THE FREUDENTHAL-HOPF THEOREM

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Abstract. In this paper, we will examine a geometric property of groups: the number of ends of a group. Intuitively, the number of ends of a group is a measure of the number of components at the boundary of the graph of a group. It turns out that any finitely generated group has a determined number of ends. The Freudenthal-Hopf Theorem shows that finitely generated groups can have either zero, one, two or infinitely many ends. This paper is directed at readers with elementary knowledge of group theory, but not necessarily of graph theory.

Contents

1. Introduction

The number of ends of groups is a geometric property of groups when represented as graphs. Naively, the ends of a group can be seen as the structure of the graph of a group at its boundary. Suppose there is a group whose graph looks like a tree, then the ends of this graph are the outer most branches of the tree; and the number of ends would be the number of the outer most branches of the tree. Of course there is a more precise definition of ends, which can be seen in Section 4.

What is very interesting about the number of ends of finitely generated groups is that it cannot be any number. The Freudenthal-Hopf Theorem shows that any finitely generated group can only have zero, one, two, or infinitely many ends. The purpose of this paper is to provide a proof of this theorem, along with defining all the prerequisite concepts and proving the lemmas needed to construct the proof of the Freudenthal-Hopf Theorem.

Date: AUGUST 30, 2013.

Since the number of ends is a geometrical property, we will first need to understand certain properties of graphs. I will dedicate the second section to defining graphs and related concepts, such as vertices, local finiteness and connectedness.

Then I will establish the relation between these two types of mathematical objects: groups and graphs. We need to show that groups can be represented by graphs in a meaningful manner, which is done by Cayley's Theorem, which states that finitely generated groups can be represented as locally finite, connected graphs, which are called Cayley graphs. And in particular, there exists a surjective map from the group to its graph. I will prove Cayley's Theorem in Section 3. I will provide a few examples of Cayley graphs of groups.

The fourth section introduces the concept of *ends* of graphs. The main objective of this section is to show that the number of ends of any group is unique. To be more precise, the number of ends of a group is independent of the generators used to construct the Cayley graph. The proof of this result requires the construction of maps among the graphs of groups with respect to different generators.

Then I will provide a proof of the Freudenthal-Hopf Theorem. The Freudenthal-Hopf Theorem shows that any finitely generated group has either zero, one, two or infinitely many ends.

I will also include a short discussion about the classification of groups with different numbers of ends. Due to the restriction on length, I will not prove those theorems, but I will provide references for further reading.

The paper is based on Chapters 1 through 3 and 11 from the published book *Groups, Graphs, and Trees* by John Meier. All definitions and statements of theorems, lemmas and corollaries are taken verbatim from the book, unless more explanation is needed. Most of the proofs I give in this paper are more detailed than the original proofs in the book. Given that I have encountered a few difficulties understanding the proofs from the book, I have added more explanation on certain statements within the proof that the author found straightforward, but could be confusing for readers who first encounter this topic. Some of the examples given in this paper are taken from the book, since I intend to provide the most representative examples for groups with different numbers of ends. I have also included some original examples that I considered helpful for understanding the theorems in a more concrete manner.

2. Graphs

This paper focuses on the geometric nature of groups in tems of graphs. Ths it is necessary first to show that groups can be mapped to graphs. In this section, we define the concepts relevant to graph, and identify certain types of graphs that are required to prove the Freudenthal-Hopf Theorem.

Definition 2.1. A *graph* Γ consists of a set $V(\Gamma)$ of *vertices* and a set $E(\Gamma)$ of *edges*, each edge being associated to an unordered pair of vertices by a function "Ends": $\text{Ends}(e) = \{v, w\}$ where $v, w \in V$. In this case we call v and w the *ends* of the edge e and we also say v and w are *adjacent*.

Graphs can be easily visualized. There are two examples of graphs on the next page.

Definition 2.2. A graph is *locally finite* if finitely many edges come out of each vertex. Or, in other words, each vertex has finite valence.

FIGURE 1. Example of two graphs

Definition 2.3. A *directed* graph consists of a vertex set V and an edge set E of ordered pairs of vertices. Thus each edge has an *initial* vertex and a *terminal* vertex. Graphically this direction is often indicated via an arrow on the edge.

Definition 2.4. An *edgepath*, or simply a *path*, in a graph consists of an alternating sequence of vertices and edges, $v_0, e_1, v_1, ..., v_{n-1}, e_n, v_n$ where $\mathrm{Ends}(e_i) = \{v_{i-1}, v_i\}$ for each i. A graph is *connected* if any two vertices can be joined by an edge path.

Example 2.5. The graphs in Figure 1 are two examples of graphs. Fig. 2.1 is connected, locally finite and undirected. Fig. 2.2 is disconnected, locally finite and directed. The union of the two graphs is, of course, disconnected.

Locally infinite graphs cannot be drawn. In such graphs, at least one vertex is connected to infinitely many edges.

3. Cayley's Theorem

We want to study the properties of groups by looking at them geometrically. This is possible when we can represent a group in terms of a graph. Cayley's Theorem shows that there is indeed a geometric way to represent finitely generated groups.

Before proving the theorem, let us first remind ourselves of some examples and properties of groups.

Definition 3.1. If X is a mathematical object such as a group, a shape, or a sequence, let $\mathrm{SYM}(X)$ denote all bijections from X to X that preserve the structure of the indicated mathematical object. SYM(X) is usually written Aut(X) and called the automorphism group of X. If the mathematical object is directed, let $\mathrm{SYM}^+(X)$ denote the group of bijections that preserves the direction.

Example 3.2. All finite dihedral groups are symmetry groups of regular polygons.

Definition 3.3. An *action* of a group G on a mathematical object X is a group homomorphism from g to $\mathrm{SYM}(X)$ (the symmetry group of X). Equivalently, it is a map $G \times X \to X$ such that

1. $e \cdot x = x$, for all $x \in X$; and
2. $(gh) \cdot x = g \cdot (h \cdot x)$, for all $g, h \in G$ and $x \in X$.

Since not all groups are abelian, we need to distinguish between left and right actions. For a group G and a mathematical object X, a left action of $g \in G$ on X can be written as $g \cdot X$, and a right action of g on X can be written as $X \cdot g$.

Definition 3.4. Given a group action on a mathematical object, the associated homomorphism as described above is a *representation* of G. The representation is *faithful* if the map is injective.

Definition 3.5. If G is a group and S is a subset of G, then S generates G if every element of G can be expressed as a product of elements from S and inverses of elements of S. A group G is *finitely generated* if it has a finite generating set.

Remark 3.6. Every finite group is finitely generated. The largest generating set in a finite group is the group itself.

Example 3.7. Groups with more than one generating sets:

(1). The generators of the group D_3 include $\{r, s\}$, that is, a rotation of $\frac{\pi}{3}$ degree and a reflection through one of the heights of the equilateral triangle, and $\{s, s*\}$, two reflections about two different heights of the triangle. Imagine D_3 to be the symmetry group of a triangle, then $s*$ stands for reflection through another axis connecting the midpoint of an edge of the triange and its opposite vertex.

(2). The generating sets for $\mathbb{Z}$ include $\{1\}$ and $\{2, 3\}$.

Theorem 3.8. *(Cayley's Theorem). Every finitely generated group can be faithfully represented as a symmetry group of a connected, directed, locally finite graph, which is called the Cayley graph of the group.*

Proof. We prove the theorem by constructing the graph of a finitely generated group.

Let G be a group and let S, a finite subset of G, be a generating set of G. Let Γ be the graph of G such that each vertex of Γ represents an element of G. And let each edge be labeled with an element of the generating set $S \cup S^{-1}$. Thus, every $s \in S$ form a directed edge with initial vertex g, and terminal vertex $g \cdot s$. And every vertex in Γ is connected with at most $2|S|$ edges.

Now we want to show that the graph is connected. This means that there exists an edge path from any $g \in G$ to any $h \in G$. To prove this, it is sufficient to prove that every $g \in G$ is connected to the identity element e, because then any two vertices can be connected by a path through e.

Let g be any element in G. Since S is a generating set, g can be written as the product of the elements in the generating set. That is, $g = s_1 s_2 s_3 ... s_n$. Since every vertex in Γ is connected with edges corresponding to every element of S, one can arrive at the element g starting from e, by first taking the left action of e act on s_1, to arrive at the vertex corresponding to s_1, that is, v_{s_1}. Then let s_2 act on $v_{s_1}$from the right, to get $v_{s_1 s_2}$... until we arrive at g. This show that the graph Γ is connected.

The graph Γ is locally finite because the generating set is finite. Each vertex in Γ is connected to edges corresponding bijectively to every element in $S \cup S^{-1}$.

The graph is directed. Let all edges associated with elements in S be in one direction. Then the edges associated with elements in S^{-1} point in the negative direction. In the case that $s = s^{-1}$ for some element $s \in S$, the two edges corresponding to s and s^{-1} coincide. The two edges form a loop that contains the vertices corresponding to v and $v \cdot s$.

FIGURE 2. Cayley Graph of D_3 with respect to two different generating sets. Any vertex can be chosen to be the identity vertex. The one on the left is the graph with respect to the set with one rotation and one reflection. The generating set of the graph on the right consists of two rotation regarding two different axes. Notice that in both graphs there are edges with two arrows. These are in fact a simplified way to representing two coincident edges pointing at opposite directions. These edges are associated with s and s^{-1}.

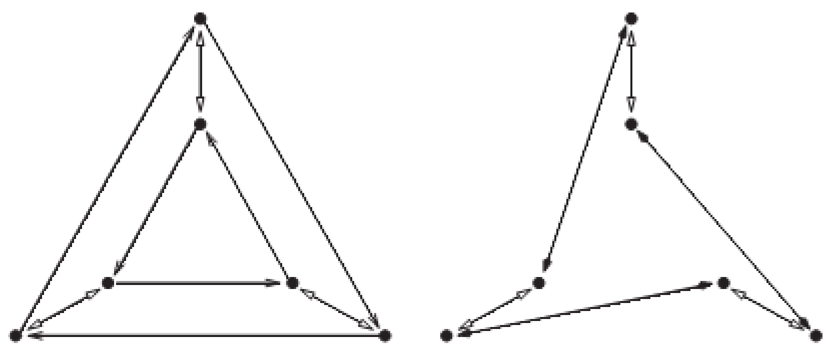
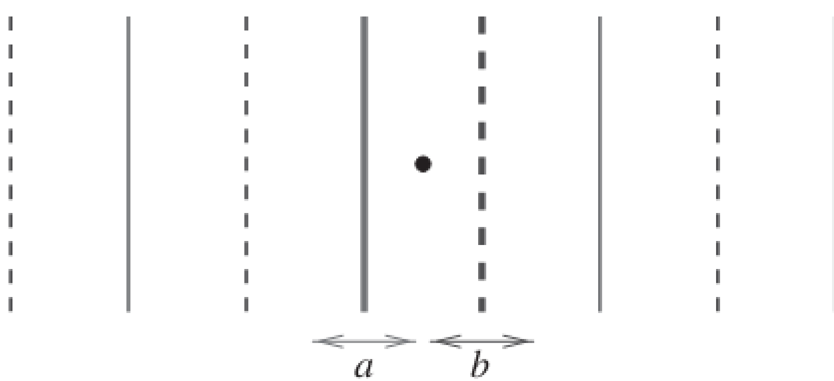


FIGURE 3. Cayley Graph of D_∞ with respect to the generator $\{a, b\}$. For those who are not familiar with this group, D_∞ can be thought of as reflections of a point with regard to two parallel lines a and b.

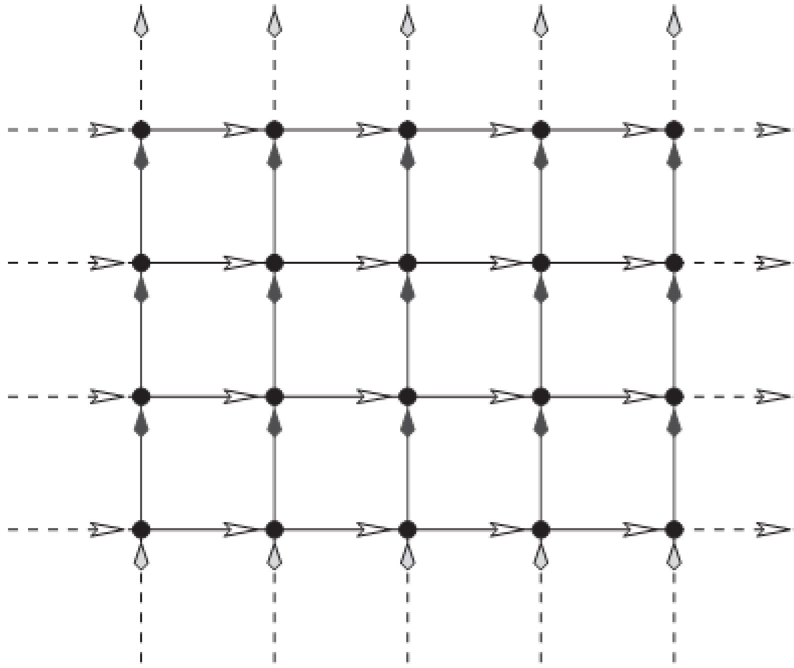


The presentation is faithful because every element in a group has a corresponding vertex.

□

Example 3.9. Figure 2, Figure 3, Figure 4, and Figure 5 are examples of Cayley Graphs for groups D_3, D_∞, $\mathbb{Z}^2$, and F_2 respectively. The explanations are within the captions of the figures.

FIGURE 4. Cayley Graph of $\mathbb{Z}^2$ with respect to the generating set $S = \{(1,0),(0,1)\}$. Any point in the graph can be considered as the identity vertex. Edges with white arrow are associated with $(0,1)$, and edges with black arrows are associated with $(1,0)$. The edges associated with S^{-1} are not labeled. But it is not hard to see that the white arrowed edge can be seen as $(-1,0)$, by letting the arrow point in the negative direction.



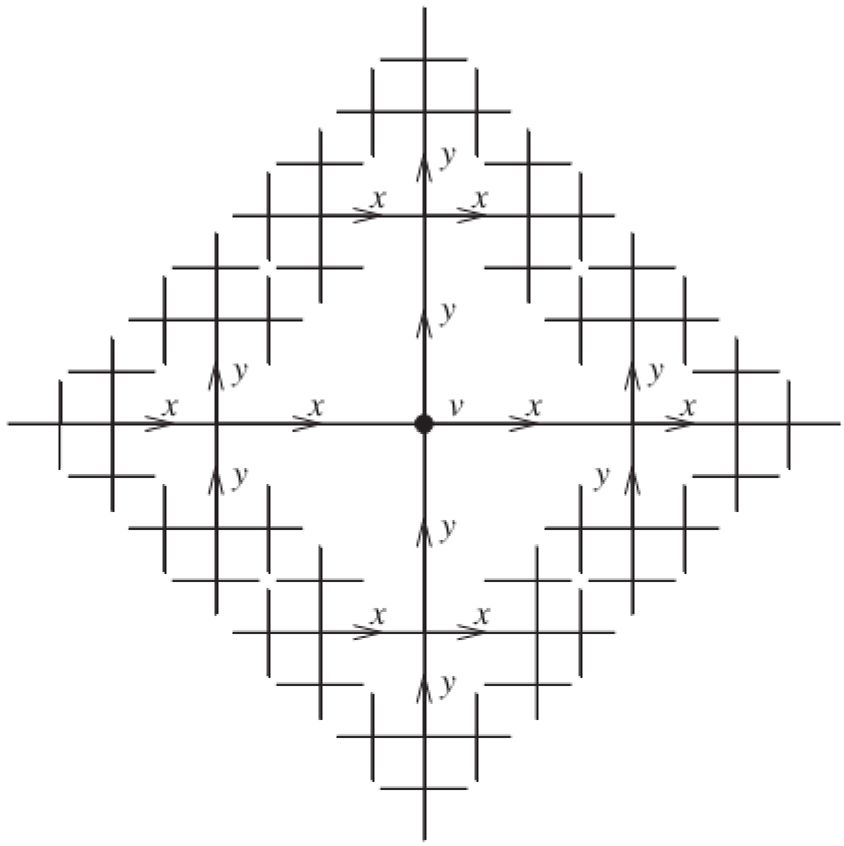
3.1. **Frucht's Theorem.** One important corollary of Cayley's Theorem is that any finitely generated group is isomorphic to the symmetry group of a labeled and directed, locally finite graph, namely, its Cayley graph.

In the examples above, it is observed that in most cases, any vertex can be associated with the identity, and the Cayley graph is isomorphic to the original graph. Let v_g be a vertex of the Cayley Graph. Say that we decide to treat this vertex as the vertex associated with the identity element of G. We know that any element $h \in G$ can be written as $e \cdot h$, where e is the identity element of G. Thus the Cayley Graph of G is isomorphic to the image of the right action of e on Γ. Now that g stands for the identity, we can obtain the new graph by letting g act on the Cayley Graph of G. But since every finitely generated group is isomorphic to the symmetry group of its Cayley Graph, each element of G corresponds to an element of $\mathrm{SYM}^+(\Gamma)$. Thus $e \cdot \Gamma$ is isomorphic to $g \cdot \Gamma$. Hence for a Cayley Graph of a finitely generated group, we can choose any vertex as the identity vertex.

Now we will state and prove the theorem mentioned above.

Theorem 3.10. *(Frucht's Theorem) Let Γ be the Cayley graph of a group G with respect to a finite generating set S. Consider Γ to be decorated with directions*

FIGURE 5. Cayley Graph of F_2 with respect to the generating set $\{x, y\}$. In this graph, the vertex v denoted the identity vertex. The edges associated with x and y are clearly labeled, and those associated with S^{-1} are easy to see. Note that in fact any vertex can be considered the identity vertex, because this infinite graph is the same no matter which vertex is placed at the center.



on its edges and labels of its edges, corresponding to its generating set S. Then $SYM^+(\Gamma) \cong G$.

Proof. In the proof of Cayley's Theorem, we constructed a left action to show that every vertex in Γ is connected to the vertex. Since it is a left action, it does not change the directions and labels on the edges of Γ (the proof of this is simple, therefore we will let the readers verify this for themselves).

To show that this homomorphism $\psi : \Gamma \to \mathrm{SYM}(\Gamma)$ is surjective, we pick an arbitrary element $\gamma \in \mathrm{SYM}^+(\Gamma)$ and show that there is a $g \in G$ such that $g = \psi(\gamma)$. Let γ be an arbitrary element of $\mathrm{SYM}^+(\Gamma)$, and let v_g be the vertex in Γ corresponding to the element $g \in G$. Then there is $g \in G$ such that $\gamma(v_e) = v_g$. Since $g \circ \Gamma$ is isomorphic to Γ, $\gamma \cdot g^{-1}$ also sends Γ to a graph isomorphic on it. Therefore, $\gamma \circ g^{-1} \in \mathrm{SYM}^+(\Gamma)$. In particular, this homomorphism maps v_e to v_e. Since v_e is fixed under this action, all edges connected to v_e are merely permuted.

And since an element of $\mathrm{SYM}^+(\Gamma)$ also fixes the direction and labeling of the edges, the vertices adjacent to v_e are also unchanged. By the Principle of Mathematical Induction, all the vertices are fixed. So $\gamma \circ g^{-1}$ is identity element in $\mathrm{SYM}^+(\Gamma)$. So it follows that γ corresponds to g. Thus the map is surjective.

Since ψ is an injective homomorphism, and we have now proven that the map is surjective, we have a bijection. Therefore $SYM^+(\Gamma) \cong G$. $\square$

4. Ends of Graphs

In this section we will introduce the ends of graphs. Namely, those of Cayley graphs. The most important result we want to achieve is to show that the number of ends of a Cayley graph of a group is unique. To do so, we will need to introduce maps between graphs, and their properties. Again, we will start with preliminary definitions.

Definition 4.1. Given a set S, a finite sequence of elements from S and S^{-1}, possibly with repetition, is called a *word* in S. Let $\{S \cup S^{-1}\}*$ be the set of words in S.

It is not hard to see that, given a finitely generated group G, there is a one-to-one correspondence between words in $\{S \cup S^{-1}\}*$ and the finite edgepaths starting from the identity vertex in the Cayley Graph of G with respect to the generating set S.

Definition 4.2. Let $g \in G$, then the *word length* of g with respect to a generating set S is the shortest length of a w in $\{S \cup S^{-1}\}*$ representing G.

4.1. Maps between Graphs.

Definition 4.3. Let Γ and Λ be two graphs. A *map* from Γ to Λ is a function ϕ taking vertices of Γ to vertices of Λ , and edges of Γ to edges of Λ, such that if v and w are vertices attached to an edge $e \in G$, then $\phi(e)$ joins $\phi(v)$ to $\phi(w)$.

Proposition 4.4. *Let S and T be two finite generating sets for a group G and let Γ_S and Γ_T be the corresponding Cayley graphs. Then there are maps $\phi_{T\leftarrow S} : \Gamma_S \to \Gamma_T$ and $\phi_{S\leftarrow T} : \Gamma_T \to \Gamma_S$ such that*

1. The compositions $\phi_{T\leftarrow S} \circ \phi_{S\leftarrow T}$ and $\phi_{S\leftarrow T} \circ \phi_{T\leftarrow S}$ induce the identity on $V(\Gamma_S)$ and $V(\Gamma_T)$ respectively.

2. There is a constant $K > 0$ such that the image of any edge $e \in \Gamma_S$ under $\phi_{S\leftarrow T} \circ \phi_{T\leftarrow S}$ is contained in the ball $B(v, K) \subset \Gamma_S$, where $v \in ENDS(e)$. This similarly holds for edges of Γ_T.

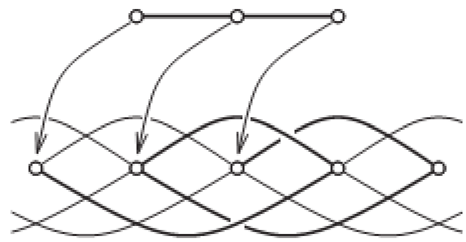
Proof. Let v_g be a vertex in Γ_S, and v'_g a vertex in Γ_T. Define $\phi_{T\leftarrow S}(v_g) = v'_g$, and $\phi_{S\leftarrow T}(v'_g) = v_g$. This proves the first claim.

Cayley's theorem shows that an edge e of a graph Γ_S is associated with a generator $s \in S$, and joins g and $g \cdot s$. For each generator $s \in S$, choose a word $w_s = t_1 t_2 ... t_k \in \{T \cup T^{-1}\}*$, and define a map $\pi : w_s \mapsto s$. This map is surjective because T is the generating set of G. Let the map $\phi_{T\leftarrow S}$ send every edge to the edgepath

$$g \to g \cdot t_1 \to gt_1 \cdot t_2 \to gt_1 t_2 ... t_k$$

in Γ_T.

FIGURE 6. Map between two Cayley graph of $\mathbb{Z}$ with respect to two different sets of generators.



Define the map $\phi_{S \leftarrow T}$ similarly, by switching the position of S and T in the maps defined above. Let k be the maximal length of the words w_t and w_s. Then $\phi_{T \leftarrow S}$ is an edge path of length $\leq k$. The image $\phi_{S \leftarrow T}$ of this path s then an edge path of length $\leq k^2$. Let the constant K in the claim of the proposition be k^2. This completes the proof of the second claim.

□

Corollary 4.5. *Let G, S and T be as above. Then there is a constant $\lambda > 1$ such that for any g and h in G,*

$$\frac{1}{\lambda} d_S(g,h) \leq d_T(g,h) \leq \lambda d_S(g,h)$$

where $d_s(g,h)$ and $d_T(g,h)$ are the distances between g and h in Γ_S and Γ_T, respectively.

Proof. Recall that w_T is a word with respect to the generating set T. Let Λ_1 be the maximum word length of $w_T \in \{S \cup S^{-1}\}*$ as defined in the proposition above. Suppose $d_T(g,h) = n$, then there is an edge path between v_g and v_h of the length n. Apply the map $\phi_{S \leftarrow T}$ on this path, then

$$d_S(g,h) \leq \Lambda_1 \cdot n = \Lambda_1 \cdot d_T(g,h)$$

The same argument works for the map $\phi_{T \leftarrow S}$. Let the maximum word length of $w_S \in \{T \cup T^{-1}\}*$ be Λ_2, we eventually get that $d_T(g,h) = \Lambda_2 \cdot d_S(g,h)$. Let $\lambda = \text{Max } (\Lambda_1, \Lambda 2)$, then we can write our result as that in the statement of the corollary. □

Example 4.6. Figure 6. shows the graph of $\mathbb{Z}$ with respect to the generating set $\{1\}$ (above) and $\{2,3\}$ (below). The curves with arrow outlines the map from one graph to another.

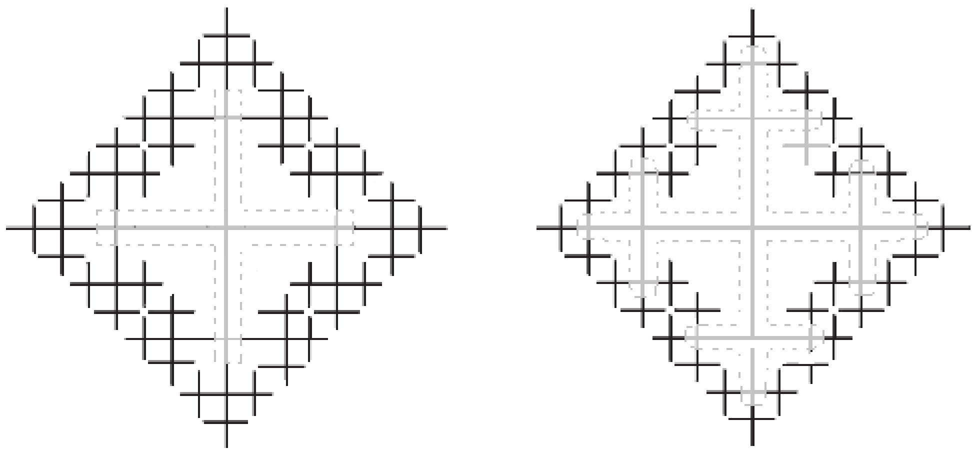
4.2. **Uniqueness of Number of Ends.**

Definition 4.7. Let $\|\Gamma \backslash B(n)\|$ be the number of connected unbounded components in the complement of $B(n)$, a ball of radius n is the number of vertices from the center to the boundary of the ball, centered around some vertex of Γ.

Lemma 4.8. *Let Γ be a locally finite graph and let $m < n$ be two positive integers. Then*

$$\|\Gamma \setminus B(m)\| \leq \|\Gamma \setminus B(n)\|$$

FIGURE 7. $\|\Gamma \setminus B(n)\|$ of F_2 with different values of n.



Proof. Let Γ' be an unbounded, connected component of $\|\Gamma \setminus B(m)\|$. By extending $B(m)$ to $B(n)$, Γ' either remains connected or it does not. Therefore $\|\Gamma \setminus B(m)\|$ has either equal of more connected unbounded components, comparing to $\|\Gamma \setminus B(n)\|$. □

Example 4.9. Figure 7 visualizes an example of the fact that, by increasing n, $\|\Gamma \setminus B(m)\|$ either remains the same or increases. The ball on the left graph has $n = 1$, which leaves the complement set of the ball with 12 unbounded, connected components. The ball in the has $n = 2$, which leaves the complement with 36 unbounded, connected components.

The Cayley graph of $\mathbb{Z}^2$ has the same number of unbounded, connected components, and this number is independent of the size of the ball taken in the graph. Now we see that, while the number of connected, unbounded components increases with every increment of the radius in certain graphs, it never decreases.

Definition 4.10. (Ends of a graph) Let Γ be a connected, locally finite graph, and let $B(n)$ be the ball of radius n about a fixed vertex $v \in V(\Gamma)$. Then the *number of ends* of Γ is

$$e(\Gamma) = \lim_{n\to\infty} \|\Gamma \setminus B(n)\| \tag{4.11}$$

This limit exists because graphs are static objects, and will have a definitive number of components outside of any subset of it. Therefore the limit is defined, though examples later shows that the limit can be infinity.

Remark 4.12. From the definition, we can immediately infer that a group has zero ends if and only if the group is finite.

Note that the number of ends of a graph is independent of the choice of the vertex as the center of $B(n)$. Intuitively, since a group acts on itself, we can take a left action on both the graphs of $B(n)$. The image of an action of an element of G on G is isomorphic to G, therefore they will have the same number of ends.

The following lemma gives a more rigorous proof for an extension for this result. Namely, we will achieve the same result by replacing $B(n)$ in Equation (4.11) by any finite subgraph of Γ.

Lemma 4.13. *Let Γ be a connected, locally finite graph and let $e(\Gamma)$ be defined as above. Let C be any finite subgraph of Γ and let $\|\Gamma \setminus C\|$ denote the number of unbounded, connected components of the complement of C. Define $e_C(\Gamma)$ to be the supremum of $\|\Gamma \setminus C\|$. Then*

$$e(\Gamma) = e_C(\Gamma)$$

Proof. For any locally finite graph Γ, the ball $B(n)$ is a finite subgraph. Since $e_C(\Gamma)$ is the supremum of $\|\Gamma \setminus C\|$, for C any finite subgraph of Γ, $e(\Gamma) \leq e_C(\Gamma)$.

Conversely, since C is a finite subset of Γ, there exists $n \in \mathbb{N}$ such that $C \subset B(n)$. Using the same argument from Lemma 4.8, $\|\Gamma \setminus C\| \leq \|\Gamma \setminus B(n)\|$. Thus $e_C(\Gamma) = e(\Gamma)$. □

Lemma 4.14. *Let S and T be two finite generating sets for a group G, and let $B_S(n)$ and $B_T(n)$ be the balls of radius n in Γ_S and Γ_T, respectively.Then there is a constant $\mu \geq 1$ such that if v_g and v_h are vertices in Γ_S that can be joined by an edge path outside of $B_S(\mu n + \mu)$ then v_g and v_h in Γ_T are outside $B_T(n)$ and can be joined by a path that stays outside of $B_T(n)$.*

Proof. Let $\phi_{T \leftarrow S}$ be the map as defined in Proposition 4.4, Corollary 4.5 shows that there exists a constant $\lambda > 1$ such that

$$\frac{1}{\lambda} d_S(g,h) \leq d_T(g,h) \leq \lambda d_S(g,h)$$

Let $\mu = \lambda^2$, and let $\{v_g = v_0, v_1, v_2, ...v_h = v_n\}$ be the vertices on the edge path from v_g to v_h that is contained in the complement of $B(\mu n + \mu)$. If v_e is the vertex corresponding to the identity element in Γ_T (also center of $B(\mu n + \mu)$), and v_i is any vertex in the edgepath from v_g to v_h, then

$$d_T(v_e, v_i) \geq \frac{1}{\lambda} d_S(v_e, v_i) \geq \frac{1}{\lambda}(\mu n + \mu) \geq n + \lambda$$

We can see from the equation above that any vertex in the edge path from v_g to v_h in Γ_T is at least $n + \lambda$ away from the v_e in Γ_T, or in other words, the entire path is outside the ball $B(n + \lambda)$, centered at the identity vertex. From the proof of Proposition 4.4 and Corollary 4.5, for each edge in the edge path defined above in Γ_S, the length of its image under $\phi_{T \leftarrow S}$ is at most λ. Thus the closest an edge in the edgepath in Γ_T can get to $B(n)$ is λ away from the identity element, that is, the center of $B(n)$. This shows that all the elements in the path are outside of $B(n)$ in Γ_T. Thus $\phi_{T \leftarrow S}$ maps the path joining v_g to v_h in $\Gamma_S \setminus B(\mu n + \mu)$ to a connected path joining v_g to v_h in $\Gamma_T \setminus B(n)$.

□

Theorem 4.15. *Let S and T be two finite generating sets for a group G, and let Γ_S and Γ_T be the corresponding Cayley graphs, respectively. Then*

$$e(\Gamma_S) = e(\Gamma_T) \tag{4.16}$$

Proof. Lemma 4.14 shows that if two vertices corresponding to elements of G are connected in $\Gamma_S \setminus B(\mu n + \mu)$, then the vertices corresponding to the same elements are connected in $\Gamma_T \setminus B(n)$. Let A be one of the unbounded, connected components of $\Gamma_S \setminus B(\mu n + \mu)$. Then the vertices corresponding to elements in A form an unbounded connected component in $\Gamma_T \setminus B(n)$. However, since the implication in Lemma 4.14 is one sided, there can also be vertices in two disconnected components of $\Gamma_S \setminus B(\mu n+\mu)$ corresponding to two connected vertices in $\Gamma_T \setminus B(n)$. Thus two or more vertices in disconnected components in A can also map to the same connected component in $\Gamma_T \setminus B(n)$. Therefore there are at least as many connected unbounded components in $\Gamma_S \setminus B(\mu n + \mu)$ compared to $\Gamma_T \setminus B(n)$, that is, $\|\Gamma_S \setminus B(\mu n + \mu)\| \geq \|\Gamma_T \setminus B(n)\|$. Now we can take the limits:

$$\lim_{n\to\infty} \|\Gamma_S \setminus B(\mu n + \mu)\| \geq \lim_{n\to\infty} \|\Gamma_T \setminus B(n)\|$$

Therefore, $e(\Gamma_S) \geq e(\Gamma_T)$.

However, we can also apply the result from Lemma 4.14 on Γ_T, and get the other inequality $e(\Gamma_S) \leq e(\Gamma_T)$. This proves $e(\Gamma_S) = e(\Gamma_T)$. □

Remark 4.17. Now that we have shown that the number of ends of the Cayley graph of a group with a finite generated set is independent of the generating set, we can define the number of ends of a group as the number of ends of any of its Cayley graphs.

5. Freudenthal-Hopf Theorem

Theorem 5.1. *(Freudenthal-Hopf Theorem). Every finitely generated group has either zero, one, two, or infinitely many ends.*

Proof. It suffices to show that there are no finitely generated groups whose number of ends is finite but is at least three. We will prove this by contradiction.

Let G be a finitely generated group with k ends, where $k \geq 3$. Let Γ be the Cayley graph of G. We know that the graph of a group has zero ends if and only if it is finite. Therefore G is an infinite group, and that there exists $n \in \mathbb{N}$ and $n > 1$ such that $\Gamma \setminus B(n)$ has k unbounded, connected components. We also know that there is an element $g \in G$ such that

$$n < d(e, g) < 2n$$

and that v_g, the vertex corresponding to g, is in one of the unbounded, connected components of $\Gamma \setminus B(n)$. And the action $g \cdot B(n)$ sends $B(n)$ at least $2n$ away from the origin. Furthermore, $g \cdot B(n)$ is also in the component containing g.

Recall that for an infinite group with a finite generating set, every element corresponds to a symmetry of its Cayley Graph. An element $h \in G$ acts on the Cayley graph by shifting v_k to v_{hk}. The image of the action is isomorphic to the original graph. And since $n < d(e, g) < 2n$, just like $B(n)$ divides Γ into k pieces, $g \cdot B(n)$ divides the unbounded, connected component of $\Gamma \setminus B(n)$ it sits in into k pieces. And amongst those, at least $k - 1$ components are unbounded. There would be a bounded piece if the component adjacent to both $B(n)$ and $g \cdot B(n)$ were connected by an edge.

Let $C = B(n) \cup g \cdot B(n)$. Then $\|\Gamma \setminus C\| \geq 2k - 2$. And since the number of ends of a group is independent of the subset we choose to cut off, $e(\Gamma) \geq \|\Gamma \setminus C\| \geq 2k - 2$. But for any finite integer $k \geq 3$, $2k - 2 > k$. Since we started by assuming $e(\Gamma) = k$, we have a contradiction. Therefore all finitely generated groups have either zero, one, two or infinitely many ends.

□

The Freudenthal-Hopf Theorem shows that the number the ends of a group can have is either zero, one, two, or infinity. But it does not classify the groups with any certain number of ends. Later, other theorems proved the properties the groups with different numbers of ends have. It turns out that a group has two ends if it has a finite index subgroup isomorphic to $\mathbb{Z}$. Stallings provided the classification of groups with infinitely many ends. We do not have space to prove all of them, since they require introducing a lot of new concepts. We will just state them. And we will point to material for further reading for readers who are interested.

Theorem 5.2. *A group G is two-ended if and only if G contains a finite index subgroup isomorphic to $\mathbb{Z}$.*

The proof of this theorem is in [Me216].

The original statement of Stallings' Theorem involved concepts that require a lot of explanation, so we will state a simplified version here. However, we will still need to define two concepts.

Definition 5.3. An element of finite order is called a *torsion* element

The statement of Stallings' Theorem also uses the concept of a free product, which is however impossible to define without introducing a few other concepts. Hence those who are interested should find the description of a free product in [Me79].

Theorem 5.4. *(Stalling's Theorem) Let G be a finitely generated, torsion-free group. Then G has infinitely many ends if and only if G is a free product, $G \approx H * K$, where neither H nor K is the trivial group.*

Acknowledgments. It is a pleasure to thank my mentor, Wouter van Limbeek, for meeting with me for two hours per week throughout the program to review the material I read and explain the parts that I failed to understand, and reading and editing drafts of this paper. I would also like to thank Peter May for organizing this program.

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