FUNDAMENTAL GROUPS AND THE VAN KAMPEN'S THEOREM

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Abstract. In this paper, we start with the definitions and properties of the fundamental group of a topological space, and then proceed to prove Van-Kampen's Theorem, which helps to calculate the fundamental groups of complicated topological spaces from the fundamental groups we know already. We also use the properties of covering space to prove the Fundamental Theorem of Algebra and Brouwer's Fixed Point Theorem.

Contents

1. Homotopies and the Fundamental Group

One of the fundamental problems in topology is to determine whether two topological spaces are homeomorphic. Though there is no general method to solve this, but there are techniques to determine this for special cases.

Showing that two topological spaces are homeomorphic is always done by constructing a homeomorphism between them and we have techniques help us to construct homeomorphisms.

However, showing that two spaces are not homeomorphic is more tricky because it is impossible to show that any continuous function is not a homeomorphism between these two spaces. Thus, to show that two spaces are not homeomorphic, we look at properties of topological spaces that are preserved under homeomorphism. One of these invariants is the fundamental group of a topological space. If two topological spaces have different fundamental groups, then they are not homeomorphic because as we will see, two homeomorphic spaces should have isomorphic fundamental groups.

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Before defining the fundamental group of a space X, we shall define an equivalence relation called *path homotopy.*

Definition 1.1. If f and f' are continuous maps of the space X into the space Y, we say that f is **homotopic** to f' if there is a continuous map $F : X \times I \to Y$ such that

$$F(x,0) = f(x) \quad \textit{and} \quad F(x,1) = f'(x) \tag{1.2}$$

for each $x \in X$. The map F is called a **homotopy** between f and f'. If f is homotopic to f', we write $f \simeq f'$. If $f \simeq f'$ and f' is a constant map, we say that f is **nulhomotopic**.

We can think of the parameter t as a representation of time, a homotopy between f and f' represents a continuous transformation from f to f'.

There is a special case when f is a path in X, connecting x_0 and x_1, i.e.

$$f : [0,1] \to X \tag{1.3}$$

is a continuous map such that $f(0) = x_0$ and $f(1) = x_1$. x_0 is called the **initial point** and x_1 is called the **final point**.

Definition 1.4. Two paths f and f' are said to be **path homotopic** if they have the same initial point x_0 and final point x_1, and if there is a continuous map $F : I \times I \to X$ such that

$$F(s,0) = f(s) \quad \textit{and} \quad F(s,1) = f'(s), \tag{1.5}$$

$$F(0,t) = x_0 \quad \textit{and} \quad F(1,t) = x_1, \tag{1.6}$$

for each $s \in I$ and each $t \in I$. We call F a **path homotopy** between f and f'. If f is path homotopic to f', we write $f \simeq_p f'$.

The first condition is simply that F is a homotopy between f and f', and the second says that for fixed t, $F(s,t)$ is a path from x_0 to x_1.

Example 1.7. One example of homotopy is the straight-line homotopy. Let f and g be two continuous maps of a space X into $\mathbb{R}^2$. Then f and g are homotopic with the following homotopy:

$$F(x,t) = (1-t)f(x) + tg(x). \tag{1.8}$$

This homotopy between f and g is called the **straight-line homotopy** for the fact that it moves from $f(x)$ to $g(x)$ along the straight line connecting them.

Example 1.9. However, in the previous example, if we replace $\mathbb{R}^2$ by the punctured plane, $\mathbb{R}^2 - \mathbf{0}$, which is not convex, the straight-line homotopy wouldn't always work. Consider the following paths in punctured plane:

$$f(s) = (cos\pi s, sin\pi s) \quad \textit{and} \quad g(s) = (cos\pi s, 2sin\pi s). \tag{1.10}$$

f and g are path homotopic and the straight-line homotopy between them is acceptable. But the straight-line homotopy between f and the path

$$h(s) = (cos\pi s, -sin\pi s) \tag{1.11}$$

is not acceptable because the straight-line would pass the origin, which is not in the punctured plane.

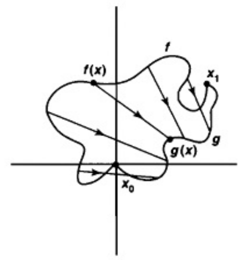


FIGURE 1. Straight-line Homotopy

Indeed, there no path homotopy between f and h but it takes some work to prove.

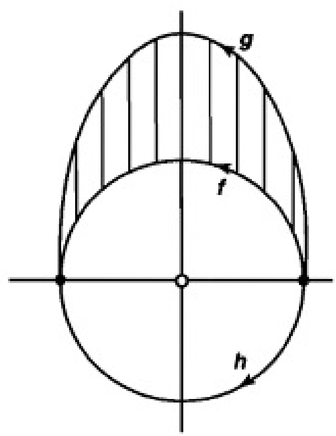


FIGURE 2. Punctured Plane

Directly from the definitions of homotopy and path homotopy, we have the following lemma, whose proof is trivial.

Lemma 1.12. *The relations* $\simeq$ *and* $\simeq_p$ *are equivalence relations.*

Because $\simeq_p$ is an equivalence relation, we can define a certain operation on path-homotopy classes as follows:

Definition 1.13. If f is a path in X from x_0 to x_1, and if g is a path in X from x_1 to x_2, we define the **product** $f * g$ of f and g to be the path h given by the equations

$$h(x) = \begin{cases} f(2s), & \text{for } s \in [0, 1/2]. \\ g(2s-1) & \text{for } s \in [1/2, 1]. \end{cases} \tag{1.14}$$

The function h is well-defined and continuous, by the pasting lemma and it is a path in X from x_0 to x_2.

This product operation on paths induces a well-defined operation on equivalence classes, defined by

$$[f] * [g] = [f * g] \tag{1.15}$$

We can prove that the operation $*$ between equivalence classes is associative with left and right identities as well as inverse for any equivalence class $[f]$.

A result that will be useful to us later is that $[f]$ can be decomposed to the product of equivalence classes of the segments composing f.

Theorem 1.16. *Let f be a path in X, and let a_0, ..., a_n be numbers such that $0 = a_0 < a_1 < ... < a_n = 1$. Let $f_i : I \to X$ be the path that equals the positive linear map of I onto $[a_{i-1}, a_i]$ followed by f. Then*

$$[f] = [f_1] * [f_2] * ... * [f_n]. \tag{1.17}$$

With this operator $*$ on equivalence classes of $\simeq_p$, we just have a groupoid instead of a group because we couldn't apply this operator on any two equivalence classes for the fact that they don't necessarily satisfy the end point condition required to apply $*$. However, if we pick out the equivalence classes with the same initial and final point, with the operator $*$, we could construct a group.

Definition 1.18. Let X be a space; let x_0 be a point of X. A path in X that begins and ends at x_0 is called a **loop** based at x_0. The set of path homotopy classes of loops based at x_0, with the operation $*$ is called the **fundamental group** of X relative to the **base point** x_0. It is denoted by $\pi_1(X, x_0)$.

Example 1.19. Let $\mathbb{R}^n$ be the euclidean n-space. Then $\pi_1(X, x_0)$ is trivial for $x_0 \in X$ because if f is a loop based at x_0, the straight-line homotopy is a path homotopy between f and the constant path at x_0. More generally, if X is any convex subset of $\mathbb{R}^n$, then $\pi_1(X, x_0)$ is the trivial group.

A well-known result is that the fundamental group of the circle is isomorphic to $\mathbb{Z}$, which can be proved by covering space theory. We state it as a theorem but omit the proof here.

Theorem 1.20. *The fundamental group of the circle is isomorphic to the additive group of the integers, i.e.*

$$\pi_1(S^1) \cong \mathbb{Z}. \tag{1.21}$$

One immediate question is that the fundamental group depends on the base point. We now consider this question.

Definition 1.22. Let α be a path in X from x_0 to x_1. We define a map

$$\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1) \tag{1.23}$$

by

$$\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha], \tag{1.24}$$

where $\bar{\alpha}(t) = \alpha(1 - t)$.

If f is path from x_0 to x_1, then the image of $[f]$ under $\hat{\alpha}$ is a loop based at x_1. Hence $\hat{\alpha}$ maps $\pi_1(X, x_0)$ to $\pi_1(X, x_1)$ as desired.

Theorem 1.25. *The map $\hat{\alpha}$ is a group isomorphism.*

Proof. To show that $\hat{\alpha}$ is a homomorphism, we compute

$$\hat{\alpha}([f]) * \hat{\alpha}([g]) = ([\bar{\alpha}] * [f] * [\alpha]) * ([\bar{\alpha}] * [g] * [\alpha]) \tag{1.26}$$
$$= [\bar{\alpha}] * [f] * [g] * [\alpha] \tag{1.27}$$
$$= \hat{\alpha}([f] * [g]). \tag{1.28}$$

To show that $\hat{\alpha}$ is an isomorphism, we show that $\hat{\hat{\alpha}}$ is an inverse for $\hat{\alpha}$.

$$\hat{\beta}([h]) = [\alpha] * [h] * [\bar{\alpha}], \tag{1.29}$$

$$\hat{\alpha}(\hat{\beta}([h])) = [\bar{\alpha}] * ([\alpha] * [h] * [\bar{\alpha}]) * [\alpha] = [h]. \tag{1.30}$$

Similarly, we have $\hat{\beta}(\hat{\alpha}[f]) = [f]$ for each $[f] \in \pi_1(X, x_0)$. And thus, $\hat{\alpha}$ is a group isomorphism. □

Using this result, we can prove that if a space X is path connected, all the fundamental groups $\pi_1(X, x)$ are isomorphic because there is a path connecting any two points in X.

Definition 1.31. A space X is said to be **simply connected** if it is a path-connected space and if $\pi_1(X, x_0)$ is trivial group for some $x_0 \in X$, and hence for every $x_0 \in X$.

Lemma 1.32. *In a simply connected space X, any two paths having the same initial and final points are path homotopic.*

This lemma is proved by using the fact that any loop is path homotopic to the constant loop and we omit the detailed proof here.
It is intuitively clear that the fundamental group is a topological invariant of the space X. We prove this fact by introducing the notion of "homomorphism induced by a continuous map."

Definition 1.33. Let $h : (X, x_0) \to (Y, y_0)$ be a continuous map. Define

$$h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0) \tag{1.34}$$

by the equation

$$h_*([f]) = [h \circ f]. \tag{1.35}$$

The map h_* is called the **homomorphism induced by** h, relative to the base point x_0.

We can show that h_* is well-defined by composing a path homotopy between f and g with h, where f and g are path homotopic loops based at x_0.
Furthermore, the induced homomorphism has two crucial properties.

Theorem 1.36. *If $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ are continuous, then $(k \circ h)_* = k_* \circ h_*$. If $i : (X, x_0) \to (X, x_0)$ is the identity map, then i_* is the identity homomorphism.*

Corollary 1.37. *If $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism of X with Y, then h_* is an isomorphism of $\pi_1(X, x_0)$ with $\pi_1(Y, y_0)$.*

Since h is a homeomorphism, it has an inverse $k : (Y, y_0) \to (X, x_0)$.And this corollary is proved by applying the previous theorem to h and k.

2. Deformation Retractions and Homotopy type

One way of computing the fundamental group of a space X is to study the covering spaces of X. One well-known result of covering space is that the fundamental group of S^1 is isomorphic to the additive group of $\mathbb{Z}$, i.e. $\pi_1(S^1) \cong \mathbb{Z}$. In this section, we will discuss another way to study the fundamental group, which involves the notion of *homotpy type*.

Definition 2.1. Let A be a subspace of X. We say that A is a **deformation retract** of X if there is a continuous map $H : X \times I \to X$ such that $H(x, 0) = x$ and $H(x, 1) \in A$ for all $x \in X$, and $H(a, t) = a$ for all $a \in A$. The homotopy H is called a **deformation retraction** of X onto A. The map $r(x) = H(x, 1)$ is a retraction of X onto A, and H is a homotopy between the identity map of X and the map $j \circ r$, where $j : A \to X$ is the inclusion map.

We can prove the following theorem:

Theorem 2.2. *Let A be a deformation retract of X; let $x_0 \in A$. Then the inclusion map*

$$j : (A, x_0) \to (X, x_0) \tag{2.3}$$

induces an isomorphism of fundamental groups.

Proof. Since A is a deformation retract of X, there exists a continuous map H defined in the previous definition. Since $H(x, 0) = x$ and $H(x, 1) \in A$, H is a homotopy between the identity map of X and the map $j \circ r$. Also, for $a \in A$, $H(a, t) = a$, then a is fixed during the homotopy and thus id_* and $j_* \circ r_*$ are equal. On the other hand, $r \circ j$ is just the identity map of A, so that $r_* \circ j_*$ is the identity homomorphism of $\pi_1(A)$. And thus, j_* is an isomorphism of fundamental groups because it has both left and right inverses. □

Example 2.4. Let B denote the z-axis in $\mathbb{R}^3$. Consider the space $\mathbb{R}^3 - B$. The punctured xy-plane $(\mathbb{R}^2 - \mathbf{0}) \times 0$ is a deformation retract of the space. The map H defined by

$$H(x, y, z, t) = (x, y, (1 - t)z) \tag{2.5}$$

is a deformation retraction. Since the punctured xy-plane has an infinite cyclic fundamental group, the space $\mathbb{R}^3 - B$ has an infinite cyclic fundamental group.

Example 2.6. Consider $\mathbb{R}^2 - \{p, q\}$, where $p, q \in \mathbb{R}^2$. One deformation retract of $\mathbb{R}^2 - p - q$ is the "theta space"

$$\theta = S^1 \cup (0 \times [-1, 1]); \tag{2.7}$$

Another deformation retract of the same space is the figure eight so figure eight and the "theta space" have isomorphic fundamental groups. But neither the figure eight nor the "theta space" is a deformation retract of the other.

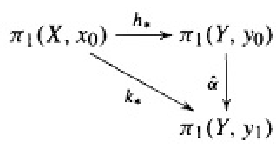
From the example above, there might be a more general way to show that two spaces have isomorphic fundamental groups than by showing that one is a deformation retract of another.

Definition 2.8. Let $f : X \to Y$ and $g : Y \to X$ be continuous maps between topological spaces. Suppose that the map $f \circ g$ is homotopic to the identity map of X and the map $g \circ f$ is homotopic to the identity map of Y. Then the maps f

and g are called **homotopy equivalences**, and each is said to be a **homotopy inverse** of the other.

It's straightforward to show the transitivity of the relation of the homotoopy equivalence and it follows that this relation is an equivalence relation. And two spaces that are homotopy equivalent are said to have the same **homotopy type**. We then proceed to show that if two spaces have the same homotopy type, then they have isomorphic fundamental groups. And we prove this fact by first studying what happens when we have a homotopy between two continuous maps of X into Y such that the base point of X does not remain fixed during the homotopy.

Lemma 2.9. *Let $h, k : X \to Y$ be continuous maps; let $h(x_0) = y_0$ and $k(x_0) = y_1$. If h and k are homotopic, there is a path α in Y from y_0 to y_1 such that $k_* = \hat{\alpha} \circ h_*$. Indeed, if $H : X \times I \to Y$ is the homotopy between h and k, then α is the path $\alpha(t) = H(x_0, t)$.*



Proof. Let f be a loop in X based at x_0. We must show that

$$k_*([f]) = \hat{\alpha}(h_*([f])). \tag{2.10}$$

which is equivalent to showing that

$$[\alpha] * [k \circ f] = [h \circ f] * [\alpha]. \tag{2.11}$$

Consider the loops f_0, f_1 and c in the space $X \times I$ given by the equations

$$f_0(s) = (f(s), 0) \quad \textit{and} \quad f_1(s) = (f(s), 1) \quad \textit{and} \quad c(t) = (x_0, t). \tag{2.12}$$

Then $H \circ f_0 = h \circ f$ and $H \circ f_1 = k \circ f$, while $H \circ c$ is the path α. See Figure 3 below.

Let $F : I \times I \to X \times I$ be the map $F(s, t) = (f(s), t)$. Consider the following

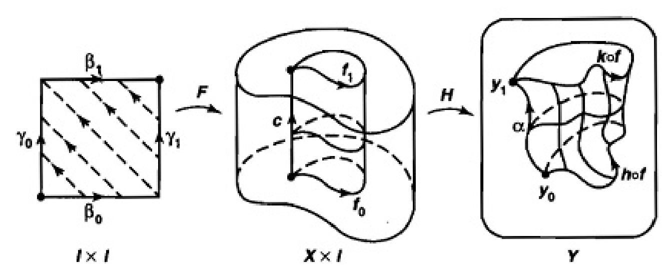


FIGURE 3

paths in $I \times I$, which run along the four edges of $I \times I$:

$$\beta_0(s) = (s, 0) \quad \textit{and} \quad \beta_1(s) = (s, 1), \tag{2.13}$$

$$\gamma_0(t) = (0, t) \quad \textit{and} \quad \gamma_1(t) = (1, t). \tag{2.14}$$

Then $F \circ \beta_0 = f_0$ and $F \circ \beta_1 = f_1$, and $F \circ \gamma_0 = F \circ \gamma_1 = c$. The paths $\beta_0 * \gamma_1$ and $\gamma_0 * \beta_1$ are path homotopic with path homotopy G. Then $F \circ G$ is a path homotopy between $f_0 * c$ and $c * f_1$. And $H \circ (F \circ G)$ is a path homotopy in Y between

$$(H \circ f_0) * (H \circ c) = (h \circ f) * \alpha \quad \textit{and} \tag{2.15}$$

$$(H \circ c) * (H \circ f_1) = \alpha * (k \circ f), \tag{2.16}$$

which is Equation (2.11). □

Corollary 2.17. *Let $h, k : X \to Y$ be homotopic continuous maps; let $h(x_0) = y_0$ and $k(x_0) = y_1$. If h_* is injective, surjective, or trivial, so is k_*.*

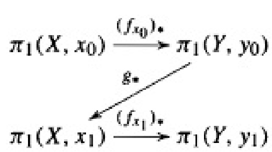
Corollary 2.18. *Let $h : X \to Y$. If h is nulhomotopic, then h_* is the trivial homomorphism.*

Theorem 2.19. *Let $f : X \to Y$ be continuous; let $f(x_0) = y_0$. If f is a homotopy equivalence, then*

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0) \tag{2.20}$$

is an isomorphism.

Proof. Let $g : Y \to X$ be a homotopy inverse for f. Consider the map $f \circ g \circ f$ and let $g(y_0) = x_1$ and $f(x_1) = y_1$. Then we have the corresponding induced homomorphisms of this composition, $(f_{x_1})_* \circ g_* \circ (f_{x_0})_*$. We need to distinguish the homomorphisms induced by f relative to two different base points, x_1 and x_0.



For the fact that $g \circ f$ is homotopic to the identity map, there is a path α in X such that

$$(g \circ f)_* = \hat{\alpha} \circ (i_X)_* = \hat{\alpha}. \tag{2.21}$$

It follows that $(g \circ f)_* = g_* \circ (f_{x_0})_*$ is an isomorphism.

Similarly, since $f \circ g$ is homotopic to the identity map i_Y, the homomorphism $(f \circ g)_*$ is also an isomorphism.

Thus g_* is both surjective and injective and thus g_* is an isomorphism.

Using Equation (2.21), we have

$$(f_{x_0})_* = (g_*)^{-1} \circ \hat{\alpha}, \tag{2.22}$$

so that $(f_{x_0})_*$ is also an isomorphism. □

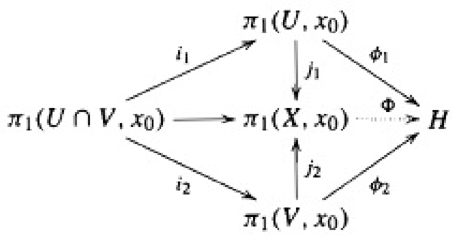
3. Van Kampen's Theorem

We now can return to our main result for this paper, to prove the van Kampen's Theorem. We should state this well-known theorem first.

Theorem 3.1. *Let $X = U \cup V$, where U and V are open in X; assume U, V and $U \cap V$ are path connected; let $x_0 \in U \cap V$. Let H be a group, and let*

$$\phi_1 : \pi_1(U, x_0) \to H \quad \textit{and} \quad \phi_2 : \pi_1(V, x_0) \to H \tag{3.2}$$

be homomorphisms. Let i_1, i_2, j_1, j_2 be the homomorphisms indicated in the following diagram, each induced by an inclusion.
If $\phi_1 \circ i_1 = \phi_2 \circ i_2$, then there is a unique homomorphism $\Phi : \pi_1(X, x_0) \to H$ such that $\Phi \circ j_1 = \phi_1$ and $\Phi \circ j_2 = \phi_2$.



Proof. We first need to show that such a Φ exists. For convenience, we introduce a new notation: Given a path f in X, we use $[f]$ to denote its path-homotopy class in X. If f lies in U, then $[f]_U$ denotes its path-homotopy class in U. Similarly, $[f]_V$ and $[f]_{U\cap V}$ denotes the path-homotopy class in V and $U \cap V$.
Step 1. We begin by defining a set map ρ that assigns an element of H to each loop f based at x_0 that lies in U or V. We define

$$\rho(f) = \phi_1([f]_U) \quad \textit{if } f \textit{ lies in } U, \tag{3.3}$$

$$\rho(f) = \phi_2([f]_V) \quad \textit{if } f \textit{ lies in } V. \tag{3.4}$$

We need to show that ρ is well defined. For f lies in both U and V,

$$\phi_1([f]_U) = \phi_1 \circ i_1([f]_{U\cap V}) \quad \textit{and} \quad \phi_2([f]_V) = \phi_2 \circ i_2([f]_{U\cap V}), \tag{3.5}$$

and by hypothesis, these two values are equal. The map ρ satisfies the following two conditions:

1. If $[f]_U = [g]_U$, or $[f]_V = [g]_V$, then $\rho(f) = \rho(g)$.
2. If both f and g lie in U, or if both lie in V, then $\rho(f * g) = \rho(f) \cdot \rho(g)$.

The first condition holds directly from the definition of ρ and the second condition holds because ϕ_1 and ϕ_2 are homomorphisms.
Step 2. We now extend ρ to a map σ that assigns an element of H to each path f lying in U or V.

We first choose a path α_x from x_0 to x for every $x \in X$. If $x \in U \cap V$, let α_x be a path in $U \cap V$. And if x is in U or V but not in $U \cap V$, let α_x be a path in U or V.

Then for any path f in U or V, we define a loop $L(f)$ in U or V based at x_0 by

$$L(f) = \alpha_x * (f * \bar{\alpha_y}), \tag{3.6}$$

where x is the initial point of f and y is the final point of f. See Figure 4. And we define the map σ by

$$\sigma(f) = \rho(L(f)). \tag{3.7}$$

We must then show that σ is an extension of ρ and satisfies condition (1) and

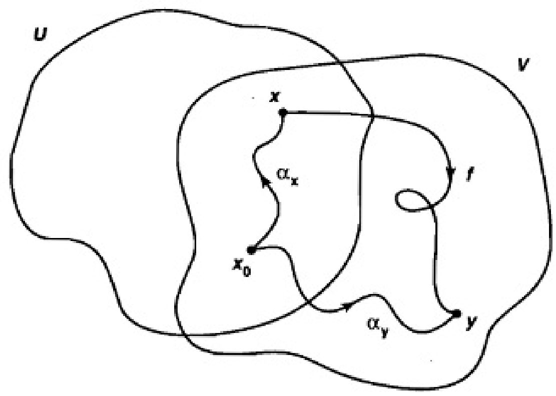


FIGURE 4

(2).

If f is a loop based at x_0 lying in either U or V, then

$$L(f) = e_{x_0} * (f * e_{x_0}) \tag{3.8}$$

because α_{x_0} is the constant path at x_0. Then $L(f)$ is path homotopic to f in U or V, so that $\rho(L(f)) = \rho(f)$ by condition (1) for ρ.

To check condition (1) for σ, let f and g be path homotopic in U or V. Then the loops $L(f)$ and $L(g)$ are also path homotopic, so condition (1) for ρ applies. To check condition (2), let f and g be arbitrary paths in U or V such that $f(1) = g(0)$. Then

$$L(f) * L(g) = (\alpha_x * (f * \bar{\alpha_y})) * (\alpha_y * (g * \bar{\alpha_z})). \tag{3.9}$$

This loop is path homotopic to $L(f * g)$ in U or V. We have

$$\rho(L(f * g)) = \rho(L(f) * L(g)) = \rho(L(f)) \cdot \rho(L(g)) \tag{3.10}$$

by condition (1) and (2) for ρ. Hence $\sigma(f * g) = \sigma(f) \cdot \sigma(g)$.

Step 3. Finally, we extend σ to a map τ that assigns an element of H to an arbitrary path f of X such that τ satisfies

(1) If $[f] = [g]$, then $\tau(f) = \tau(g)$.
(2) $\tau(f * g) = \tau(f) \cdot \tau(g)$ if $f * g$ is defined.

By Lebesgue Number Lemma, given f, there exists a subdivision $s_0 \leq ... \leq s_n$ of $[0, 1]$ such that f maps each of the subintervals $[s_{i-1}, s_i]$ into U or V. Let f_i denote the positive linear map of $[0, 1]$ onto $[s_{i-1}, s_i]$, followed by f. Then f_i is a path in U or V, and

$$[f] = [f_1] * ... * [f_n]. \tag{3.11}$$

If τ is an extension of σ, which satisfies (1) and (2), we should have

$$\tau(f) = \sigma(f_1) \cdot \sigma(f_2) ... \sigma(f_n). \tag{3.12}$$

So we define τ with this equation.

We need to show that this definition is independent of the choice of subdivision.It suffices to show that the value of $\tau(f)$ remains unchanged if we adjoin a single additional point p to the subdivision. Let $s_{i-1} \leq p \leq s_i$. If we compute $\tau(f)$ after adding p into the subdivision, the only change is that $\sigma(f_i)$ is replaced by $\sigma(f_i') \cdot \sigma(f_i'')$, where f_i' and f_i'' equal the positive linear maps of $[0,1]$ to $[s_{i-1}, p]$ and to $[p, s_i]$ followed by f. $\sigma(f_i) = \sigma(f_i') \cdot \sigma(f_i'')$ because f_i is path homotopic to $f_i' * f_i''$ in U or V and by conditions (1) and (2) for σ. Hence, τ is well-defined because it is independent of the choice of subdivision.
If f lies in U or V, we can use the trivial partition of $[0,1]$ to compute $\tau(f)$; then $\tau(f) = \sigma(f)$ by definition of τ.
Step 4. We shall prove condition (1) for τ in this step.

We first prove this condition under an additional hypothesis. Suppose f and g are paths in X from x to y, and F is a path homotopy between f and g. We assume the additional hypothesis that there exists a subdivision $s_0, ..., s_n$ of $[0,1]$ such that F carries each rectangle $R_i = [s_{i-1}, s_i] \times I$ into either U or V. We show that under this hypothesis, $\tau(f) = \tau(g)$.

Given i, consider paths f_i and g_i, the positive linear map of $[0,1]$ onto $[s_{i-1}, s_i]$ followed by f and g. The restriction of F to R_i is a homotopy between f_i and g_i in either U or V, but the restriction is not a path homotopy because the end points could move during the homotopy. We define $\beta_i(t) = F(s_i, t)$, the path traced out by end point during the homotopy. Then β_i is a path in X from $f(s_i)$ to $g(s_i)$. See Figure 5. We show that for each i,

$$f_i * \beta_i \simeq_p \beta_{i-1} * g_i, \tag{3.13}$$

with the path homotopy in U or V.

In the rectangle R_i, take the broken-line path that runs along the bottom and

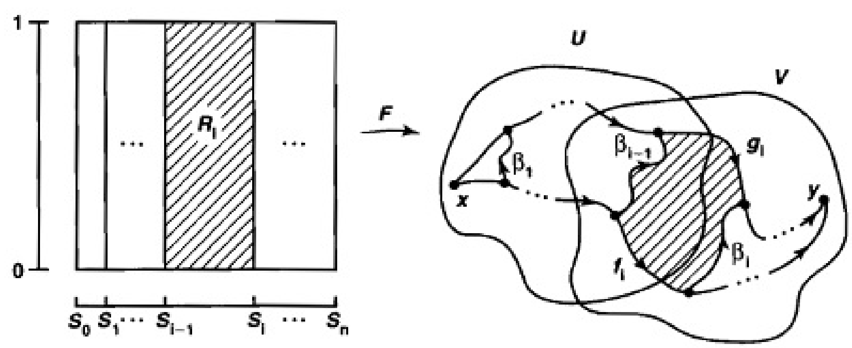


FIGURE 5

right edges of R_i, from $s_{i-1} \times 0$ to $s_i \times 0$ to $s_i \times 1$; we get the path $f_i * \beta_i$ by following this path by F. Similarly, take the broken-line path that runs along the left and top edges of R_i followed by F, we get the path $\beta_{i-1} * g_i$. R_i is convex, then the straight-line homotopy is a path homotopy between the two broken-line paths; if we follow by F, we obtain a path homotopy between $f_i * \beta_i$ and $\beta_{i-1} * g_i$ in either U or V, as desired.

The conditions (1) and (2) for σ imply that

$$\sigma(f_i) \cdot \sigma(\beta_i) = \sigma(\beta_{i-1}) \cdot \sigma(g_i), \tag{3.14}$$

so that

$$\sigma(f_i) = \sigma(\beta_{i-1}) \cdot \sigma(g_i) \cdot \sigma(\beta_i)^{-1}. \tag{3.15}$$

Similarly, since β_0 and β_n are constant paths, $\sigma(\beta_0) = \sigma(\beta_n) = 1$. Then we have

$$\tau(f) = \sigma(f_1) \cdot \sigma(f_2) ... \sigma(f_n) \tag{3.16}$$
$$= \sigma(g_1) \cdot \sigma(g_2) ... \sigma(g_n) \tag{3.17}$$
$$= \tau(g). \tag{3.18}$$

Now we can prove condition (1) without the hypothesis. Given f and g and a path homotopy F between them, there exist subdivisions $s_0, ..., s_n$ and $t_0, ..., t_m$ of $[0,1]$ such that F maps each subrectangle $[s_{i-1}, s_i] \times [t_{j-1}, t_j]$ into either U or V. Let $f_j(s) = F(s, t_j)$; then the pair of paths f_{j-1} and f_j satisfy the requirements of special case, so that $\tau(f_{j-1}) = \tau(f_j)$ for each j. Hence $\tau(f) = \tau(g)$ as desired.

Step 5. Now we prove the condition (2) for the map τ. Given a path $f * g$ in X, we can choose a subdivision $s_0 \leq ... \leq s_n$ of $[0,1]$ with the point $1/2$ as a subdivision point, such that $f * g$ carries each subinterval into either U or V. Let $s_k = 1/2$.

For $i = 1, ..., k$, let f_i be the positive linear map of $[0,1]$ to $[s_{i-1}, s_i]$, followed by $f * g$. f_i is the same as the positive linear map of $[0,1]$ to $[2s_{i-1}, 2s_i]$ followed by f. Similarly, for $i = k+1, ..., n$, let g_{i-k} be the positive linear map of $[0,1]$ to $[s_{i-1}, s_i]$ followed by $f * g$. g_{i-k} is the same as the positive linear map of $[0,1]$ to $[2s_{i-1} - 1, 2s_i - 1]$ followed by g. Thus

$$\tau(f * g) = \sigma(f_1) ... \sigma(f_k) \cdot \sigma(g_1) ... \sigma(g_{n-k}). \tag{3.19}$$

Using the subdivision $2s_0, ..., 2s_k$ for the path f, we have

$$\tau(f) = \sigma(f_1) ... \sigma(f_k). \tag{3.20}$$

Similarly, using the subdivision $2s_k - 1, ..., 2s_n - 1$ for g, we have

$$\tau(g) = \sigma(g_1) ... \sigma(g_{n-k}). \tag{3.21}$$

Thus, the condition (2) holds for the map τ.

Step 6. Now we can define the map Φ to prove the theorem. For each loop f in X based on x_0, we define

$$\Phi([f]) = \tau(f). \tag{3.22}$$

Condition (1) implies that Φ is well-defined and condition (2) implies that Φ is a homomorphism.

We show that $\Phi \circ j_1 = \phi_1$. If f is a loop in U, then

$$\Phi(j_1([f]_U)) = \Phi([f]) \tag{3.23}$$
$$= \tau(f) \tag{3.24}$$
$$= \rho(f) \tag{3.25}$$
$$= \phi_1([f]_U). \tag{3.26}$$

Similarly, we can prove that $\Phi \circ j_2 = \phi_2$.

At last, we prove uniqueness. We know that $\pi_1(X, x_0)$ is generated by the images of j_1 and j_2. We also have $\Phi(j_1(g_1)) = \phi_1(g_1)$ and $\Phi(j_2(g_2)) = \phi_2(g_2)$. Hence Φ is completely determined by ϕ_1 and ϕ_2.

□

4. Applications of van Kampen's Theorem

In this section, we apply the van Kampen's Theorem proved in the previous section to compute the fundamental groups for several topological spaces.

We start with the fundamental group of a wedge of circles.

Definition 4.1. Let X be a Hausdorff space that is the union of the subspaces $S_1, ..., S_n$, each of which is homeomophic to the unit circle S^1. Assume that there exists a point p of X such that $S_i \cap S_j = \{p\}$ whenever $i \neq j$. Then X is called the **wedge of the circles** $S_1, ..., S_n$.

We note that S_i is compact and thus closed in X.

Theorem 4.2. *Let X be the wedge of circles $S_1, ..., S_n$; let p be the common point of these circles. Then $\pi_1(X, p)$ is a free group. If f_i is a loop in S_i tat represents a generator of $\pi_1(S_i, p)$, then the loops $f_1, ..., f_n$ represent a system of free generators for $\pi_1(X, p)$.*

Proof. The result is trivial for $n = 1$. We proceed by induction on the number of circles n.

Let X be the wedge of circles $S_1, ..., S_n$ with common point p. Choose $q_i \in S_i$ such that $q_i \neq p$ for each i. Let $W_i = S_i - q_i$, and let $U = S_1 \cup W_2 \cup ... \cup W_n$ and $V = W_1 \cup S_2 \cup ... \cup S_n$. See Figure 6. Then we have

$$U \cap V = W_1 \cup W_2 \cup ... \cup W_n \quad and \quad U \cup V = X. \tag{4.3}$$

Note that U, V and $U \cap V$ are path connected because they are the union of

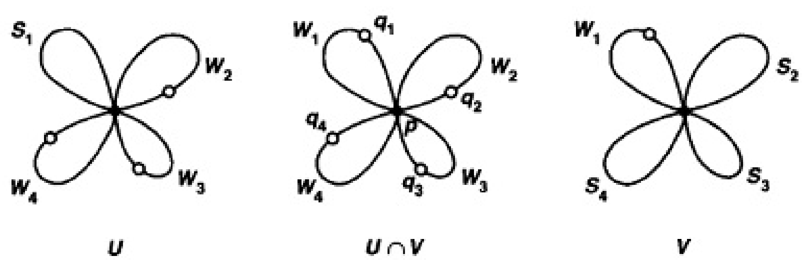


Figure 6

path-connected spaces with a common point.

The space W_i is homeomorphic to an open interval and thus has the point p as its deformation retract; let $F_i : (U \cap V) \times I \to U \cap V$ be the deformation retraction of $U \cap V$ onto p. Then F_i extends to a map $F : (U \cap V) \times I \to U \cap V$ that is a deformation retraction of $U \cap V$ onto p. Hence, the space $U \cap V$ is simply connected and thus has trivial fundamental group. So $\pi_1(X, p)$ is the free product of the groups $\pi_1(U,p)$ and $\pi_1(V,p)$, relative to the monomorphisms induced by inclusion.

Similarly, S_1 is a deformation retract of U and $S_2 \cup S_3 \cup ... \cup S_n$ is a deformation retract of V. Then we have f_1 represents a generator of the infinite cyclic group $\pi_1(U,p)$ and by induction, $\pi_1(V,p)$ is a free group with free generators $f_2, ..., f_n$. And thus $\pi_1(X,p)$ is a free group with free generators $f_1, ..., f_n$. □

We also study the fundamental group of the n-dimensional unit sphere, S^n.

Theorem 4.4. *If $n \geq 2$, the n-sphere S^n is simply connected.*

Proof. Let $p = (0, ..., 0, 1) \in \mathbb{R}^{n+1}$ and $q = (0, ..., 0, -1)$ be the "north pole" and "south pole" of S^n.
Step 1. We show that if $n \geq 1$, the punctured sphere $S^n - p$ is homeomorphic to $\mathbb{R}^n$.
Define $f : (S^n - p) \to \mathbb{R}^n$ by the equation

$$f(x) = f(x_1, ..., x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, ..., x_n). \tag{4.5}$$

We show that f is a homeomorphism by showing that the map $g : \mathbb{R}^n \to (S^n - p)$ defined by

$$g(y) = g(y_1, ..., y_n) = (t(y) \cdot y_1, ..., t(y) \cdot y_n, 1 - t(y)), \tag{4.6}$$

where $t(y) = 2/(1 + |y|^2)$, is a right and left inverse of f. Note that the reflection map on the last coefficient is a homeomorphism of $S^n - p$ with $S^n - q$, so the latter is also homeomophic to $\mathbb{R}^n$.
Step 2. We prove the theorem by using the van Kampen's theorem. Let U and V be the open sets $U = S^n - p$ and $V = S^n - q$. For $n \geq 1$, the sphere S^n is path connected and thus U and V are path connected.
$U \cap V$ is path connected because it is homeomorphic to $\mathbb{R}^n - \mathbf{0}$ under the stereographic projection. Then by van Kampen's theorem, S^n has trivial fundamental group because U and V have trivial fundamental groups. Thus S^n is simply connected. □

5. Fundamental Theorem of Algebra

We prove the Fundamental Theorem of Algebra using Theorem 1.20.

Theorem 5.1 (The fundamental theorem of algebra)**.** *A polynomial equation*

$$x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 = 0 \tag{5.2}$$

of degree $n > 0$ with real coefficients has at least one real root.

Proof. Step 1. Consider the map $f : S^1 \to S^1$ given by $f(z) = z^n$, where z is a complex number. We show that the induced homomorphism f_* of fundamental groups is injective.

Let $p_0 : I \to S^1$ be a loop in S^1 such that,

$$p_0(s) = e^{2\pi i s} = (cos 2\pi s, sin 2\pi s). \tag{5.3}$$

Then we have

$$f(p_0(s)) = (e^{2\pi i s})^n = (cos 2n\pi s, sin 2n\pi s). \tag{5.4}$$

This is a loop and lifts to the path $s \to ns$ in the covering space $\mathbb{R}$. Therefore the loop $f \circ p_0$ corresponds to the integer n under the isomorphism from $\pi_1(S^1)$

to S^1. Thus f_* can be considered as a "multiplication by n" in the fundamental group of S^1 and then f_* is injective.
Step 2. We want to show that if $g : S^1 \to \mathbb{R}^2 - \mathbf{0}$ is the map $g(z) = z^n$, then g is not nulhomotopic.

Let $j : S^1 \to \mathbb{R}^2 - \mathbf{0}$ be the inclusion map. Then we have $g = f \circ j$. From step 1 we know that f_* is injective and j_* is injective because S^1 is the retract of $\mathbb{R}^2 - \mathbf{0}$. Therefore $g_* = j_* \circ f_*$ is also injective. Thus g is not nulhomotopic.
Step 3. Now we want to prove a special case of the fundamental theorem of algebra under the condition that

$$\sum_{i=0}^{n-1} |a_i| < 1, \tag{5.5}$$

and we want to show that the equation has a root lying in the unit ball B^2.
Assume that there's no such root. We can define $k : B^2 \to \mathbb{R}^2 - \mathbf{0}$ by

$$k(z) = z^n + a_{n-1}z^n + ... + a_1 z + a_0. \tag{5.6}$$

Let h be the restriction of k to S^1. Because h extends to a map of the unit ball into $\mathbb{R}^2 - \mathbf{0}$, h is nulhomotopic. On the other hand, if we define $F : S^1 \times I \to \mathbb{R}^2 - \mathbf{0}$ by

$$F(z,t) = z^n + t(a_{n-1}z^{n-1} + ... + a_0), \tag{5.7}$$

F is a homotopy between h and g. Then g is nulhomotopic, which is a contradiction with the result in step 2.
Step 4. Now we can prove the general case. Let $c > 0$ be any real number and we substitute $x = cy$ and divide both side of the equation by c^n, we obtain the equation

$$y^n + \frac{a_{n-1}}{c}y^{n-1} + ... + \frac{a_1}{c^{n-1}}y + \frac{a_0}{c^n} = 0. \tag{5.8}$$

For c large enough we have

$$\left| \frac{a_{n-1}}{c} \right| + ... + \left| \frac{a_1}{c^{n-1}} \right| + \left| \frac{a_0}{c^n} \right| < 1. \tag{5.9}$$

Then from step 3, this equation has a root y_0 in the unit ball, and $x_0 = cy_0$ is a root of the original equation. This completes the proof of the fundamental theorem of algebra.

Note that the similar results hold if we replace the word "real" by "complex" in the initial statement. □

6. Brouwer Fixed Point Theorem

Definition 6.1. A **vector field** on B^2 is an ordered pair $(x, v(x))$, where x is in B^2 and v is a continuous map of B^2 into $\mathbb{R}^2$.

Lemma 6.2. *Given a nonvanishing vector field on B^2, there exists a point of S^1 where the vector field points directly inward and a point of S^1 where it points directly outward.*

Proof. Let $(x, v(x))$ be a nonvanishing vector field on B^2, then we have $v(x) \neq 0$ for every x, and thus v maps B^2 to $\mathbb{R}^2 - \mathbf{0}$.

We first suppose that $v(x)$ does not point directly inward at any point x of S^1 and derive a contradiction. Let w be the restriction of v to S^1. Since w extends to a map of B^2 into $\mathbb{R}^2 - \mathbf{0}$, w is nulhomotopic.

On the other hand, w is homotopic to the inclusion map $j : S^1 \to \mathbb{R}^2 - \mathbf{0}$ by the straight-line homotopy

$$F(x,t) = tx + (1-t)w(x), \tag{6.3}$$

for $x \in S^1$. We must show that $F(x,t) \neq 0$. For $t = 1$ and $t = 0$, this is trivial. If $F(x,t) = 0$ for some $0 < t < 1$, then we have $tx + (1-t)w(x) = 0$, so that $w(x) = -\frac{t}{1-t}x$. And thus $w(x)$ points directly inward at x, which is a contradiction. Hence $F \neq 0$.

It follows that j is nulhomotopic, which contradicts the fact that j is the homotopy equivalence and $\pi_1(S^1) \cong \mathbb{Z}$.

To show that v points directly outward at some point of S^1, we simply let w be the restriction of $-v$ on S^1. And this ends the proof of the lemma. □

With this lemma, we now prove the Brouwer fixed-point theorem for the disc.

Theorem 6.4 (Brouwer fixed-point theorem for the disc)**.** *If $f : B^2 \to B^2$ is continuous, then there exists a point $x \in B^2$ such that $f(x) = x$.*

Proof. We prove this theorem by contradiction. Suppose $f(x) \neq x$ for any $x \in B^2$. Then let $v(x) = f(x) - x$, and thus the vector field $(x, v(x))$ is nonvanishing on B^2. But v cannot point directly outward at any $x \in S^1$, for that would have

$$f(x) - x = ax \tag{6.5}$$

for $a > 0$, so that $f(x) = (1 + a)x$ lies outside the unit ball. This is a contradiction. □

The fixed-point theorems are of interest in mathematics because many problems, such as problems concerning existence of solutions of systems of equations can be formulated as fixed-point problems. For example, a classical theorem of Frobenius can be proved using the Brower Fixed Point Theorem.

Corollary 6.6. *Let A be a 3 by 3 matrix of positive real numbers. Then A has a positive real eigenvalue (characteristic value).*

The proof of this corollary requires some knowledge of linear algebra and we skip the proof here. Readers could refer to **Corollary 55.7** in Munkres' book.

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References

[1] James Munkres. Topology (2nd Edition). Pearson. 2000.