AN ELEMENTERY PROOF OF THE PRIME NUMBER THEOREM

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Abstract. This paper will present a proof of the prime number theorem which is elementery in that it does not make use of analytic techniques. The prime number theorem states that the number of primes less than x asymptotically approaches $x/ln(x)$. This theorem was originally proven in 1896 by Jacques Hadamard and Charles Jean de la Valle-Poussin. Until Selberg and Erdös produced a new proof in 1949 it was not known whether the proof was possible without analytic techniques. My paper follows their proof closely

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1. Introduction

The prime number theorem is an old and important theorem in number theory. It states that the number of primes less than a number x approaches $\frac{x}{\log x}$ as x goes to infinity. It was first formulated by Legendre in 1798. Tchebychef made important progress towards solving this problem in 1852 when he showed that $\pi(x)$ was the same order of magnitude as $\frac{x}{\log x}$ using elementary techniques. Many improvements were given to the bounds of $\pi(x)$ but the elementary methods employed were largely ad hoc and gave little hope of actually settling the problem.

The theorem was first proven in 1896 by Jacques Hadamard and Charles Jean de la Valle-Poussin using integral theory and the Riemann zeta function $\zeta(s)$ defined by,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The proof was concluded by use of a trigonometric identity. Later simplified proofs were developed using similar techniques.

It was quite surprising to the mathematical world when Erdös announced that he and Selberg had developed a completely elementery proof of the theorem. Unfortunately a very bitter dispute over credit followed between the two mathematicions but that is not the concern of this paper. My paper follows Selberg's proof closely.

The fact that the prime number theorem may be proven using only elementary methods is something of an intellectual curiosity but sadly the results did not lead to any future notable results, though other elementary techniques in number theory that Selberg and Erdös developed are of great importance.

2. An equivalent statement of the prime number theorem, $\psi(x)\log x \sim \pi(x)$

First we begin by shifting focus from a function that explicitly counts primes to a function with more useful asymptotic behaviour which is still intimately connected to the prime number theorem.

Definition 2.1. For each $x \geq 0$, we define $\pi(x) = \textit{the number of primes} \leq x$.

Definition 2.2. For each $x \geq 0$, we define $\psi(x) = \sum_{p\leq x} \log p$.

Definition 2.3. We will also need the notation $\sim$. We say $g(x) \sim f(x)$ iff $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

Using definition (2.3) we may now formally state the prime number theorem which we will work on proving throughout this paper,

Theorem 2.4. $\pi(x) \sim \frac{x}{\log x}$.

Theorem 2.5. $\psi(x) \sim \pi(x)\log(x)$

Proof. $\psi(x) = \sum_{p\leq x} \log p \lfloor \frac{\log(x)}{\log(p)} \rfloor \leq \sum_{p\leq x} \log(x) \leq \pi(x)\log(x)$

we have for any ε

$\psi(x) \geq \sum_{x^{1-\varepsilon}\leq p\leq x} \log(p) \geq \sum_{x^{1-\varepsilon}\leq p\leq x} (1-\varepsilon)\log(x) = (1-\varepsilon)(\pi(x) + O(x^{1-\varepsilon}))\log(x)$

$\Rightarrow \psi(x) \to \pi(x)\log(x)$ as x $\to \infty$ because ε is arbitrary. □

This theorem implies that $\psi(x) \sim x$ is an equivalent statement of the PNT. This paper will prove the PNT in this way.

3. Deriving $\psi(x)\log x + \sum_{p\leq x} \log p\psi(x/p) = 2x\log x + O(x)$

This is the primary contribution of Selberg from which several proofs of the prime number theorem are possible. Proving the equality takes the bulk of the paper.

First we consider the functions

$$\lambda_d = \lambda_{d,x} = \mu(d)\log^2(\frac{x}{d}) \tag{3.1}$$

and

$$\theta_n = \theta_{n,x} = \sum_{d/n} \lambda_d. \tag{3.2}$$

Theorem 3.3.

$$\theta_n = \begin{cases} \log^2(x) & \textit{if } n = 1 \\ \log(p) log(\frac{x^2}{p}) & \textit{if } n = p^\alpha, \alpha \geq 1 \\ 2\log(p)\log(q) & \textit{if } n = p^\alpha q^\beta, \alpha, \beta \geq 1 \\ 0 & \textit{for all other } n. \end{cases}$$

Proof. The first case follows immediately from (3.1).

To show the second case first observe $\mu(p^\alpha) = 0$ for $\alpha \geq 2$. We then have

$$\theta_n = \log^2(x) - \log^2(\frac{x}{p}) = \log^2(x) - (\log(x) - \log(p))^2 = 2\log(x)\log(p) - \log^2(p) = \log(p)\log(\frac{x^2}{p}).$$

The third case follows from the same properties of μ and log. We have

$$\theta_n = \log^2(\frac{x}{1}) - \log^2(\frac{x}{p}) - \log^2(\frac{x}{q}) + \log^2(\frac{x}{pq}) = \log(p) log(q).$$

The fourth case may be proven by induction. We may consider n square free as if powers of some prime p divide n all d in the summation which are divisible by powers of p will have $\mu(d) = 0$. Let $n = p_1 p_2 ... p_k$.

We have $\theta_{n,x} = \theta_{n/p_k,x} - \theta_{n/p_k,x/p_k}$ from (3.2). Observe the 3rd statement is independent of x and the result follows. ☐

We now calculate $\sum_{n \leq x} \theta_n$ in 2 ways: first by use of the definition, (3.2), and then by use of the theorem, (3.3). Observe that,

$$\sum_{n \leq x} \theta_n = \sum_{n \leq x} \sum_{d/n} \lambda_d = \sum_{d \leq x} \lambda_d \lfloor \frac{x}{d} \rfloor = x \sum_{d \leq x} \frac{\lambda_d}{d} + O(\sum_{d \leq x} |\lambda_d|) = \tag{3.4}$$

$$x \sum_{d \leq x} \frac{\mu(d)}{d} \log^2(\frac{x}{d}) + O(\sum_{d \leq x} \log^2(\frac{x}{d}) = x \sum_{d \leq x} \frac{\mu(d)}{d} \log^2(\frac{x}{d}) + O(x).$$

Next the sum will be evaluated using the results $\sum_{p \leq x} \frac{\log(p)}{x} = \log(x) + O(1)$ and $\psi(x) = O(x)$ to approximate the remainders. We have,

$$\sum_{n \leq x} \theta_n = \log^2(x) + \sum_{p^\alpha \leq x} \log(p)\log(\frac{x^2}{p}) + 2 \sum_{p^\alpha q^\beta \leq x, p < q} = \tag{3.5}$$

$$\sum_{p \leq x} \log^2(p) + \sum_{pq \leq x} \log(p)\log(q) + O(\sum_{p \leq x} \log(p)\log(\frac{x}{p})) + O(\sum_{p^\alpha \leq x, \alpha \geq 1} \log^2(x)) + O(\sum_{`p^\alpha q^\beta \leq x, \alpha > 1} + \log^2(x) =$$

$$= \sum_{p \leq x} \log^2(p) + \sum_{pq \leq x} \log(p)\log(q) + O(x).$$

Combining the two yields:

$$\sum_{p \leq x} \log^2(p) + \sum_{pq \leq x} \log(p)\log(q) = x \sum_{d \leq x} \frac{\mu(d)}{d} \log^2(\frac{x}{d}) + O(x). \tag{3.6}$$

The next step is to approximate the right hand sum. To this end we use an asymptotic formula for $\log^2(x)$ derived from known formulas for $\sum_{n\leq x}\frac{1}{n}$ and $\sum_{n\leq x}\sigma(n)$ where σ is the divisor function, $\sigma(x) = the\ number\ of\ divisors\ of\ x$. We have,

$$\sum_{n\leq x}\frac{1}{n} = \log(x) + c_1 + O(z^{-\frac{1}{4}}) \tag{3.7}$$

and

$$\sum_{n\leq x}\sigma(n) = x\log(x) + c_2x + O(\sqrt{x}). \tag{3.8}$$

We may use these two equations along with partial summation to derive,

$$\sum_{n\leq x}\frac{\sigma(n)}{n} = \frac{1}{2}\log^2(x) + c_3log(x) + O(z^{\frac{1}{4}}). \tag{3.9}$$

From (3.7) and (3.9) we obtain,

$$\log^2(x) = 2\sum_{n\leq x}\frac{\sigma(n)}{n} + c_4\sum_{n\leq x}\frac{1}{n} + c_5 + O(z^{\frac{1}{4}}). \tag{3.10}$$

Going back to the sum in equation (3.6) and using the well known results, $\sum_{d/n}\mu(d)\sigma(\frac{n}{d}) = 1$ and $\sum_{d\leq x}\frac{\mu(d)}{d} = O(1)$ we can derive,

$$\sum_{d\leq x}\frac{\mu(d)}{d}\log^2(\frac{x}{d}) = 2\sum_{d\leq x}\frac{\mu(d)}{d}\sum_{r\leq x/d}\frac{\sigma(r)}{r} + c_4\sum_{d\leq x}\frac{\mu(d)}{d}\sum_{r\leq x/d}\frac{1}{r}+ \tag{3.11}$$

$$+c_5\sum_{d\leq x}\frac{\mu(d)}{d} + O(x^{\frac{1}{4}}\sum_{d\leq x}d^{-\frac{3}{4}}) = 2\sum_{dr\leq x}\frac{\mu(d)\sigma(r)}{dr} + c_4\sum_{dr\leq x}\frac{\mu(d)}{dr}+$$

$$+c_5\sum_{d\leq x}\frac{\mu(d)}{d} + O(1) = 2\sum_{n\leq x}\frac{1}{n}\sum_{d/n}\mu(d)\sigma(\frac{n}{d}) + c_4\sum_{n\leq x}\frac{1}{n}\sum_{d/n}\mu(d) + O(1) =$$

$$= 2\sum_{n\leq x}\frac{1}{n} + c_4 + O(1) = 2\log(x) + O(1).$$

We now have

$$\sum_{p\leq x}\log^2(p) + \sum_{pq\leq x}\log(p)\log(q) = 2x\log(x) + O(x) \tag{3.12}$$

and by using the fact that

$$\sum_{p\leq x}\log^2(p) = \psi(x)\log(x) + O(x) \tag{3.13}$$

we get the form of the equation we are looking for,

$$\psi(x)\log x + \sum_{p\leq x}\log p\psi(x/p) = 2x\log x + O(x). \tag{3.14}$$

One more form of this equation is of interest in later sections. We start deriving it with the following equation obtained by partial summation from (3.12),

$$\sum_{p\leq x}\log(p)+\sum_{pq\leq x}\frac{\log(p)\log(q)}{\log(pq)}=2x+O(\frac{x}{\log(x)}) \tag{3.15}$$

which gives,

(3.16)
$$\sum_{pq\leq x}\log(p)\log(q)=\sum_{p\leq x}\log(p)\sum_{q\leq x/p}\log(q)=2x\sum_{p\leq x}\frac{\log(p)}{p}-\sum_{p\leq x}\log(p)\sum_{qr\leq x/p}\frac{\log(q)\log(r)}{\log(qr)}+$$
$$+O(x\sum_{p\leq x}\frac{\log(p)}{p(1+\log(\frac{x}{p}))})=2x\log(x)-\sum_{qr\leq x}\frac{\log(q)\log(r)}{\log(qr)}\psi(\frac{x}{qr})+O(x\log(\log(x))$$

which when plugged into (3.12) yields,

$$\psi(x)\log(x)=\sum_{pq\leq x}\frac{\log(p)\log(q)}{\log(pq)}\psi(\frac{x}{pq})+O(x\log(\log(x))). \tag{3.17}$$

4. The form of the function $R(x)=\psi(x)-x$

It is our goal to show that $\frac{\psi(x)}{x}\to 1$ as $x\to\infty$ so it is useful to consider $\psi(x)=x+R(x)$ and determine the properties of the remainder term $R(x)$. First we may obtain from (3.14),

$$R(x)\log(x)=-\sum_{p\leq x}\log(p)R(\frac{x}{p})+O(x) \tag{4.1}$$

and from (3.16) we obtain,

$$R(x)\log(x)=\sum_{pq\leq x}\frac{\log(p)\log(q)}{\log(pq)}R(\frac{x}{pq})+O(x\log(\log(x))). \tag{4.2}$$

Combining (4.1) and (4.2) yields the inequality,

(4.3)
$$2|R(x)|\log(x)\leq\sum_{p\leq x}\log(p)|R(\frac{x}{p})|+\sum_{pq\leq x}\frac{\log(p)\log(q)}{\log(pq)}|R(\frac{x}{pq})|+O(x\log(\log(x))).$$

This inequality implies the following less strict inequality,

(4.4)
$$2|R(x)|\log(x)\leq\sum_{n\leq x}\{\sum_{p\leq n}\log(p)+\sum_{pq\leq x}\frac{\log(p)\log(q)}{\log(pq)}\}\{|R(\frac{x}{n})|-|R(\frac{x}{n+1})|\}+O(x\log(\log(x))).$$

This inequality follows from the previous one, because all terms in the summation are strictly positive. We are simply adding more terms to the series by allowing n that cannot be written as the product of two primes. We may now use (3.15) and partial summation to arrive at the following equation, (4.5) $$|R(x)| \leq \frac{1}{\log(x)} \sum_{n\leq x} |R(\frac{x}{n})| + O(x\frac{\log(\log(x))}{\log(x)}).$$

5. Other properties of $R(x)$

Partial summation from $\sum_{p\leq x} \frac{\log(p)}{x} = \log(x) + O(1)$ yields,

(5.1) $$\sum_{n\leq x} \frac{\psi(n)}{n} = \log(x) + O(1).$$

Using $\psi(n) = n + R(n)$ and $\sum_{n\leq x} \frac{1}{n} = \log(n) + O(1)$ it is clear that,

(5.2) $$\sum_{n\leq x} \frac{R(n)}{n^2} = O(1).$$

This implies that for $x > 4$ and $x' > x$ there exists a constant K_1 such that,

(5.3) $$|\sum_{x\leq n\leq x'} \frac{R(n)}{n^2}| < K_1.$$

From this we observe:

$$\inf_{x\leq y\leq x'} |\frac{R(y)}{y}|(\log(\frac{x'}{x}) + O(1)) = \inf_{x\leq y\leq x'} |\frac{R(y)}{y}| \sum_{x\leq n\leq x'} \frac{1}{n} \leq$$

$$\leq \sum_{x\leq n\leq x'} \inf_{x\leq y\leq x'} |\frac{R(y)}{y}|\frac{1}{n} \leq |\sum_{x\leq n\leq x'} \frac{R(n)}{n^2}| < K_1.$$

Which shows for a different constant $K_2 \geq 0$,

(5.4) $$|\frac{R(y)}{y}| < \frac{K_2}{\log(\frac{x'}{x})}.$$

or in other words, for a fixed $0 < \delta < 1$ and $x > 4$, there will exist a y in the interval $x \leq y \leq e^{K_2/\delta}$, with

(5.5) $$|R(y)| < \delta y.$$

From (3.15) it is clear that $y < y'$,

$$\sum_{y\leq p\leq y'} \log(p) \leq 2(y' - y) + O(\frac{y'}{\log(y')}),$$

from which it follows by the definition of $\psi(x)$ that,

$$|R(y') - R(y)| \leq y' - y + O(\frac{y'}{\log(y')})$$

or

$$|R(y')| \leq |R(y)| + |y' - y| + \frac{K_3 y'}{\log(y')}.$$

Where K_3 is a constant whose existence follows from the definition of big O notation.

Now consider an interval $(x, e^{K_2/\delta})$. We know by, (5.5) that there exists a y in this interval such that,

$$|R(y)| < \delta y.$$

Therefore for any y' in the interval $y/2 \leq y' \leq 2y$, we have

$$|R(y')| \leq \delta y + |y' - y| + \frac{K_3 y'}{\log(x)},$$

or

$$\frac{|R(y')|}{y'} \leq 2\delta + |1 - \frac{y'}{y}| + \frac{K_3}{\log(x)}.$$

Therefore if we have $x > e^{K_2/\delta}$ and $e^{-\delta/2} \leq y'/y \leq e^{\delta/2}$, it follows that,

$$\frac{|R(y')|}{y'} \leq 2\delta + (e^{\delta/2} - 1) + \delta < 4\delta.$$

In other words we have determined that for $x > e^{K_2/\delta}$ the interval $(x, e^{K_2/\delta}x)$ will contain a sub-interval $(y_1, e^{\delta/2}y_1)$, with the property $|R(z)| < 4\delta$ for z in the sub-interval. We now have all the tools we need to prove the prime number theorem.

6. $R(x)/x \to 0$

We know $\psi(x) = O(x)$ which implies $R(x) = O(x)$ as well. This means for some constant K_4 and $x > 1$ we have,

$$|R(x)| < K_4 x. \tag{6.1}$$

Now assume for some $0 < \alpha < 8$,

$$|R(x)| < \alpha x, \tag{6.2}$$

is true for all $x > x_0 > e^{K_2/\delta}$. Setting $\delta = \alpha/8$, we have from section (5) that intervals of the form $(x, e^{K_2/\delta}x)$ contain an interval $(y, e^{\delta/2}y)$ with,

$$|R(z)| < \alpha z/2. \tag{6.3}$$

The inequalities (3.14) and (6.1) yield,

$$|R(x)| \leq \frac{1}{\log(x)} \sum_{n \leq x} |R(\frac{x}{n})| + O(\frac{x}{\sqrt{\log(x)}}) <$$

$$< K_4 \frac{x}{\log(x)} \sum_{x/x_0 < x \leq x} \frac{1}{n} + \frac{x}{\log(x)} \sum_{n \leq x/x_0} \frac{1}{n} |\frac{n}{x} R(\frac{x}{n})| + O(\frac{x}{\sqrt{\log(x)}}).$$

Setting $\rho = e^{K_2/\delta}$, and using (6.2) and (6.3) we get,

$$\frac{\alpha x}{\log(x)} \sum_{n \leq (x/x_0)} \frac{1}{n} - \frac{\alpha x}{2\log(x)} \sum_{1 \leq r \leq (\log(x/x_0)/\log(\rho)} \sum_{y_r \leq n \leq y_r e^{\delta/2}, \rho^{r-1} < y_r \leq \rho^r e^{-(\delta/2)}} \frac{1}{n}$$

$$+O(\frac{x}{\sqrt{\log(x)}}) = \alpha x - \frac{\alpha x}{2\log(x)} \sum_{1 \leq r \leq (\log(x/x_0)/\log(\rho)} \frac{\delta}{2} + O(\frac{x}{\sqrt{\log(x)}}) = \alpha x - \frac{\alpha\delta}{4\log(\rho)} x + O(\frac{x}{\sqrt{\log(x)}}) =$$

$$= \alpha(1 - \frac{\alpha^2}{256K_2})x + O(\frac{x}{\sqrt{\log(x)}}) < \alpha(1 - \frac{\alpha^2}{300K_2})x.$$

And now we're done because the sequence,

$$\alpha_{n+1} = \alpha_n(1 - \frac{\alpha_n^2}{300K_2})$$

converges to 0 as it is monotonically decreasing and only has a fixed point at $\alpha = 0$. This implies that $R(x)/x \to 0$.

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