THE DENSITY OF PRIMES OF THE FORM $a + km$

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Abstract. The Dirichlet's theorem on arithmetic progressions states that the number of prime numbers less than x of the form $a + km$ is approximately $\frac{x}{\phi(m)\log x}$ when x goes to infinity. In this paper, we will present a proof of the theorem and discuss the notion of natural density and analytic density.

Contents

The Dirichlet's theorem on arithmetic progressions describes the density of the prime numbers. Specifically, denote $\pi_a(x)$ to be the number of primes p lesser than x of the form $a+km$, then $\pi_a(x) \sim \frac{x}{\phi(m)\log x}$. In this paper, we will present a proof of Dirichlet's theorem on arithmetic progressions by analyzing the singularities of certain L-functions. Some preliminary backgrounds will be presented in the first section. In the second section some basic properties of the L-function $L(s, \chi)$ will be established. Section 3 contains the proof of Dirichlet's theorem using Wiener-Ikehara theorem, whose proof is postponed to section 5. In section 4 we will compare the natural density and the analytic density of a set.

1. Preliminary

Let us recall some definitions that will be used throughout the paper in this section. Set a, m to be natural numbers such that $m \geq 2$ and $gcd(a, m) = 1$. Denote $\zeta(s)$ to be the Riemann Zeta function, i.e.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1.1}$$

for $Re(s) > 1$.

Facts 1.2.

Date: AUGUST 30, 2013.

- When $Re(s) > 1$, we have
$$\zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}}$$
where the product is taken over all prime numbers.
- $\zeta(s)$ is holomorphic and is nonzero when $Re(s) > 1$.
- We have
$$\zeta(s) = \frac{1}{s-1} + \phi(s)$$
where $\phi(s)$ is holomorphic when $Re(s) > 0$.
- $\zeta(s)$ can be extended to a meromorphic function on $\mathbb{C}$ with a unique simple pole at $s = 1$, and the residue of $\zeta(s)$ at $s = 1$ is 1.

We omit the proofs of these facts. See [1] or [7] for details.

Definition 1.3. Let

$$\pi_a(x) = \sum_{\substack{p \le x \\ p \equiv a (mod m)}} 1 \tag{1.4}$$

Definition 1.5. Let

$$\psi_a(x) = \sum_{\substack{p^n \le x \\ p \equiv a (mod m)}} \log p^n = \sum_{k \le x} \Lambda_a(k)$$

where

$$\Lambda_a(n) = \begin{cases} \log n, & \text{if } n = p^k, \ and \ p \equiv a \ (mod \ m) \\ 0, & \text{otherwise} \end{cases}$$

Definition 1.6. Let

$$\theta_a(x) = \sum_{\substack{p \le x \\ p \equiv a (mod m)}} \log p$$

Let χ be a group homomorphism from the multiplicative group $((\mathbb{Z}/m\mathbb{Z})^\times, \cdot)$ to $(\mathbb{C}^\times, \cdot)$ such that $\chi(ab) = \chi(a)\chi(b)$. We extend the function to all of $\mathbb{Z}$ by putting $\chi(c) = 0$ if $gcd(c, m) > 1$. we call χ the Dirichlet character mod m.

Lemma 1.7. *(Orthogonality) Let G be the multiplicative group $(\mathbb{Z}/m\mathbb{Z})^\times$ Then*

$$\sum_{g \in G} \chi(g) = \begin{cases} \phi(m), & if \ \chi = \mathbb{1} \\ 0, & otherwise \end{cases}$$

Proof. Let $\chi \neq \mathbb{1}$. Choose $h \in G$ such that $\chi(h) \neq 1$.
Then,

$$\chi(h) \sum_{g \in G} \chi(g) = \sum_{g \in G} \chi(hg) = \sum_{g \in G} \chi(g)$$

Hence,

$$(\chi(h) - 1) \sum_{g \in G} \chi(g) = 0$$

As $\chi(h) \neq 1$,

$$\sum_{g \in G} \chi(g) = 0$$

□

Corollary 1.8. *Fix $g \in G$. Then*

$$\sum_{\chi \in \hat{G}} \chi(g) = \begin{cases} \phi(m), & \text{if } g = 1 \\ 0, & \text{otherwise} \end{cases}$$

Proof. We apply lemma 1.5 to the dual group $\hat{G}$. □

Next we will present Abel's summation formula which will be used extensively throughout the paper.

Theorem 1.9. *(Abel's summation formula) Let $0 \leq \lambda_1 \leq \lambda_2 \leq ...$ be a sequence of real numbers such that $\lambda_n \to \infty$ when $n \to \infty$, and let $A(x) = \sum_{\lambda_n \leq x} a_n$ where (a_n) is a sequence of complex numbers. If $\rho : \mathbb{R} \to \mathbb{C}$ is a function(not necessarily continuous), we have*

$$\sum_{n=1}^{k} a_n \rho(\lambda_n) = A(\lambda_k)\rho(\lambda_k) - \sum_{n=1}^{k-1} A(\lambda_n)(\rho(\lambda_{n+1}) - \rho(\lambda_n)) \tag{1.10}$$

Furthermore, if ρ has a continuous derivative in $(0, \infty)$, and $x \geq \lambda_1$, then the above equation can be written as

$$\sum_{\lambda_n \leq x} a_n \rho(\lambda_n) = A(x)\rho(x) - \int_{\lambda_1}^{x} A(t)\rho'(t)dt \tag{1.11}$$

Proof. For convenience, let $A(\lambda_0) = 0$. Then, we have

$$\sum_{n=1}^{k} a_n \rho(\lambda_n) = \sum_{n=1}^{k} (A(\lambda_n) - A(\lambda_{n-1})\rho(\lambda_n)$$
$$= A(\lambda_k)\rho(\lambda_k) - \sum_{n=1}^{k-1} A(\lambda_n)(\rho(\lambda_{n+1}) - \rho(\lambda_n))$$

which proves (1.10). Now, assume that ρ has a continuous derivative ρ' in $(0, \infty)$. Let $k = max\{k | \lambda_k \leq x\}$. Then,

$$\sum_{n=1}^{k} a_n \rho(\lambda_n) = \sum_{n=1}^{k-1} A(\lambda_n) \int_{\lambda_n}^{\lambda_{n+1}} \rho'(t)dt$$

And

$$A(\lambda_k)\rho(\lambda_k) = A(x)\rho(x) - \int_{\lambda_n}^{x} A(t)\rho'(t)dt$$

as $A(t)$ is a step function that remains constant in the interval $[\lambda_k, \lambda_{k+1})$. Hence, (1.11) follows from (1.10). □

Corollary 1.12. *With the same notations as in theorem 1.9, if $A(x)\rho(x) \to 0$ when $x \to \infty$, we have*

$$\sum_{n=1}^{\infty} a_n \rho(\lambda_n) = -\int_{\lambda_1}^{x} A(t)\rho'(t)dt$$

Proof. Send $x \to \infty$ in the equation (1.11). □

Corollary 1.13. *(Abel's lemma) Let* (a_n) (b_n) *be two sequences. Define*

$$A_{k,l} = \sum_{i=k}^{l} a_i \text{ and } S_{k,n} = \sum_{i=k}^{n} a_i b_i$$

.
Then,

$$S_{k,n} = \sum_{i=k}^{n-1} A_{k,i}(b_i - b_{i+1}) + A_{k,n} b_n \tag{1.14}$$

Proof. Put $a_i = A_{n,i} - A_{n,i-1}$ and the result is immediate. □

2. The L-functions $L(s, \chi)$

In this section, we will present some facts about L-functions associated to Dirichlet characters.

Definition 2.1. Let χ be a Dirichlet character mod m and $s \in \mathbb{C}$. Define

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

for $Re(s) > 1$.

Immediately we have the following lemma:

Lemma 2.2.

$$L(s, \mathbb{1}) = \zeta(s) \prod_{p|m} (1 - p^{-s})$$

We want to understand the behavior of $L(s, \chi)$ when s tends to 1. It turns out that if $\chi = \mathbb{1}$, then $L(s, \chi)$ diverges when $s \to 1$ and has a simple pole there. On the other hand, when $\chi \neq \mathbb{1}$, we have that $L(1, \chi) \neq 0$.

Corollary 2.3. $L(s, \mathbb{1})$ *extends to a meromorphic function for* $Re(s) > 0$
It has a unique simple pole at $s = 1$ *with* $res(L(s, \mathbb{1}), 1) = \frac{1}{\phi(m)}$.

Proof. This follows from the fact that $\zeta(s)$ extends to a meromorphic function for $Re(s) > 0$ with a unique simple pole at s = 1. We also know that $res(\zeta, 1) = 1$. Applying Lemma 2.2, we get

$$res(L(s, \mathbb{1}), 1) = \prod_{p|m} (1 - p^{-1}) \cdot res(\zeta, 1) = \frac{1}{\phi(m)}$$

□

Let $G(s)$ denote the *ordinary Dirichlet series*, which means that

$$G(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Note that L-functions are ordinary Dirichlet series with positive coefficients.

We could use Landau's theorem as a very powerful tool to determine the radius of convergence for an ordinary Dirichlet series with positive coefficients.

Theorem 2.4. *(Landau) Let $G(s)$ be an ordinary Dirichlet series with positive coefficients. Then, the domain of convergence of $G(s)$ is bounded by a singularity of $G(s)$ located on the real axis.*

In other words, if $\sigma > 0$ is the abscissa of convergence of $G(s)$, then $G(s)$ has a singularity at σ.

Proof. Let

$$G(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

where $a_n \geq 0$ for all $n \in \mathbb{N}$. Suppose that $G(s)$ converges for $Re(s) > \rho$ with $\rho \in \mathbb{R}$. Assume that $G(s)$ can be extended analytically to a function holomorphic in a neighborhood of the point $z = \rho$ in the complex plane. We will show that there exists $\epsilon > 0$ such that $G(s)$ converges for $Re(s) > \rho - \epsilon$.

Replace s by $s - \rho$ in the equation. Now assume $\rho = 0$.

We see that $G(s)$ is holomorphic in $Re(s) > 0$ plus a neighborhood of $s = 0$, so it is holomorphic in a disk $|s - 1| \leq 1 + \epsilon$ for some $\epsilon > 0$. Moreover, the Taylor series of $G(s)$ converges in the disk.

From complex analysis, we know that the k^{th} derivative of $G(s)$ can be expressed as

$$G^{(k)}(s) = \sum_{n \in \mathbb{N}} (-\log n)^k \frac{a_n}{n^s}$$

when $Re(s) > 0$.

Hence,

$$G^{(k)}(1) = (-1)^k \sum_{n \in \mathbb{N}} (\log n)^k \frac{a_n}{n}$$

The Taylor series expansion of G is :

$$G(s) = \sum_{k=0}^{\infty} \frac{1}{k!}(s-1)^k G^{(k)}(1), \text{ where } |s - 1| \leq 1 + \epsilon.$$

For $s = -\epsilon$, we have

$$G(-\epsilon) = \sum_{k=0}^{\infty} \frac{1}{k!}(1+\epsilon)^k (-1) G^{(k)}(1).$$

We know that

$$(-1)^k G^{(k)}(1) = \sum_{n} (\log n)^k \frac{a_n}{n}$$

is a convergent series with positive terms.

Thus,

$$G(-\epsilon) = \sum_{k,\ n \in \mathbb{N}} \frac{1}{k!}(1+\epsilon)^k (\log n)^k \frac{a_n}{n}$$

converges. We get

$$\begin{aligned} G(-\epsilon) &= \sum_n \frac{a_n}{n} \sum_{k=0}^{\infty} \frac{1}{k!}(1+\epsilon)^k (\log n)^k \\ &= \sum_n \frac{a_n}{n} n^{1+\epsilon} = \sum_n a_n n^{\epsilon} \end{aligned}$$

Hence, the series converges for $s = -\epsilon$, and it also converges for $Re(s) > -\epsilon$. □

Now, let us consider $L(s, \chi)$ when $\chi \neq \mathbb{1}$.

Lemma 2.5. *Let*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

where $a_n \geq 0$. *If* $A_{k,l} = \sum_{k}^{l} a_n$ *are bounded, then* f *converges for* $Re(s) > 0$.

Proof. Assume that $|A_{k,l}| < M \in \mathbb{R}^+$. From (1.14), we see that

$$\Big| \sum_{i=l}^{m} \frac{a_i}{i^s} \Big| \leq M \Big(\sum_{l}^{m-1} \Big| \frac{1}{i^s} - \frac{1}{(i+1)^s} \Big| + \Big| \frac{1}{m^s} \Big| \Big)$$

and thus,

$$| \sum_{i=l}^{m} \frac{a_i}{i^s} | \leq \frac{M}{l^s}.$$

Hence f converges. □

Proposition 2.6. *If* $\chi \neq \mathbb{1}$, $L(s, \chi)$ *converges when* $Re(s) > 0$ *and converges absolutely when* $Re(s) > 1$. *If* $Re(s) > 1$, *we have*

$$L(s, \chi) = \prod_p \frac{1}{1 - \frac{\chi(p)}{p^s}}$$

Proof. For $Re(s) > 1$, the absolute convergence follows from the fact that $\sum_{n=1}^{\infty} \frac{1}{n^k}$ converges absolutely when $k > 1$.

In order to show the convergence of the series for $Re(s) > 0$, we use the above lemma 2.5

Using the lemma, it is enough to show

$$A_{l,n} = \sum_{l}^{n} \chi(i)$$

are bounded for all l, n.

From Lemma 1.7 (orthogonality), we know that $\sum_{i=l}^{l+m-1} \chi(i) = 0$. Hence, it is enough to consider $l \leq n$ such that $n - l < m$.

As $|\chi(i)| = 1$ if $gcd(i, m) = 1$ and $|\chi(i)| = 0$ otherwise, we see that

$$|A_{l,n}| = \Big| \sum_{l}^{n} \chi(i) \Big| \leq \sum_{l}^{n} |\chi(i)| \leq \phi(m)$$

The partial sums are bounded, so $L(s, \chi)$ converges.

□

Next we will prove that $L(s, \chi) \neq 0$ when $\chi \neq \mathbb{1}$. In order to show this, we need to define a function ζ_m and inspect how the function behaves near $s = 1$.

Definition 2.7. Let

$$\zeta_m(s) = \prod_{\chi} L(s, \chi) \tag{2.8}$$

Let p be a prime not dividing m. I will simply write p as the image of p in the group $G = (\mathbb{Z}/m\mathbb{Z})^{\times}$. Denote $f(p)$ to be the order of p in G. Let $g(p) = \frac{\phi(m)}{f(p)}$.

We can rewrite ζ_m with respect to prime numbers that do not divide m.

Proposition 2.9.

$$\zeta_m(s) = \prod_{p \nmid m} \frac{1}{\left(1 - \frac{1}{p^{f(p)s}}\right)^{g(p)}}$$

In order to prove the proposition, we need a lemma.

Lemma 2.10. *If p is a prime not dividing m, we have*

$$\prod_{\chi} (1 - \chi(p)T) = (1 - T^{f(p)})^{g(p)}.$$

Proof. Let W be the set of all $f(p)^{th}$ roots of unity. We have

$$\prod_{w \in W} (1 - wT) = 1 - T^{f(p)}.$$

For all $w \in W$, we have $g(p)$ characters χ in $\hat{G}$ such that $\chi(p) = w$, as $g(p)$ is th order of the quotient of G by the cyclic subgroup $< p >$. Hence, the lemma follows. □

Proof. (of Proposition 2.9)

$$\zeta_m(s) = \prod_{\chi} L(s, \chi).$$

Replace $L(s, \chi)$ using Proposition 2.6, we have

$$\zeta_m(s) = \prod_{\chi \in \hat{G}} \frac{1}{\prod_{p} (1 - \frac{\chi(p)}{p^s})} = \prod_{p \nmid m} \frac{1}{\prod_{\chi} (1 - \frac{\chi(p)}{p^s})}.$$

By Lemma 2.10, we get

$$\prod_{\chi} (1 - \frac{\chi(p)}{p^s}) = \left(1 - \frac{1}{p^{f(p)s}}\right)^{g(p)}.$$

Thus the proposition follows.

□

Proposition 2.11. $L(1, \chi) \neq 0$ *if* $\chi \neq \mathbb{1}$.

Proof. We will prove this result by contradiction.

We know that $L(s, \chi \neq \mathbb{1})$ is holomorphic in a neighborhood of $s = 1$(Proposition 2.6), and $L(s, \mathbb{1})$ extends to a meromorphic function for $Re(s) > 0$ with a unique simple pole at $s = 1$ (Corollary 2.3).

Now, suppose that the proposition is not true. There exists $\chi \neq \mathbb{1}$ such that $L(1, \chi) = 0$, and we conclude that $\zeta_m(s) = \prod_{\chi} L(s, \chi)$ is holomorphic at $s = 1$.

As ζ_m is an ordinary Dirichlet series with positive coefficients, Landau's theorem (2.4) shows that the abscissa of convergence $\sigma \leq 0$. We will show that this is impossible.

The p^{th} factor of ζ_m is

$$\frac{1}{(1 - p^{f(p)s})^{g(p)}} = (1 + p^{-f(p)s} + p^{-2f(p)s} + \dots)^{g(p)}$$

and it dominates the series

$$1 + p^{-\phi(m)s} + p^{-2\phi(m)s} + \dots$$

Hence, all the coefficients of ζ_m are greater than those of the series

$$F(s) = \sum_{(n,m)=1} n^{-\phi(m)s}$$

A problem occurs because the series F diverges at $s = \frac{1}{\phi(m)}$. We see that $\zeta_m(s) \geq F(s)$ for all $s \geq \sigma$. Hence, ζ_m diverges at $s = \frac{1}{\phi(m)}$.

We have shown that ζ_m cannot be holomorphic for the entire half plane $Re(s) > 0$. Therefore $L(1, \chi) \neq 0$ if $\chi \neq \mathbb{1}$.

□

Now we are ready to prove the theorem of arithmetic progressions.

3. The Proof of the Dirichlet's theorem on Arithmetic Progressions

Recall that a, m are natural numbers with $m \geq 2$ and $gcd(a, m) = 1$. The aim of this section is to prove the following theorem.

Theorem 3.1. *There exist infinitely many prime numbers p such that $p \equiv a$ (mod p). Moreover $\pi_a(x) \sim \frac{\log x}{\phi(m)x}$ as $x \to \infty$.*

Showing an asymptotic behavior of π_a is quite strenuous, so we instead try to observe the behavior of ψ_a when $x \to \infty$.

Lemma 3.2. *When $x \to \infty$, the following are equivalent:*

- $\pi_a(x) \sim \frac{\log x}{\phi(m)x}$
- $\psi_a(x) \sim \frac{x}{\phi(m)}$
- $\theta_a(x) \sim \frac{x}{\phi(m)}$

Proof. By Abel's summation formula (Corollary 1.8), we have

$$\pi_a(x) = \sum \log p \frac{1}{\log p} = \frac{\theta_a(x)}{\log x} + \int_2^x \frac{\theta_a(t)}{t \log^2 t} dt$$

and

$$\int_2^x \frac{\theta_a(t)}{t\log^2 t} dt = O(\frac{x}{\log^2 x}).$$

Also, we see that

$$\psi_a(x) = \theta_a(x) + \theta_a(x^{1/2}) + \theta_a(x^{1/3}) + \dots$$

Hence $\pi_a(x) \sim \frac{\log x}{\phi(m)x} \Leftrightarrow \psi_a(x) \sim \frac{x}{\phi(m)} \Leftrightarrow \theta_a(x) \sim \frac{x}{\phi(m)}$ when $x \to \infty$.

□

Now we are reduced to showing that

$$\psi_a(x) \sim \frac{x}{\phi(m)}. \tag{3.3}$$

Let

$$F_a(s) = \sum_{n=1}^{\infty} \frac{\Lambda_a(n)}{n^s} \tag{3.4}$$

.

For $Re(s) > 1$ we define $\log L(s,\chi)$ as

$$\log L(s,\chi) = \sum_p \log \frac{1}{1-\chi(p)p^{-s}} = \sum_{n,p} \frac{\chi(p)^n}{np^{ns}}.$$

The series $\sum_{n,p} \frac{\chi(p)^n}{np^{ns}}$ is obviously convergent. Then we can set

$$\frac{L'(s,\chi)}{L(s,\chi)} = \frac{d}{ds}\Big(\log L(s,\chi)\Big) = -\sum_{n=1}^{\infty} \frac{\chi(n)\Lambda(n)}{n^s}.$$

Lemma 3.5. *For $s \in \mathbb{C}$ and $Re(s) > 1$,*

$$F_a(s) = -\frac{1}{\phi(m)} \sum_{\chi} \overline{\chi(a)} \frac{L'(s,\chi)}{L(s,\chi)}. \tag{3.6}$$

Proof. In essence, this is Fourier analysis on the finite abelian group $G = (\mathbb{Z}/m\mathbb{Z})^{\times}$.

We see that

$$\sum_{\chi} \overline{\chi(a)} \frac{L'(s,\chi)}{L(s,\chi)} = \sum_{\chi} \overline{\chi(a)} \Big(-\sum_{n=1}^{\infty} \frac{\chi(n)\Lambda(n)}{n^s}\Big) = -\sum_{n=1}^{\infty} \Big(\sum_{\chi} \chi(na^{-1})\Big) \frac{\Lambda(n)}{n^s}$$

Using Corollary 1.8, we see that the above equation equals to

$$\sum_{\chi} \overline{\chi(a)} \frac{L'(s,\chi)}{L(s,\chi)} = -\phi(m) \sum_{n=1}^{\infty} \frac{\Lambda_a(n)}{n^s}.$$

□

We will relate F_a and ψ_a by a specific formula, and prove (3.3) from this formula using the Wiener-Ikehara theorem (Theorem 3.13).

Let us first recall a simple lemma from complex analysis:

Lemma 3.7. *Let f be a meromorphic function. If f has at most a pole at z_0, then $res(f'/f, z_0) = ord(f, z_0)$.*

Proof. Translate if necessary, we may take $z_0 = 0$. Assume that it has at most a pole at 0. Then f has only a finite number of negative terms. Hence, let us write

$$f(z) = a_m z^m + (\text{higher terms}) \qquad a_m \neq 0, m \in \mathbb{Z}.$$

Then,

$$f(z) = a_m z^m (1 + h(z)) \tag{3.8}$$

where $h(z)$ is a power series with no constant term.

Differentiating both sides, we get

$$f'(z) = m a_m z^{m-1} + a_m z^m h'(z) \tag{3.9}$$

Divide (3.9) by (3.8), we see that

$$\frac{f'}{f} = \frac{m}{z} + \frac{h'(z)}{1 + h(z)}.$$

As $h(z)$ has no constant term, $\frac{h'(z)}{1+h(z)}$ is holomorphic at 0.

Hence, $res(f'/f, 0) = ord(f, 0)$.

□

Lemma 3.10. *The function $F_a(s)$ can be extended to a meromorphic function on an open set $\mathcal{O}$ containing $\{s \in \mathbb{C} | Re(s) \geq 1, s \neq 1\}$. Moreover, F_a has a simple pole at $s = 1$ with residue $\frac{1}{\phi(m)}$.*

Proof. From Proposition 2.11, if $\chi \neq \mathbb{1}$, we see that $L(s, \chi)$ is holomorphic and is nonzero on an open set $\mathcal{O}$ containing $\{s \in \mathbb{C} | Re(s) \geq 1, s \neq 1\}$. Hence $L'(s, \chi)/L(s, \chi)$ is holomorphic.

On the other hand, $L(s, \mathbb{1})$ has a unique simple pole at $s = 1$. From Lemma 3.7, we see that $L'(s, \mathbb{1})/L(s, \mathbb{1})$ has a simple pole at $s = 1$ with residue 1.

Hence, using the relation (3.6) from Lemma 3.5, we see that F_a has a simple pole with

$$res(F_a, 1) = \frac{1}{\phi(m)}$$

□

Using Corollary 1.12 (Abel's summation formula), we see that

$$F_a(s) = \sum_{n=1}^{\infty} \frac{\Lambda_a(n)}{n^s} = s \int_1^{\infty} \frac{\psi_a(t)}{t^{s+1}} dt.$$

If we put $t = e^x$, we get

$$\frac{F_a(s)}{s} = \int_0^{\infty} \psi_a(x) e^{-xs} dx. \tag{3.11}$$

Lemma 3.10 tells that

$$res\Big(\frac{F_a(s)}{s}, 1\Big) = \frac{1}{\phi(m)}. \tag{3.12}$$

Theorem 3.13. *(Wiener-Ikehara) Let $A(x)$ be a non-negative, non-decreasing function in an interval $[0, \infty)$. Assume that for $\sigma > 1$, the integral*

$$\int_0^\infty A(x)e^{-xs}dx,\ s = \sigma + it$$

converges to the function F(s), where F is holomorphic for $\sigma \geq 1$ except for a simple pole at $s = 1$ with residue γ. Then,

$$\lim_{x\to\infty} e^{-x}A(x) = \gamma \tag{3.14}$$

The proof of the theorem is long and requires techniques from functional analysis, so we will present it later in Section 5.

Proof. *(of Theorem 3.1)* We will show that $\psi_a(x) \sim \frac{x}{\phi(m)}$.

Set $A(x) = \psi_a(e^x)$. We know that ψ_a is non-decreasing and non-negative.

From (3.11), we see that

$$F(s) = \frac{F_a(s)}{s} = \int_0^\infty A(x)e^{-xs}dx$$

and (3.12) tells us that the residue of F at $s = 1$ is $\frac{1}{\phi(m)}$.

Hence, by Theorem 3.13, we conclude that

$$e^{-x}\psi_a(e^x) \sim \frac{1}{\phi(m)} \text{as } x \to \infty.$$

In other words, we have $\psi_a(x) \sim \frac{x}{\phi(m)}$ as $x \to \infty$.

From Lemma 3.2, we conclude that $\pi_a(x) \sim \frac{\log x}{\phi(m)x}$ as $x \to \infty$. This completes the proof of the Dirichlet's theorem on Arithmetic Progressions.

□

Corollary 3.15. *(Dirichlet) There are infinitey many primes of the form $a + km$ where $gcd(a, m) = 1$.*

Proof. The number of the primes of the form $a + km$ is given as $\pi_a(x) \sim \frac{\log x}{\phi(m)x}$, and it is not bounded above. □

Corollary 3.16. *Primes are evenly distributed among the congruence classes modulo m.*

Proof. $\pi_a(x) \sim \frac{\log x}{\phi(m)x}$, and $\frac{\log x}{\phi(m)x}$ does not depend on a. □

We will finish this section by presenting another proof showing $\psi_a(x) \sim \frac{x}{\phi(m)}$ by contour integral instead of Theorem 3.1.

$$\text{Claim: } \psi_a(x) \sim \frac{x}{\phi(m)}.$$

Proof. Fix $x \in \mathbb{R}$ and define

$$g(s) = F_a(s)\frac{x^{s+1}}{s(s+1)}.$$

From Lemma 3.10, we see that $g(s)$ is meromorphic on an open set $\mathcal{O}$ containing $\{s \in \mathbb{C} | Re(s) \geq 1, s \neq 1\}$ with a simple pole at $s = 1$ and

$$res(g,1) = \frac{x^2}{1\cdot 2}\cdot res(F_a,1) = \frac{x^2}{2\phi(m)}.$$

Then, consider

$$\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} g(s)ds. \tag{3.17}$$

We want to express (3.17) in two different ways.

First, let us use the fact that $F_a(s) = \sum \Lambda_a(n)/n^s$.

Hence, (3.17) becomes

$$\sum_0^\infty \Lambda_a(n)\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{x^{s+1}ds}{n^s s(s+1)}. \tag{3.18}$$

Note that we need to show the termwise integration is valid so that (3.17) is equivalent to (3.18). We can rewrite $g(s)$ as

$$g(s) = \left(x\sum_1^\infty \Lambda_a(n)\big(\frac{x}{n}\big)^s\frac{1}{s}\right) - \left(x\sum_1^\infty \Lambda_a(n)\big(\frac{x}{n}\big)^{s+1}\frac{n}{s+1}\right).$$

Von Mangoldt's method from [3] shows that each of the two terms can be integrated termwise. Therefore, the termwise integration is valid.

Now, using the fact that

$$\frac{x^{s+1}}{n^s s(s+1)} = \frac{x}{s}\big(\frac{x}{n}\big)^s - \frac{n}{s+1}\big(\frac{x}{n}\big)^{s+1},$$

we have

$$\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\frac{x^{s+1}ds}{n^s s(s+1)} = \frac{x}{2\pi i}\int_{c-\infty i}^{c+\infty i}\big(\frac{x}{n}\big)^s\frac{ds}{s} - \frac{n}{2\pi i}\int_{c+1-\infty i}^{c+1+\infty i}\big(\frac{x}{n}\big)^v\frac{dv}{v}.$$

$$\Leftrightarrow \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\frac{x^{s+1}ds}{n^s s(s+1)} = \begin{cases} x-n & \text{if } n\le x\\ 0, & \text{if } n\ge x\end{cases} \tag{3.19}$$

Thus (3.18) becomes

$$(3.18) = \sum_{n\le x}\Lambda_a(n)(x-n) = \int_0^x \psi_a(t)dt \tag{3.20}$$

The second way of expressing (3.17) is to do contour integral of a rectangle defined by $Re(s) = c$ and $Re(s) = 1$ with a small protuberance near $s = 1$, and two edges in the infinity.

We know that $g(s)$ is meromorphic on an open set $\mathcal{O}$ with a unique simple pole at $s = 1$. Hence, by the residue theorem,

$$\int_\gamma g(s)ds = res(g,1) = \frac{x^2}{2\phi(m)}.$$

Hence,

$$\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} g(s)ds \sim \int_{\gamma} g(s)ds = \frac{x^2}{2\phi(m)} \tag{3.21}$$

as we see that integral over other edges of the protuberated rectangle is $o(x^2)$. Hence, putting (3.21) and (3.18) together, we get

$$\int_0^x \psi_a(t)dt \sim \frac{x^2}{2\phi(m)} \tag{3.22}$$

.

We are reduced to showing that (3.22) implies

$$\psi_a(x) \sim \frac{x}{\phi(m)}.$$

The reader is encouraged to look at [3] for a rigorous proof. Although in [3] the author shows that $\int_0^x \psi(t)dt \sim \frac{x^2}{2}$ implies $\psi(x) \sim x$, we can almost exactly follow the argument to prove our result. Here, we will provide a simpler argument that

$$(3.22) \;\Rightarrow \psi_a(x) \sim \frac{x}{\phi(m)}$$

from the fundamental theorem of calculus:

$$\int_0^x \psi_a(t)dt \sim \frac{x^2}{2\phi(m)}$$

$$\Leftrightarrow \int_0^x \psi_a(t)dt = \frac{x^2}{2\phi(m)} + o(x^2).$$

Hence, from the fundamental theorem of calculus we get

$$\psi_a(x) = \frac{x}{\phi(m)} + o(x)$$

by differentiating both sides. The above equation is equivalent to

$$\psi_a(x) \sim \frac{x}{\phi(m)}.$$

□

4. Natural Density and Analytic Density

In this section, we will discuss two different notions of *density*. We will present another proof of Dirichlet's theorem on arithmetic progressions (i.e. Corollary 3.15) using *analytic density*.

Let P be the set of all prime numbers. Let A be any subset of P. Denote $A_n = \{a \in A | a \leq n\}$ and $P_n = \{p \in P | p \leq n\}$.

Definition 4.1. The *natural density* $N(A)$ of a subset $A \subset P$ is defined to be

$$N(A) = \lim_{n\to\infty} \frac{|A_n|}{|P_n|}$$

if the limit exists.

Definition 4.2. The *analytic density*, or *Dirichlet density* of $A \subset P$ is defined to be

$$D(A) = \lim_{s \to 1^+} \frac{\sum\limits_{a \in A} a^{-s}}{\sum\limits_{p \in P} p^{-s}}$$

if the limit exists.

We can extend the definitions of two kinds of density for two sets $A \subset B \subset \mathbb{N}$.

Theorem 4.3. *Let A be a subset of $B \subset \mathbb{N}$. Assume that*

$$\sum_{m=1}^{\infty} \frac{\chi_B(m)}{m^s} \to \infty \text{ when } s \to 1.$$

If $N(A)$ (with respect to B) exists and is equal to $k \in \mathbb{R}$, then $D(A)$ also exists and equals k.

Proof. Let χ_A and χ_B be characteristic functions of the sets A and B.

Set $s \in \mathbb{C}$ with $Re(s) \geq 1$. Then we have that

$$k = \lim_{n \to \infty} \frac{\sum\limits_{m=1}^{n} \chi_A(m)}{\sum\limits_{m=1}^{n} \chi_B(m)}.$$

By Corollary 1.13, we know

$$\sum_{m=1}^{n} \frac{\chi_A(m)}{m^s} = \sum_{m=1}^{n-1} (\frac{1}{m^s} - \frac{1}{(m+1)^s}) A_m + \frac{A_n}{n^s},$$

where

$$A_m = \sum_{i=1}^{m} \chi_A(i) < m.$$

Hence, when $n \to \infty$, we have

$$\sum_{m=1}^{\infty} \frac{\chi_A(m)}{m^s} = \sum_{m=1}^{\infty} \Big(\frac{1}{m^s} - \frac{1}{(m+1)^s}\Big) A_m,$$

as $0 \leq \frac{A_n}{n^s} \leq \frac{n}{n^s} = n^{s-1} \to 0$ when n $\to \infty$.

By assumption, for any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that if $n \geq N$,

$$k - \epsilon < \frac{\sum\limits_{m=1}^{n} \chi_A(m)}{\sum\limits_{m=1}^{n} \chi_B(m)} < k + \epsilon$$

$$\Leftrightarrow k - \epsilon < \frac{A_n}{B_n} < k + \epsilon$$

where B_n is defined similar to A_n.

Hence,

$$(k - \epsilon) B_n < A_n < (k + \epsilon) B_n \text{ for all } n \geq N. \tag{4.4}$$

Choose suitably large $l \in \mathbb{N}$ such that

$$(k-\epsilon)B_n - l < A_n < (k+\epsilon)B_n + l \text{ for all } n \in \mathbb{N}. \tag{4.5}$$

This is possible because there are at most finitely many terms that do not satisfy the inequality (4.4).

We see that

$$\sum_{m=1}^{\infty} \frac{\chi_A(m)}{m^s} = \sum_{m=1}^{\infty} \Big(\frac{1}{m^s} - \frac{1}{(m+1)^s}\Big) A_m.$$

Using the former inequality (4.5), we get

$$\sum_{m=1}^{\infty} \frac{\chi_A(m)}{m^s} < (k+\epsilon) \sum_{m=1}^{\infty} \Big(\frac{1}{m^s} - \frac{1}{(m+1)^s}\Big) B_m + l \sum_{m=1}^{\infty} \Big(\frac{1}{m^s} - \frac{1}{(m+1)^s}\Big).$$

We see that

$$(k+\epsilon) \sum_{m=1}^{\infty} \Big(\frac{1}{m^s} - \frac{1}{(m+1)^s}\Big) B_m = (k+\epsilon) \sum_{m=1}^{\infty} \frac{\chi_B(m)}{m^s}$$

.

Moreover,

$$l \sum_{m=1}^{\infty} \Big(\frac{1}{m^s} - \frac{1}{(m+1)^s}\Big) = l$$

.

We get

$$(k-\epsilon) \sum_{m=1}^{\infty} \frac{\chi_B(m)}{m^s} - l < \sum_{m=1}^{\infty} \frac{\chi_A(m)}{m^s} < (k+\epsilon) \sum_{m=1}^{\infty} \frac{\chi_B(m)}{m^s} + l$$

$$k - \epsilon - \frac{l}{\sum \frac{\chi_B(m)}{m^s}} < \frac{\sum \frac{\chi_A(m)}{m^s}}{\sum \frac{\chi_B(m)}{m^s}} < k + \epsilon + \frac{l}{\sum \frac{\chi_B(m)}{m^s}} \tag{4.6}$$

From the assumption we have

$$\sum_{m=1}^{\infty} \frac{\chi_B(m)}{m^s} \to \infty \text{ when } s \to 1,$$

so

$$\frac{l}{\sum_{m=1}^{\infty} \frac{\chi_B(m)}{m^s}} \to 0 \text{ when} s \to 1.$$

Since $\epsilon > 0$ is arbitrary, we can send $\epsilon \to 0$ in (4.6), and send $s \to 1$. Consequently, we get the desired result.

Therefore, *(Analytic Density) = (Natural Density) = k.*

□

Remark 4.7. Note that the converse is not always true.

Example 4.8. Let $A \subset \mathbb{N}$ be the set of natural numbers which have first digit 1.

Proof. For simplicity denote

$$N(n) = \frac{|A_n|}{|\mathbb{N}_n|}, \text{ where } \mathbb{N}_n = \{k \in \mathbb{N} | k \leq n\}.$$

We see that A has an analytic density, but it does not have a natural density, as

$$\limsup_{n\to\infty} N(n) \neq \liminf_{n\to\infty} N(n).$$

For simplicity denote $N(m) = \frac{|A_n|}{|\mathbb{N}_n|}$.
A simple counting argument shows that

$$|A_n| = \frac{10^k - 1}{9} \text{ if } n = 10^k - 1.$$

Hence, we see that

$$\liminf_{n\to\infty} N(n) \leq \frac{1}{9} \text{ (in fact equality holds here).}$$

If $n = 2 \cdot 10^k - 1$, we see that

$$|A_n| = \frac{10^k - 1}{9} + 10^k = \frac{10^{k+1} - 1}{9}, \text{ which implies}$$

$$\limsup_{n\to\infty} N(n) \geq \frac{5}{9}$$

Therefore, natural density does not exist.

Now, let us show that the analytic density exists nevertheless.
I claim that

$$\frac{\sum\limits_{a\in A} a^{-s}}{\sum\limits_{n\in\mathbb{N}} n^{-s}} \to \log_{10} 2 \text{ when } s \to 1.$$

From facts 1.2, we know that $\zeta(s) = \sum\limits_{n\in\mathbb{N}} n^{-s} \sim \frac{1}{s-1}$ when $s \to 1$. Hence, the claim is equivalent to

$$\lim_{s\to1} \frac{\sum\limits_{a\in A} a^{-s}}{\frac{1}{s-1}} = \lim_{s\to1} (s-1) \sum_{a\in A} \frac{1}{a^s} = \log_{10} 2.$$

For $s > 1$, we see from the figure below that

$$\sum_{k=0}^{\infty} \left(\int_{10^k}^{2\cdot10^k} \frac{1}{x^s} dx \right) \leq \sum_{a\in A} \frac{1}{a^s} \leq \sum_{k=0}^{\infty} \left(\int_{10^k}^{2\cdot10^k} \frac{1}{x^s} dx \right) + 1$$

since (Blue areas) ≤ 1: shift all the blue regions above $[10^k, 2 \cdot 10^k]$ to the rectangle above $[1, 2]$. Obviously the sum of all blue areas above $[10^k, 2 \cdot 10^k]$ is less than that of the rectangle above $[1, 2]$, which has area 1.
We can calculate

$$\sum_{k=0}^{\infty} \left(\int_{10^k}^{2\cdot10^k} \frac{1}{x^s} dx \right)$$

using basic integration and facts about geometric progression.

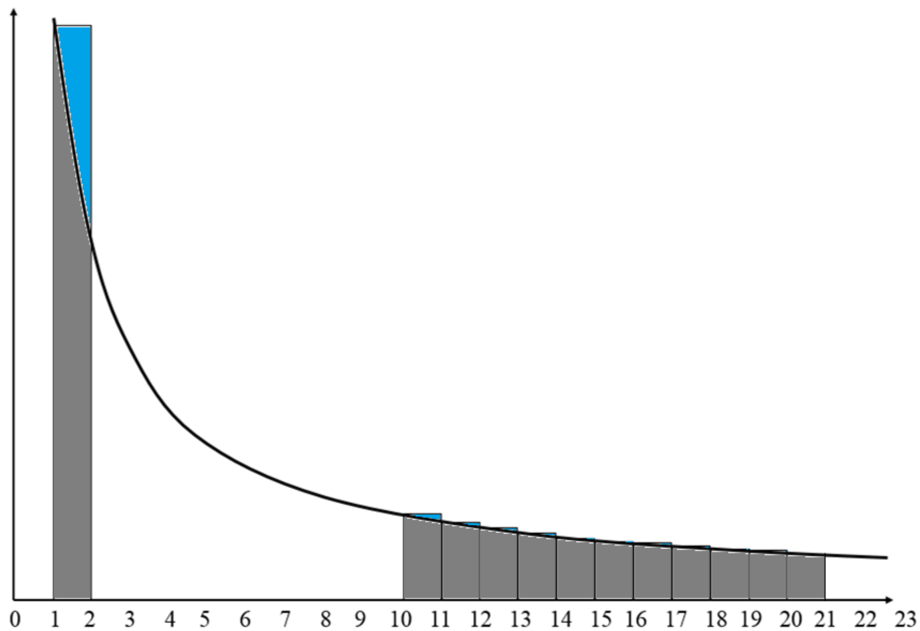

FIGURE 1. $y = 1/x^s$ and $\sum_{a\in A} a^{-s}$

$$\sum_{k=0}^{\infty}\left(\int_{10^k}^{2\cdot 10^k}\frac{1}{x^s}dx\right) = \sum_{k=0}^{\infty}\frac{1}{1-s}\left[\frac{1}{(2\cdot 10^k)^{s-1}} - \frac{1}{(10^k)^{s-1}}\right]$$
$$= \frac{1}{1-s}\sum_{k=0}^{\infty}\frac{2-2^s}{2^s}\frac{1}{(10^k)^{s-1}} = \frac{1}{1-s}\frac{2-2^s}{2^s}\frac{10^{s-1}}{10^{s-1}-1}$$

Hence, we have

$$\frac{1}{1-s}\frac{2-2^s}{2^s}\frac{10^{s-1}}{10^{s-1}-1} \leq \sum_{a\in A}\frac{1}{a^s} \leq \frac{1}{1-s}\frac{2-2^s}{2^s}\frac{10^{s-1}}{10^{s-1}-1} + 1$$

multiplying by $s-1>0$ and sending $s \to 1^+$ gives

$$\frac{\log 2}{\log 10} \leq D(A) \leq \frac{\log 2}{\log 10},$$

which shows that

$$D(A) = \log_{10} 2.$$

Similarly, if we define $A_k = \{n \in \mathbb{N} \mid n \text{ has first digit k}\}$, we can show $D(A_k) = \log_{10}\left(1+\frac{1}{k}\right)$. Note that

$$\sum_{k=1}^{9}\log_{10}\left(1+\frac{1}{k}\right) = \log_{10}\prod_{k=1}^{9}\left(1+\frac{1}{k}\right) = \log_{10} 10 = 1.$$

□

Surprisingly, the same holds for the set of prime numbers which have first digit 1: it also has analytic density $\log_{10} 2$ but it does not have natural density. For more information, see [1] page 76, [5], and [6].

We also note that the conclusion aligns with *Benford's Law*, or *the first-digit law* in statistics which claims that numbers with the leading digit $k \in \{1, \cdots, 9\}$ occur with probability $\log_{10}\left(1 + \frac{1}{k}\right)$. We should note that the probability of finding the numbers with first digit k is not $\log_{10}\left(1 + \frac{1}{k}\right)$ (As we just noted in the example, natural density of A_k does not exist). Nonetheless, the analytic density of each A_k is $\log_{10}\left(1 + \frac{1}{k}\right)$.

We will show that $D(P_a) = \frac{1}{\phi(m)}$ following [1], which shows that the number of primes congruent to a modulo m is infinite. Of course, we can use theorem 4.3 and the fact that $N(P_a) = \frac{1}{\phi(m)}$ (one could use the prime number theorem $\pi(x) = \frac{\log x}{x}$ together with theorem 3.1). However, there is a direct way to see that $D(P_a) = \frac{1}{\phi(m)}$. The proof is mainly based on the facts that $L(s, \chi) \neq 0$ when $\chi \neq \mathbb{1}$ (proposition 2.11) and $L(\mathbb{1}, s)$ has a simple pole at $s = 1$(corollary 2.3).

Lemma 4.9.
$$\sum_{p \in P} p^{-s} \sim \log \frac{1}{s-1} \ \textit{when } s \to 1.$$

Proof. From the facts 1.2, we can express ζ as
$$\zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}} = \frac{1}{s-1} + \phi(s).$$

Hence, if we take the logarithm, we get
$$\log \zeta(s) = \sum_{\substack{p \in P \\ k \geq 1}} \frac{1}{kp^{ks}} = \sum_{p \in P} \frac{1}{p^s} + \mu(s),$$

where $\mu(s) = \sum_{\substack{p \in P \\ k \geq 2}} \frac{1}{kp^{ks}}$ is bounded. As ζ has a simple pole in $s = 1$, the lemma follows.

□

Proposition 4.10. *If $D(A)$ exists, then*
$$D(A) = \lim_{s \to 1^+} \frac{\sum_{a \in A} a^{-s}}{\log \frac{1}{s-1}}$$

Proof. Lemma 4.9 indicates that
$$\sum_{p \in P} p^{-s} \sim \log \frac{1}{s-1} \text{ when } s \to 1.$$

□

Let $P_a = \{p \in P | \quad p \equiv a \pmod{\text{m}}\}$.

The goal now is to show that P_a has analytic density $\frac{1}{\phi(m)}$. In order to prove this, we will first establish three lemmas.

Let us define

$$f_\chi(s) = \sum_{p \nmid m} \frac{\chi(p)}{p^s}.$$

Note that f_χ obviously converges for $Re(s) > 1$.

Lemma 4.11. *If $\chi = \mathbb{1}$, then $f_\chi \sim \log \frac{1}{s-1}$ when $s \to 1$.*

Proof. Observe that f_χ differs from the series $\sum \frac{1}{p^s}$ at finite number of terms (i.e. prime numbers p such that $p|m$. The result immediately follows from Lemma 4.9. ☐

Lemma 4.12. *If $\chi \neq \mathbb{1}$, then f_χ remains bounded when $s \to 1$.*

Proof. Recall that we defined $\log L(s,\chi)$ in the proof of lemma 3.2 as follow:

$$\log L(s,\chi) = \sum_p \log \frac{1}{1-\chi(p)p^{-s}} = \sum_{n,p} \frac{\chi(p)^n}{np^{ns}}.$$

Now, we can rewrite the above equation as

$$\log L(s,\chi) = \sum_{n,p} \frac{\chi(p)^n}{np^{ns}} = \sum_p \frac{\chi(p)}{p^s} + \sum_{n,p \geq 2} \frac{\chi(p)^n}{np^{ns}}.$$

We see that

$$\sum_p \frac{\chi(p)}{p^s} = f_\chi.$$

Proposition 2.11 shows that $\log L(s,\chi)$ remains bounded when $s \to 1$.

$$\sum_{n,p \geq 2} \frac{\chi(p)^n}{np^{ns}}$$

also remains bounded when $s \to 1$.

Therefore, f_χ remains bounded when $s \to 1$.

☐

Define

$$g_a(s) = \sum_{p \in P_a} \frac{1}{p^s}.$$

We see that the analytic density of P_a is

$$D(P_a) = \lim_{s \to 1} \frac{g_a(s)}{\log \frac{1}{s-1}}. \tag{4.13}$$

Let us examine the behavior of g_a when $s \to 1$.

Lemma 4.14. *We have*

$$g_a(s) = \frac{1}{\phi(m)} \sum_\chi \overline{\chi(a)} f_\chi(s).$$

Proof. The proof is similar to that of Lemma 3.5.

If we substitute f_χ in the given formula, we get

$$\sum_{\chi} \overline{\chi(a)} f_\chi(s) = \sum_{p \nmid m} \Big(\sum_{\chi} \chi(pa^{-1}) \Big) / p^s.$$

From Corollary 1.8, we see that

$$\sum_{\chi} \chi(pa^{-1}) = \begin{cases} \phi(m), & \text{if } pa^{-1} \equiv 1 \pmod{m} \\ 0, & \text{if not} \end{cases}$$

$$pa^{-1} \equiv 1 \pmod{m} \Leftrightarrow p \equiv a \pmod{m}.$$

Therefore,

$$\sum_{\chi} \overline{\chi(a)} f_\chi(s) = \phi(m) g_a(s).$$

□

Now we are ready to prove that the analytic density of $P_a = \{p \in P | \quad p \equiv a \pmod{m}\}$ is $\frac{1}{\phi(m)}$.

Theorem 4.15.

$$D(P_a) = \frac{1}{\phi(m)}$$

Proof. From lemma 4.11,

$$f_{\mathbb{1}} \sim \log \frac{1}{s-1} \text{ when } s \to 1$$

Lemma 4.12 tells that all other $f_{\chi \neq \mathbb{1}}$ are bounded.

Hence, we conclude that

$$g_a(s) \sim \frac{1}{\phi(m)} \log \frac{1}{s-1}$$

from lemma 4.14.

From the equation (4.13), we see that

$$D(P_a) = \frac{1}{\phi(m)}.$$

□

Note that we cannot directly tell that the natural density of P_a is $\frac{1}{\phi(m)}$ from this proof. The proof only tells that $D(P_a) = \frac{1}{\phi(m)} > 0$.

5. The Proof of the Wiener-Ikehara Theorem

This section is entirely devoted to prove the Wiener-Ikehara theorem (theorem 3.13) we used in section 3. We will follow [4] with some additional explanation and clarification if applicable. Let us first recall the Wiener-Ikehara Theorem.

Theorem 5.1. *(Wiener-Ikehara) Let $A(x)$ be a non-negative, non-decreasing function in an interval $[0, \infty)$. Assume that for $\sigma > 1$, the integral*

$$\int_0^\infty A(x)e^{-xs}dx,\ s = \sigma + it$$

converges to the function F(s), where F is holomorphic for $\sigma \geq 1$ except for a simple pole at $s = 1$ with residue γ. Then,

$$\lim_{x\to\infty} e^{-x}A(x) = \gamma \tag{5.2}$$

Example 5.3. An obvious (but uninteresting) example would be $A(x) = e^x$. It is non-negative, non-decreasing function of $x \in [0, \infty)$. Also, we see that

$$\int_0^\infty A(x)e^{-xs}dx = \frac{1}{s-1}.$$

and $\frac{1}{s-1}$ is holomorphic for $\sigma \geq 1$ except for $s = 1$ where it has a simple pole with a residue 1. Evidently,

$$\lim_{x\to\infty} e^{-x}e^x = 1 = res(\frac{1}{s-1}, 1).$$

Let us normalize $A(x)$ to make $\gamma = 1$. (Replace $A(x)$ by $\frac{A(x)}{\gamma}$ if necessary). Set $B(x) = e^{-x}A(x)$.

We will first show that for all $\lambda > 0$,

$$\lim_{y\to\infty}\int_{-\infty}^{\lambda y} B(y - \frac{v}{\lambda})\frac{\sin^2 v}{v^2}dv = \pi,$$

and then deduce $\lim_{x\to\infty} B(x) = 1$ by showing that

$$\limsup_{x\to\infty} B(x) \leq 1 \leq \liminf_{x\to\infty} B(x).$$

Lemma 5.4. *For all $\lambda > 0$,*

$$\lim_{y\to\infty}\int_{-\infty}^{\lambda y} B(y - \frac{v}{\lambda})\frac{\sin^2 v}{v^2}dv = \pi.$$

Proof. For $\sigma > 1$, we have

$$f(s) = \int_0^\infty A(x)e^{-xs}dx, \text{ and } \frac{1}{s-1} = \int_0^\infty e^{-(s-1)s}dx$$

Hence,

$$f(s) - \frac{1}{s-1} = \int_0^\infty (B(x) - 1)e^{-(s-1)x}dx,\ \sigma > 1.$$

Set

$$g(s) = f(s) - \frac{1}{s-1}, \text{ and } g_\epsilon(t) = g(1 + \epsilon + it) \text{ for } \epsilon > 0.$$

Then $g(s)$ is analytic for $Re(s) = \sigma \geq 1$ as f is a meromorphic function with a unique simple pole at $s = 1$ with residue 1.

For $\lambda > 0$, we have

$$\frac{1}{2}\int_{-2\lambda}^{2\lambda} g_\epsilon(t)\Big(1-\frac{|t|}{2\lambda}\Big)e^{iyt}dt$$

$$= \frac{1}{2}\int_{-2\lambda}^{2\lambda}\Big(1-\frac{|t|}{2\lambda}\Big)e^{iyt}\Big(\int_0^\infty (B(x)-1)e^{-(\epsilon+it)x}dx\Big)dt. \tag{5.5}$$

The order of integration in the above equation can be interchanged by Fubini's theorem.

Since $A(x)$ is nonnegative and nondecreasing, if $s\in\mathbb{R}$ and $x>0$,

$$f(s) = \int_0^\infty A(x)e^{-xs}dx \geq A(x)\int_x^\infty e^{-us}du = \frac{A(x)e^{-xs}}{s}.$$

Hence $A(x)\leq sf(s)e^{xs}$. As $f(s)$ is holomorphic for $\sigma>1$, we see that $sf(s)$ is a constant number for $\sigma>1$. Consequently $A(x)=O(e^{xs})$ for any $s>1$, and $\frac{A(x)}{e^{xs}}\to 0$ as $x\to\infty$, or equivalently $A(x)=o(e^{xs})$ (if not, we will have that $f(s)=\int_0^\infty A(x)e^{-xs}dx$ diverges).

Therefore, for any $\delta>0$, we have $B(x)e^{-\delta x}=A(x)e^{-(1+\delta)x}=o(1)$.

As a result, the integral

$$\int_0^\infty (B(x)-1)e^{-(\epsilon+it)x}dx$$

converges uniformly in $t\in[-2\lambda,2\lambda]$. Fubini's theorem tells that the order of integration in (5.5) is interchangeable.

We now have

(5.6)

$$\begin{aligned}\frac{1}{2}\int_{-2\lambda}^{2\lambda} g_\epsilon(t)\Big(1-\frac{|t|}{2\lambda}\Big)e^{iyt}dt &= \int_0^\infty (B(x)-1)e^{-\epsilon x}\bigg(\int_{-2\lambda}^{2\lambda}\frac{1}{2}\Big(1-\frac{|t|}{2\lambda}\Big)e^{i(y-x)t}dt\bigg)dx\\ &= \int_0^\infty (B(x)-1)e^{-\epsilon x}\frac{\sin^2\lambda(y-x)}{\lambda(y-x)^2}dx.\end{aligned}$$

Because g is analytic if $\sigma\geq 1$, and $g_\epsilon(t)\to g(1+it)$ uniformly in an interval $t\in[-2\lambda,2\lambda]$ when $\epsilon\to 0$. Hence,

$$\lim_{\epsilon\to 0}\int_0^\infty e^{-\epsilon x}\frac{\sin^2\lambda(y-x)}{\lambda(y-x)^2}dx = \int_0^\infty \frac{\sin^2\lambda(y-x)}{\lambda(y-x)^2}dx.$$

Therefore, the limit

$$\lim_{\epsilon\to 0}\int_0^\infty B(x)e^{-\epsilon x}\frac{\sin^2\lambda(y-x)}{\lambda(y-x)^2}dx$$

exists.

As the integrand is nonnegative and monotonically increasing as $\epsilon\to 0$, we can apply the monotone convergence theorem and get

$$\lim_{\epsilon\to 0}\int_0^\infty B(x)e^{-\epsilon x}\frac{\sin^2\lambda(y-x)}{\lambda(y-x)^2}dx = \int_0^\infty B(x)\frac{\sin^2\lambda(y-x)}{\lambda(y-x)^2}dx.$$

This implies that

$$\int_{-2\lambda}^{2\lambda} g(t)\Big(1-\frac{|t|}{2\lambda}\Big)e^{iyt}dt = \int_0^\infty B(x)\frac{\sin^2\lambda(y-x)}{\lambda(y-x)^2}dx - \int_0^\infty \frac{\sin^2\lambda(y-x)}{\lambda(y-x)^2}dx$$

by (5.6).

If we let $y \to \infty$, then (LHS) $\to 0$ due to the Riemann-Lesbegue lemma.

On (RHS), second term gives

$$\lim_{y\to\infty}\int_0^\infty \frac{\sin^2\lambda(y-x)}{\lambda(y-x)^2}dx = \lim_{y\to\infty}\int_{-\infty}^{\lambda y}\frac{\sin^2 v}{v^2}dv = \pi.$$

Hence,

$$\lim_{y\to\infty}\int_{-\infty}^{\lambda y} B(y-\frac{v}{\lambda})\frac{\sin^2 v}{v^2}dv = \pi \tag{5.7}$$

and we finish the proof of the lemma.

□

Now, let us show that $\lim_{x\to\infty} B(x) = 1$ using Lemma 5.4.

Lemma 5.8. $\lim_{x\to\infty} B(x) = 1$

Note that this is the Wiener-Ikehara theorem.

Proof. We will first show that

$$\limsup_{x\to\infty} B(x) \le 1 \tag{5.9}$$

and then show

$$1 \le \liminf_{x\to\infty} B(x). \tag{5.10}$$

For (5.9), choose a and $\lambda \in \mathbb{R}^+$. Let $y > \frac{a}{\lambda}$. Then, from (5.7), we have

$$\limsup_{y\to\infty}\int_{-a}^{a} B(y-\frac{v}{\lambda})\frac{\sin^2 v}{v^2}dv \le \pi$$

since the intergrand is nonnegative.

Because $A(u) = B(u)e^u$ is nondecreasing, we have

$$e^{y-a/\lambda}B\Big(y-\frac{a}{\lambda}\Big) \le e^{y-v/\lambda}B\Big(y-\frac{v}{\lambda}\Big) \text{ for } v \in [-a,a].$$

Then,

$$B\Big(y-\frac{v}{\lambda}\Big) \ge B\Big(y-\frac{a}{\lambda}\Big)e^{(v-a)/\lambda} \ge B\Big(y-\frac{a}{\lambda}\Big)e^{-2a/\lambda}.$$

Hence,

$$\limsup_{y\to\infty}\int_{-a}^{a} B(y-\frac{a}{\lambda})e^{-2a/\lambda}\frac{\sin^2 v}{v^2}dv \le \pi$$

i.e.,

$$\limsup_{y\to\infty} B(y-\frac{a}{\lambda})e^{-2a/\lambda}\int_{-a}^{a}\frac{\sin^2 v}{v^2}dv \le \pi.$$

As we fixed a and λ, we have $\limsup_{y\to\infty} B(y - a/\lambda) = \limsup_{y\to\infty} B(y)$. We then conclude

$$e^{-2a/\lambda} \limsup_{y\to\infty} B(y) \int_{-a}^{a} \frac{\sin^2 v}{v^2} dv \le \pi$$

for all $a > 0$ and $\lambda > 0$. Let $a \to \infty$ and $\lambda \to \infty$ while $a/\lambda \to 0$. Then, as the above inequality holds for all a, $\lambda > 0$

$$\limsup_{y\to\infty} B(y) \int_{-\infty}^{\infty} \frac{\sin^2 v}{v^2} dv \le \pi$$

That is,

$$\pi \limsup_{y\to\infty} B(y) \le \pi \Leftrightarrow \limsup_{y\to\infty} B(y) \le 1,$$

so the inequality (5.9) holds.

We will now show that (5.10) also holds, which completes the proof.

Inequality (5.9) implies that $|B(s)| \le c$, for suitably large c. Let's fix $a, \lambda > 0$ as before. If y is large enough, we have

$$\begin{aligned} \int_{-\infty}^{\lambda y} B(y - \frac{v}{\lambda}) \frac{\sin^2 v}{v^2} dv \ \le & c\Big(\int_{-\infty}^{-a} \frac{\sin^2 v}{v^2} dv + \int_{a}^{\infty} \frac{\sin^2 v}{v^2} dv \Big) \\ & + \int_{-a}^{a} B\Big(y - \frac{v}{\lambda}\Big) \frac{\sin^2 v}{v^2} dv. \end{aligned} \tag{5.11}$$

As before, if $v \in [-a, a]$ we have

$$B\big(y - \frac{v}{\lambda}\big) \le B\big(y + \frac{a}{\lambda}\big) e^{2a/\lambda},$$

which implies that

$$\int_{-a}^{a} B(y - \frac{v}{\lambda}) \frac{\sin^2 v}{v^2} dv \le B\big(y + \frac{a}{\lambda}\big) e^{2a/\lambda} \int_{-a}^{a} \frac{\sin^2 v}{v^2} dv. \tag{5.12}$$

From (5.7), (5.11), and (5.12) we conclude

$$\pi \le c\Big(\int_{-\infty}^{-a} \frac{\sin^2 v}{v^2} dv + \int_{a}^{\infty} \frac{\sin^2 v}{v^2} dv \Big) + \liminf_{y\to\infty} B\big(y + \frac{a}{\lambda}\big) e^{2a/\lambda} \int_{-a}^{a} \frac{\sin^2 v}{v^2} dv.$$

i.e,

$$\pi \le c\Big(\int_{-\infty}^{-a} \frac{\sin^2 v}{v^2} dv + \int_{a}^{\infty} \frac{\sin^2 v}{v^2} dv \Big) + \liminf_{y\to\infty} B(y) e^{2a/\lambda} \int_{-a}^{a} \frac{\sin^2 v}{v^2} dv.$$

Again, send $a, \lambda \to \infty$ while $\frac{a}{\lambda} \to 0$. Then,

$$\pi \le \pi \liminf_{y\to\infty} B(y). \Leftrightarrow 1 \le \liminf_{y\to\infty} B(y).$$

Hence, (5.10) is verified. We have proved $\lim_{y\to\infty} B(y) = 1$, which is the Wiener-Ikehara theorem.

□

Acknowledgments. It is my pleasure to thank my mentor, Tianqi Fan for her help and helpful advice throughout this project. I also thank for Professor Emerton and Professor Narasimhan for their helpful comments for the project. Without their help, the project must have gone astray. I also thank Professor May for organizing REU this year.

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