unit disc. The results from earlier sections will allow us to prove the following.

Theorem 5.1 (Riemann Mapping Theorem). *If $U \subset \mathbb{C}$ is a simply connected open set which is not the whole plane, then U is biholomorphically equivalent to the unit disc.*

In order to prove this, we will consider a family F of holomorphic functions $f : U \longrightarrow D$ which are injective and take a fixed $p \in U$ to 0. First, of course, we must show that F is nonempty. Once we have shown that it is nonempty, we can use results from Section 4 to pick a function in $f_0 \in F$ which maximizes

$|f'(p)|$, for $f \in F$. Surprisingly, this function will be a biholomorphic function which maps U onto the unit disc.

We can use Hurwitz's theorem to show injectivity. The proof of surjectivity is the least obvious part of the proof. If our function is not surjective, then we will be able to construct another function (dependent on some point in $D \setminus f_0(U)$) in F whose derivative at p exceeds that of f_0, thus proving surjectivity by contradiction. At that point, we will have shown that f_0 is biholomorphic, proving the theorem.

Proof of the Riemann Mapping Theorem. Fix $p \in U$. Consider the family F of holomorphic, injective functions $f : U \longrightarrow D$ such that $f(p) = 0$. First, we must show that F is nonempty. Pick some $q \in \mathbb{C} \setminus U$ and set $\phi(z) = z - q$. We showed in Lemma 2.1 that we can find a holomorphic function h such that $h^2 = \phi$. Pick some $y \in h(U)$. Since h is holomorphic and nonconstant, we can apply the open mapping theorem. Then there exists some $r > 0$ such that $D(y, r) \subset h(U)$. Since ϕ is injective, if $h(z_1) = h(z_2)$ or if $h(z_1) = -h(z_2)$, then $z_1 = z_2$. This implies that $D(-y, r)$ is disjoint from the image of h. Then we can define

$$f(z) = \frac{r}{2[h(z) + y]}.$$

We must have $|h(z) + y| \geq r$ for all $z \in U$, so f maps U to D. We also know that h is injective, so f must be as well. Then we can compose f with a Möbius transformation to create a function which maps p to zero (see Section 4) and is thus in F.

We now know that F is nonempty, so Theorem 3.2 applies. This yields a sequence $\{f_j\}$ in F which converges normally to a holomorphic function $f_0 : U \longrightarrow D$ (which clearly takes p to zero) such that

$$|f_0'(p)| = \sup_{f \in F} |f'(p)|. \tag{5.2}$$

Pick an arbitrary $z_0 \in U$. We will now show that $f_0(z) - f_0(z_0)$ is nonvanishing, which will prove injectivity. For each j, define

$$g_j(z) = f_j(z) - f_j(z_0)$$

on $U \setminus \{z_0\}$. Since each f_j is injective, each g_j is nowhere vanishing on $U \setminus \{z_0\}$. Therefore, we can apply Hurwitz's theorem, which states that the limit function $f_0(z) - f_0(z_0)$ is either nowhere vanishing or identically zero. If it is identically zero, then f_0 must be constant, which implies that $f_0'(p) = 0$. But (5.2) tells us that if $f_0'(p) = 0$, then $f'(p) = 0$ for all $f \in F$, which is a contradiction because each $f \in F$ is injective. Therefore, $f_0(z) - f_0(z_0)$ is nowhere vanishing on $U \setminus \{z_0\}$. Since our choice of z_0 was arbitrary, this means that f_0 is injective.

Now, in order to prove that U is biholomorphically equivalent to the unit disc, it suffices to show that f_0 is surjective and therefore biholomorphic. Suppose that f_0 is not surjective; that is, there exists some $d \in D$ which is not contained in the image of f_0. Then we can show that f_0 cannot satisfy the derivative-maximizing condition from earlier. Consider the function ψ on U such that

$$\psi(z) = \frac{f_0(z) - d}{1 - \bar{d} f_0(z)}.$$

Note that ψ maps to the unit disc and is both injective and nonvanishing. Since U is simply connected, we can again apply Lemma 2.1 to obtain a function $\sigma : U \longrightarrow D$ such that $\sigma^2 = \psi$. We can compose this with a Möbius transformation, as follows:

$$\tau(z) = \frac{\sigma(z) - \sigma(p)}{1 - \overline{\sigma(p)}\sigma(z)}.$$

Now τ maps U to D, is injective and holomorphic, and takes p to zero, so we have $\tau \in F$. We will now demonstrate that $|\tau'(p)| > |f_0'(p)|$, giving us the desired contradiction. We have

$$|\tau'(p)| = \left| \frac{\sigma'(p)[1 - |\sigma(p)|]^2}{[1 - |\sigma(p)|^2]^2} \right| = \left| \frac{\sigma'(p)}{1 - |\sigma(p)|^2} \right|.$$

Note that we have $\psi = \sigma^2$ and $\psi' = 2\sigma \cdot \sigma'$. We also know $f_0(p) = 0$, $\psi(p) = -d$, and $|\sigma(p)| = \sqrt{|d|}$. Thus we have the following:

$$\begin{aligned}
\left|\frac{\sigma'(p)}{1-|\sigma(p)|^2}\right| &= \left|\frac{\psi'(p)}{(1-|\psi(p)|)2\sigma(p)}\right| \\
&= \left|\frac{f_0'(p)[1-\overline{d}f_0(p)] + \overline{d}f_0'(p)[f_0(p)-d]}{[1-\overline{d}f_0(p)]^2 \cdot 2\sigma(p) \cdot (1-|\psi(p)|)}\right| \\
&= \left|f_0'(p) \cdot \frac{1-|d|^2}{2\sigma(p)(1-|d|)}\right| \\
&= \left|f_0'(p) \cdot \frac{1+|d|}{2\sqrt{|d|}}\right|.
\end{aligned}$$

We can use elementary algebra to show that

$$\frac{1+|d|}{2\sqrt{|d|}} > 1,$$

so $|\tau'(p)| > |f_0'(p)|$. This is a contradiction, so f_0 must be surjective. Therefore, f_0 is a biholomorphic function from U to the unit disc, proving the theorem.

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