REPRESENTATIONS OF LIE GROUPS AND LIE ALGEBRAS

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Abstract. This paper studies the relationship between representations of a Lie group and representations of its Lie algebra. We will make the correspondence in two steps: First we shall prove that a given representation of a Lie group will provide us with a corresponding representation of its Lie algebra. Second, we shall go backwards and see whether a given representation of a Lie algebra will have a corresponding representation of its Lie group.

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1. INTRODUCTION TO LIE GROUPS AND LIE ALGEBRAS

In this section, we shall introduce the notion of Lie group and its Lie algebra. Since a Lie group is a smooth manifold, we shall also introduce some basic theory of smooth manifolds here.

Definition 1.1. A Lie group is a smooth manifold $G$ that also has a group structure, with the property that the multiplication map $m : G \times G \to G$ and the inversion map $i : G \to G$, given respectively by

$$m(g, h) = gh, \quad i(g) = g^{-1},$$

are both smooth maps.

One of the most important examples of Lie groups is the group GL($V$). Suppose $V$ is some real or complex vector space. Then the group GL($V$) denotes the set of all invertible linear transformations from $V$ to itself. The group multiplication is just composition. If $V$ is finite-dimensional, then any basis for $V$ will induce an isomorphism of GL($V$) with GL($n, \mathbb{R}$) or GL($n, \mathbb{C}$), with $n = \dim V$. (Here, GL($n, \mathbb{R}$) is the general linear group consisting of all $n \times n$ matrices with real entries. Analogously, GL($n, \mathbb{C}$) is the complex general linear group).

Therefore, once we pick a basis for $V$, we get a chart $\phi$ on GL($V$) which sends an element of GL($V$) to its matrix in the chosen basis (which can be thought as an element

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\footnote{Both GL($n, \mathbb{R}$) and GL($n, \mathbb{C}$) are equipped with a standard smooth structure. For details, please refer to chapter 1 of Lee’s Introduction to Smooth Manifolds.}
in $\mathbb{R}^{2n}$. Obviously, this single chart covers the entirety of $GL(V)$, thus forming an atlas. This makes $GL(V)$ a Lie group. If we choose another basis for $V$, then the transition map between the two charts is given by a map of the form $A \to BAB^{-1}$ with $B$ the transition matrix between the two bases, it follows that the transition maps are smooth. Thus, this smooth manifold structure on $GL(V)$ is independent of the choice of basis. This Lie group $GL(V)$ will play an important role in the later sections of this paper.

First we review the concept of tangent vectors on a manifold. There are in fact several different ways to define tangent vectors to a smooth manifold. Here, we will define a tangent vector at a point $p$ in a smooth manifold $M$ as a derivation at $p$.

**Definition 1.2.** Let $M$ be a smooth manifold and $p$ be a point of $M$. A derivation at $p$ is a linear map $X : C^\infty(M) \to \mathbb{R}$ that satisfies

$$X(fg) = f(p)Xg + g(p)Xf$$

for any $f, g \in C^\infty(M)$.

The intuition behind such definition is that we can regard a derivation as a directional derivative encountered in multivariable calculus theory. The set of all derivations of $C^\infty(M)$ is purely a local construction. Suppose $M$ is a smooth manifold, $p \in M$, and $X \in T_p M$. If both functions agree on some neighborhood of $p$, then $Xf = Xg$.

**Proposition 1.3.** Suppose $M$ is a smooth manifold, $p \in M$, and $X \in T_p M$. If $f, g \in C^\infty(M)$ and both functions agree on some neighborhood of $p$, then $Xf = Xg$.

**Proof.** Let $h = f - g$. Then by linearity of derivations, it is sufficient to show that $Xh = 0$ whenever $h$ vanishes in a neighborhood of $p$. Let $\psi \in C^\infty(M)$ be a smooth bump function that is equal to 1 on the support of $h$ and is supported in $M \setminus \{p\}$. Since $\psi \equiv 1$ where $h$ is not zero, then the product $\psi h$ is identically equal to $h$. Since $h(p) = \psi(p) = 0$, then $Xh = X(\psi h) = 0$, which follows from the definition of a derivation. \qed

Having defined tangent vectors on a manifold, we then need to explore how tangent vectors behave under smooth maps. Suppose $M$ and $N$ are two smooth manifolds and $F : M \to N$ is a smooth map. For each $p \in M$, there is a natural map from $T_p M$ to $T_{F(p)} N$, which is called the push-forward of $F$.

**Definition 1.4.** If $M$ and $N$ are smooth manifolds and $F : M \to N$ is a smooth map, for each $p \in M$ we define a map $F_* : T_p M \to T_{F(p)} N$, called the push-forward of $F$, by

$$(F_* X)(f) = X(f \circ F).$$

for all $f \in C^\infty(N)$.

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2The proof can be found in chapter 2 of Lee’s *Introduction to Smooth Manifolds*. 
It is straightforward to show that the map $F^* : T_pM \to T_{F(p)}N$ is a linear map between the two vector spaces $T_pM$ and $T_{F(p)}N$. Moreover, it is not hard to verify the following lemma by simply following the definition of push-forward.

**Lemma 1.5.** Let $F : M \to N$ and $G : N \to K$ be smooth maps, and let $p \in M$.

(a) $F^* : T_pM \to T_{F(p)}N$ is linear.

(b) $(G \circ F)^* = G^* \circ F^*$

(c) $(\text{Id}_M)^* = \text{Id}_{T_pM}$

(d) If $F$ is a diffeomorphism, then $F^* : T_pM \to T_{F(p)}N$ is an isomorphism

With the concept of push-forward, we can now construct a basis for $T_pM$ for any $p \in M$ by the following theorem:

**Theorem 1.6.** Let $M$ be a smooth $n$-manifold. For any $p \in M$, $T_pM$ is an $n$-dimensional vector space. If $(U, \phi = (x^i))$ is any smooth chart containing $p$, the coordinate vectors $(\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p)$ form a basis for $T_pM$, where

$$
\frac{\partial}{\partial x^i}|_p \equiv (\phi^{-1})_* \left( \frac{\partial}{\partial x^i} \right)_{\phi(p)}.
$$

With the concept of push-forward, we can now extend the familiar notion of tangent vectors to a smooth curve in $\mathbb{R}^n$ to smooth curves in manifolds.

**Definition 1.7.** If $M$ is a smooth manifold, we define a **smooth curve** in $M$ to be a smooth map $\gamma : J \to M$, where $J \subseteq \mathbb{R}$ is an interval. (In most situations, we shall let $J$ be an open interval.)

Since we can regard $J$ as an open submanifold of $\mathbb{R}$ with the standard smooth structure, then it is natural to define the tangent vector to a smooth curve in $M$ as the push-forward of $d/dt|_{t_0} \in T_{t_0}\mathbb{R}$.

**Definition 1.8.** If $\gamma$ is a smooth curve in a smooth manifold $M$, then the tangent vector to $\gamma$ at $t_0 \in J$ is the vector

$$
\gamma'(t_0) = \gamma_* \left( \frac{d}{dt} \big|_{t_0} \right) \in T_{\gamma(t_0)}M,
$$

where $d/dt|_{t_0}$ is the standard coordinate basis for $T_{t_0}\mathbb{R}$.

Vector fields on $\mathbb{R}^n$ are familiar objects to us. Now, we wish to define a vector field on an abstract smooth manifold $M$. In order to make a precise definition, we shall first introduce the concept of the **tangent bundle** of $M$.

**Definition 1.9.** Given a smooth manifold $M$, the **tangent bundle** of $M$, denoted by $TM$, is the disjoint union of the tangent spaces at all points of $M$:

$$
TM = \coprod_{p \in M} T_pM,
$$

together with the smooth structure defined in Theorem 1.10. An element of $TM$ will be written as an ordered pair $(p, X)$.

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3We have omitted the proof here. Readers can find the proof in chapter 3 of Lee’s *Introduction to Smooth Manifolds*. 
The tangent bundle is equipped naturally with a projection map \( \pi : TM \to M \) by \( \pi(p, X) = p \). From our definition, the tangent bundle seems to be merely a collection of vector spaces on \( M \). However, with the following theorem, we see that \( TM \) has a smooth manifold structure on it such that the projection map \( \pi \) defined before is smooth with respect to this structure.

**Theorem 1.10.** For any smooth \( n \)-manifold \( M \), the tangent bundle \( TM \) has a topology and smooth structure that make it into a \( 2n \)-dimensional smooth manifold. With this structure, \( \pi : TM \to M \) is a smooth map.

However, we shall not provide the proof here. The smooth charts in the smooth structure defined by the theorem are \((\pi^{-1}(U), \phi)\), where \((U, \phi)\) are smooth charts for \( M \) and the map \( \tilde{\phi} : \pi^{-1}(U) \to \mathbb{R}^{2n} \) is defined by

\[
\tilde{\phi} \left( v^i \frac{\partial}{\partial x^i} \right) = (x^1(p), \ldots, x^n(p), v^1, \ldots, v^n).
\]

Now, we are ready to define a vector field on a smooth manifold \( M \).

**Definition 1.11.** If \( M \) is a smooth manifold, a vector field on \( M \) is a continuous map \( Y : M \to TM \), usually denoted as \( p \mapsto Y_p \), satisfying

\[
(1.12) \quad \pi \circ Y = \text{Id}_M
\]

A smooth vector field on \( M \) is then a smooth map \( Y : M \to TM \) which satisfies (1.12). A rough vector field on \( M \) is a map (the map does not need to be continuous) \( Y : M \to TM \) satisfying (1.10). We will use the notation \( \mathcal{X}(M) \) to denote the set of all smooth vector fields on \( M \). It becomes a vector space under pointwise addition and scalar multiplication by:

\[
(aY + bZ)_p = aY_p + bZ_p.
\]

In addition, if \( f \in C^\infty(M) \) and \( Y \in \mathcal{X}(M) \), then \( fY : M \to TM \) defined by

\[
(fY)_p = f(p)Y_p
\]

is another smooth vector field. We can also regard \( Y \in \mathcal{X}(M) \) as a map \( C^\infty(M) \to C^\infty(M) \) by \( f \mapsto Yf \), where \( Yf \) is defined by

\[
Yf(p) = Y_pf.
\]

From the product rule for tangent vectors, it follows that for \( f, g \in C^\infty(M) \) and \( Y \in \mathcal{X}(M) \),

\[
(1.13) \quad Y(fg) = fYg + gYf.
\]

In general, we call a map from \( C^\infty(M) \) to \( C^\infty(M) \) which is linear over \( \mathbb{R} \) and satisfies equation (1.13) as a derivation. Actually, the derivations of \( C^\infty(M) \) can be identified with smooth vector fields on \( M \). We shall merely quote the statement of the following proposition here.

**Proposition 1.14.** Let \( M \) be a smooth manifold. A map \( \mathcal{Y} : C^\infty(M) \to C^\infty(M) \) is a derivation if and only if it is of the form \( \mathcal{Y}f = Yf \) for some \( Y \in \mathcal{X}(M) \).

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4 For proof, see chapter 4 of Lee’s *Introduction to Smooth Manifolds.*

5 We have used Einstein summation convention in the following equation.

6 For proof, see chapter 4 of Lee’s *Introduction to Smooth Manifolds.*
Since we have defined the push-forward of a tangent vector, it is natural to extend the idea to the vector fields.

**Definition 1.15.** Suppose \( F : M \to N \) is a smooth map between the smooth manifolds \( M \) and \( N \). Let \( Y \in \mathfrak{X}(M) \) and \( Z \in \mathfrak{X}(N) \). We say that the vector fields \( Y \) and \( Z \) are \( F \)-related if \( F_*Y_p = Z_{F(p)} \) for each \( p \in M \).

Given two smooth vector fields \( V \) and \( W \), we can get a third smooth vector field, called *Lie bracket* of \( V \) and \( W \), defined by

\[
[V, W] f = VWf - WVf.
\]

We shall show that \([V, W]\) defined as above is indeed a smooth vector field.

**Lemma 1.16.** The Lie bracket \([V, W]\) of \( V, W \in \mathfrak{X}(M) \) is also a smooth vector field.

**Proof.** By Proposition 1.14, it is sufficient to show that \([V, W]\) is a derivation of \( C^\infty(M) \). For any \( f, g \in C^\infty(M) \), we have

\[
[V, W] (fg) = V(W(g)) - W(V(g)) = V(fWg + gWf) - W(fVg + gVf) = f(VWg) + VfWg + g(VWf) + VeWf - f(WVg) - WfVg - g(WVf) - WgVf = f([V, W]g) + g([V, W]f).
\]

\( \square \)

From the definition, it is not hard to see that Lie bracket has following properties.

**Lemma 1.17.** The Lie bracket satisfies following properties for all \( V, W, Y \in \mathfrak{X}(M) \):

(a) Bi-linearity: For \( a, b \in \mathbb{R} \),

\[
[aV + bW, Y] = a[V, Y] + b[W, Y],
\]

\[
\]

(b) Anti-symmetry:

\[
[V, W] = -[W, V].
\]

(c) Jacobi Identity:

\[
\]

**Proof.** Bi-linearity and anti-symmetry follow simply from our definition of the Lie bracket. For the proof of Jacobi identity:

\[
([V, [W, Y]] + [W, [Y, V]] + [Y, [V, W]])f
= VWYf - VYWf - WYVf + YWVf + WYVf - WVYf
= -YVWf + YVWf + YVWf - YWVf - VWYf + WVYf
= 0
\]

\( \square \)
Another important property of Lie bracket is expressed in the next theorem.

**Theorem 1.18.** Let $F : M \to N$ be a smooth map, and let $V_1, V_2 \in \mathfrak{X}(M)$ and $W_1, W_2 \in \mathfrak{X}(N)$. If $V_i$ is $F$-related to $W_i$ for $i = 1, 2$, then $[V_1, V_2]$ is $F$-related to $[W_1, W_2]$.

**Proof.** Fix $p \in M$. Then for any $f \in C^\infty(N)$, we have

$$F_*[V_1, V_2]_p(f) = [V_1, V_2]_p(f \circ F)$$

$$= (V_1)_p((V_2)(f \circ F)) - (V_2)_p((V_1)(f \circ F))$$

$$= (V_1)_p((W_2f) \circ F) - (V_2)_p((W_1f) \circ F)$$

$$= (W_1)_p(W_2f) - (W_2)_p(W_1f)$$

$$= [W_1, W_2]_p(f)$$

\[ \square \]

The most important application of Lie brackets actually occurs in the context of Lie groups. Suppose $G$ is a Lie group. According to the definition, for any $g \in G$, we have the map $L_g : G \to G$, which is called the left translation, defined by

$$L_g(h) = gh$$

for all $h \in G$, is a smooth map. In fact, it is a diffeomorphism of $G$ because the smooth map $L_{g^{-1}}$ is clearly the inverse for $L_g$.

A vector field $X$ on $G$ is said to be left-invariant if it is $L_g$-related to itself for every $g \in G$. This means

$$(L_g)_* X_h = X_{gh}, \text{ for all } g, h \in G.$$ 

Since $L_g$ is also a diffeomorphism, then it follows that $(L_g)_* X = X$ for every $g \in G$.

Suppose $X, Y$ are both smooth left-invariant vector fields on $G$, then it follows that

$$(L_g)_*(aX + bY) = a(L_g)_* X + b(L_g)_* Y = aX + bY,$$

for all $a, b \in \mathbb{R}$, so the set of all smooth left-invariant vector fields on $G$ is a linear sub-space of $\mathfrak{X}(G)$. More importantly, this sub-space is closed under Lie brackets.

**Theorem 1.19.** Let $G$ be a Lie group. If $X$ and $Y$ are both smooth left-invariant vector fields on $G$, then $[X, Y]$ is also a smooth left-invariant vector field.

**Proof.** Fix some $g \in G$. By assumption, we have $(L_g)_* X = X$ and $(L_g)_* Y = Y$. Then by Theorem 1.18, we have $[X, Y]$ is $L_g$-related to $[(L_g)_* X, (L_g)_* Y] = [X, Y]$, so we have $(L_g)_* [X, Y] = [X, Y]$. Therefore, $[X, Y]$ is a smooth left-invariant vector field on $G$. \[ \square \]

Now, we are ready to define the Lie algebra of a given Lie group. First, we will give the definition for a general Lie algebra.

**Definition 1.20.** A real Lie algebra is a real vector space $\mathfrak{g}$ endowed with a bilinear map called the bracket $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which is also anti-symmetric and satisfies the Jacobi identity for all $X, Y, Z \in \mathfrak{g}$:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$
From Theorem 1.19, it follows that if $G$ is a Lie group, the set of all smooth left-invariant vector fields on $G$ is a Lie algebra under the Lie bracket. We call this Lie algebra the Lie algebra of $G$, and denote it by $\mathfrak{g}$.

Having defined the Lie algebra of a Lie group, we shall first explore some basic properties of $\mathfrak{g}$.

**Theorem 1.21.** Let $G$ be a Lie group. The evaluation map $\varepsilon : \mathfrak{g} \to T_e G$, defined by $\varepsilon(X) = X_e$, is a vector space isomorphism. Therefore, $\dim \mathfrak{g} = \dim T_e G = \dim G$.

**Proof.** Clearly by our definition of $\varepsilon$, it is a linear map. Therefore, to prove the theorem, we only need to find a linear inversion for $\varepsilon$. For each $V \in T_e G$, we shall define a vector field $\tilde{V}$ on $G$ by

\[(1.22) \quad \tilde{V}_g = (L_g)_* V.\]

Clearly, if $X$ is a left-invariant vector field on $G$ such that $X_e = V$, then $X$ has to be given by the equation $1.22$.

Now, we shall check that $\tilde{V}$ is smooth. To verify this, it is sufficient to show that $\tilde{V}f$ is smooth for any $f \in C^\infty(U)$ for any open subset $U$ of $G$.

Let $\gamma : (-\epsilon, \epsilon) \to G$ be such that $\gamma(0) = e$ and $\gamma'(0) = V$. Then for any $g \in U$, we have

\[(\tilde{V}f)(g) = \tilde{V}_g f = ((L_g)_* V)f = V(f \circ L_g) = \gamma'(0)(f \circ L_g) = \frac{d}{dt}
\big|_{t=0} (f \circ L_g \circ \gamma)(t).\]

(1.23)

Let $\psi : (-\epsilon, \epsilon) \times G \to \mathbb{R}$ be the map defined by $\psi(t, g) = f \circ L_g \circ \gamma(t)$. Since the multiplication on $G$, $f$ and $\gamma$ are all smooth maps, then it follows that $\psi$ is also smooth. By the equation (1.23), we can see that $(\tilde{V}f)(g) = \partial \psi / \partial t|_{(0, g)}$. Since $\psi$ is smooth, $\partial \psi / \partial t|_{(0, g)}$ depends smoothly on $g$. So $\tilde{V}f$ is smooth.

It is easy to verify that $\tilde{V}$ is left-invariant. For all $h, g \in G$, by definition of $\tilde{V}$, we have

\[(L_h)_* \tilde{V}_g = (L_h)_* (L_g)_* V = (L_{hg})_* V = \tilde{V}_{hg},\]

where the second equality follows from Lemma 1.5.

Now, let $\tau : V \mapsto \tilde{V}$ be defined as above. The linearity of $\tau$ follows from its definition. For any $V \in T_e G$, we have

\[\varepsilon(\tau(V)) = \varepsilon(\tilde{V}) = (\tilde{V})_e = (L_e)_* V = V.\]

So $\varepsilon \circ \tau = Id_{T_e G}$. On the other hand, for any vector field $X \in \mathfrak{g}$, we have

\[(\tau(\varepsilon(X)))_g = (\tau(X_e))_g = (L_g)_* X_e = X_g.\]

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\(^7\)For the details, check Lemma 4.6 which is proved in chapter 4 of Lee’s *Introduction to Smooth Manifolds*.

\(^8\)The existence of such a smooth curve is not hard to prove. Let $(U, \phi)$ be a smooth coordinate chart centered at $e$, and let $V = V^i \partial / \partial x^i|^e$ in terms of the coordinate basis. Define the map $\gamma : (-\epsilon, \epsilon) \to U$ by $\gamma(t) = (tV^1, \ldots, tV^n)$; then this $\gamma$ is a smooth curve with $\gamma(0) = e$ and $\gamma'(0) = V$. 


Thus, $\tau \circ \varepsilon = \text{Id}_g$. □

From the proof of Theorem 1.21, we see that if $V$ is a left-invariant rough vector field on the Lie group $G$, we then have $V = \tilde{V}$. This shows that every left-invariant vector field on a Lie group is a smooth vector field.

Before finishing this section, we shall prove an important property of connected Lie groups, as stated in the following theorem.

**Theorem 1.24.** Suppose $G$ is a connected Lie group. Let $U \subseteq G$ be any open neighborhood of the identity. Then every element of $G$ can be written as a finite product of elements in $U$, that is, $U$ generates $G$.

**Proof.** Let $U$ be an open neighborhood of $e$. For every $n \in \mathbb{N}$, we let

$$U^n = \{u_1u_2 \cdots u_n : u_1, u_2, \ldots, u_n \in U\}.$$  

Fix some $g \in U^n$. Then there exist some $u \in U$ and $v \in U^{n-1}$ such that $g = vu = L_vu$. Since $L_v$ is a diffeomorphism, thus a homeomorphism, it then follows that $g \in L_v(U) \subseteq U^n$ where $L_v(U)$ is open. Therefore, $U^n$ is an open set. Let

$$W = \bigcup_{n \in \mathbb{N}} U^n.$$  

Then $W$ is also an open set. Now, we want to show that $W$ is also closed.

Let $g \in \overline{W}$, the closure of $W$. Since the inversion map $i : G \rightarrow G$ is smooth and clearly $i \circ i = \text{Id}$, then the inversion map $i$ is a diffeomorphism of $G$, thus a homeomorphism. So $U^{-1} = i(U)$ is also an open neighborhood of $e$. It follows that $gU^{-1} = L_g(U^{-1})$ is an open neighborhood of $g$. Since $g \in \overline{W}$, then $gU^{-1} \cap W \neq \emptyset$.

Let $h \in gU^{-1} \cap W$. Then, there exists some $u \in U$ such that $h = gu^{-1}$. So $g = hu$. Since $h \in W$, then $h \in U^k$ for some $k \in \mathbb{N}$. Then it follows that $g \in U^{k+1} \subseteq W$. So $\overline{W} = W$, which implies that $W$ is closed.

Therefore, we have shown that $W$ is both closed and open. Since $G$ is connected and clearly $W \neq \emptyset$, then we have $G = W = \bigcup_{n \in \mathbb{N}} U^n$. This implies that $U$ generates $G$. □

2. FROM REPRESENTATIONS OF LIE GROUPS TO LIE ALGEBRAS

Having introduced the concept of a Lie group and its Lie algebra, we shall now explore the connection between the two. In both mathematics and physics, the study of the representations of a Lie group plays a major role in the understanding of continuous symmetry. One of the basic tools in studying the representations of Lie groups is the use of the corresponding representations of Lie algebras. In this section, we shall focus on how to get a corresponding representation of the Lie algebra, given a representation of its Lie group. First, we will give the definition of a representation of a Lie group and that of a representation of a Lie algebra.

**Definition 2.1.** A representation of a Lie group $G$ on a finite-dimensional vector space $V$ is a smooth group homomorphism $\rho : G \rightarrow \text{GL}(V)$.

Namely, for any $h, g \in G$ and $v \in V$, we have $\rho(hg)(v) = (\rho(h) \circ \rho(g))(v)$. 

Definition 2.2. A representation of a Lie algebra \( \mathfrak{g} \) on a finite-dimensional vector space \( V \) is a Lie algebra homomorphism
\[
\rho : \mathfrak{g} \to \text{gl}(V) = \text{End}(V).
\]
Namely, for any \( X, Y \in \mathfrak{g} \) and \( v \in V \), we have
\[
\rho([X,Y])(v) = (\rho(X) \circ \rho(Y))(v) - (\rho(Y) \circ \rho(X))(v).
\]
The important aspect of the Lie algebra of a Lie group arises from the fact that each Lie group homomorphism induces a Lie algebra homomorphism, as stated in the next theorem.

Theorem 2.3. Suppose \( G \) and \( H \) are Lie groups and let \( \mathfrak{g}, \mathfrak{h} \) be their Lie algebras. If \( F : G \to H \) is a Lie group homomorphism, then for every \( X \in \mathfrak{g} \), there is a unique vector field \( Y \in \mathfrak{h} \) that is \( F \)-related to \( X \). Let this vector field be denoted by \( F_*X \), then the map \( F_* : \mathfrak{g} \to \mathfrak{h} \) defined in this way is a Lie algebra homomorphism.

Proof. If there exists a vector field \( Y \in \mathfrak{h} \) that is \( F \)-related to \( X \), then we must have \( Y_e = F_*X_e \). Then according to the remark we have made after Theorem 1.21, \( Y \) is uniquely determined by
\[
y = Y_e = \widetilde{F_*X_e}.
\]
Now, we need to show that the vector field \( Y \) determined this way is indeed \( F \)-related to \( X \). Since \( F \) is a Lie group homomorphism, we have
\[
(F \circ L_g)(g') = F(g g') = F(g)F(g') = (L_{F(g)} \circ F)(g'),
\]
So by Lemma 1.5, we get
\[
F_* (L_g)_* = (F \circ L_g)_* = (L_{F(g)} \circ F)_* = (L_{F(g)})_* \circ F_*,
\]
It then follows that
\[
F_* X_g = F_* (L_g)_* X_e = (L_{F(g)})_* \circ F_* X_e = (L_{F(g)})_* Y_e = Y_{F(g)}.
\]
This proves that \( X \) and \( Y \) are \( F \)-related.

Thus, we define a map \( F_* : \mathfrak{g} \to \mathfrak{h} \) by \( X \mapsto F_*X = Y \), where \( X \in \mathfrak{g} \) and \( Y \) is the unique vector field in \( \mathfrak{h} \) that is \( F \)-related to \( X \) as we have shown above. Now what remains to show is that \( F_* \) is a Lie algebra homomorphism. This is an immediate consequence of Theorem 1.18, which states precisely that
\[
F_*[X,Y] = [F_*X,F_*Y].
\]

The map \( F_* : \mathfrak{g} \to \mathfrak{h} \) defined in the previous theorem will be called the induced Lie algebra homomorphism of \( F \). In some literature, \( F_* \) is called the differential of map \( F \).

Now consider a representation \( \rho : G \to \text{GL}(V) \) of a Lie group \( G \) on a finite-dimensional (real or complex) vector space \( V \). By definition, the map \( \rho \) is actually a Lie group homomorphism between the Lie group \( G \) and the Lie group \( \text{GL}(V) \). Thus, by applying Theorem 2.3, we get a induced Lie algebra homomorphism
\[
\rho_* : \mathfrak{g} \to \text{Lie}(\text{GL}(V)),
\]
where \( \text{Lie}(\text{GL}(V)) \) denotes the Lie algebra of the Lie group \( \text{GL}(V) \). In fact, the map \( \rho_* \) is indeed a representation of the Lie algebra \( \mathfrak{g} \) on the finite-dimensional vector
space $V$. This is a consequence of the important fact that $\text{Lie}(\text{GL}(V))$ is isomorphic to $\mathfrak{gl}(V)$ in the Lie algebra sense, which is stated in the following theorem.

**Theorem 2.4.** The composition of the natural maps

$$\text{Lie}(\text{GL}(V)) \rightarrow T_{Id}\text{GL}(V) \rightarrow \mathfrak{gl}(V)$$

gives a Lie algebra isomorphism between $\text{Lie}(\text{GL}(V))$ and $\mathfrak{gl}(V)$.

**Proof.** Without loss of generality, we will assume that $V$ is a finite-dimensional complex vector space. Fix a basis for $V$. Then according to the remark made in section one, $\text{GL}(V)$ can be then identified with $\text{GL}(n, \mathbb{C})$. Analogously, $\mathfrak{gl}(V)$ can be identified with $\mathfrak{gl}(n, \mathbb{C})$. This provides us with globally defined coordinates on $\text{GL}(V)$ by considering the corresponding matrix entries $X_{ij}^k$ of $X \in \text{GL}(V)$ respecting the basis we have previously chosen. Then we can get a natural isomorphism $T_{Id}\text{GL}(V) \leftrightarrow \mathfrak{gl}(V)$ which takes the form

$$(A_j^i \frac{\partial}{\partial X_j^i})_{| Id} \leftrightarrow (A_j^i)$$

where $(A_j^i)$ denotes the element $A \in \mathfrak{gl}(V)$ with respect to the basis we have chosen. It then follows that any $A = (A_j^i) \in \mathfrak{gl}(V)$ determines a left-invariant vector field $\tilde{A} \in \text{Lie}(\text{GL}(V))$ as defined by equation 1.22, which in this case becomes

$$\tilde{A}_X = (L_X)_* \left( A_j^i \frac{\partial}{\partial X_j^i} \right)_{| Id}$$

for any $X \in \text{GL}(V)$. Since $L_X$ is the linear map $Y \mapsto XY$ for $Y \in \text{GL}(V)$, then it follows that the coordinate representation of the push-forward $(L_X)_*$ is given by

$$(2.5) \quad (L_X)_* \left( A_j^i \frac{\partial}{\partial X_j^i} \right)_{| Id} = X_j^k A_k^j \frac{\partial}{\partial X_k^j} \bigg|_{X}$$

where $(X_k^j)$ is the matrix representation of $X$ under the chosen basis. Note that Einstein summation convention is used in equation 2.5. It follows that for any $A \in \mathfrak{gl}(V)$, the left-invariant vector field $\tilde{A} \in \text{Lie}(\text{GL}(V))$ determined by $A$ is given by

$$\tilde{A} = X_j^k A_k^i \frac{\partial}{\partial X_k^j} \bigg|_{X}$$

Clearly, the map

$$A \in \mathfrak{gl}(V) \mapsto \tilde{A} \in \text{Lie}(\text{GL}(V))$$

and the composition map

$$X \in \text{Lie}(\text{GL}(V)) \mapsto X_{Id} \in T_{Id}\text{GL}(V) \mapsto (X_j^i) \in \mathfrak{gl}(V)$$

are linear inversions of each other. Thus, we have a natural isomorphism between $\text{Lie}(\text{GL}(V))$ and $\mathfrak{gl}(V)$ in the sense of vector spaces. What remains to be shown is that this natural isomorphism is indeed a Lie algebra isomorphism. For any
A, B ∈ gl(V), we have

\[
[A, B]_X = \left[ X^i_j A^j_k \frac{\partial}{\partial X^i_k}, X^p_q B^q_r \frac{\partial}{\partial X^p_r} \right]_X
\]

\[
= X^i_j A^j_k \frac{\partial}{\partial X^i_k} \left( X^p_q B^q_r \right) \frac{\partial}{\partial X^p_r} - X^p_q B^q_r \frac{\partial}{\partial X^p_r} \left( X^i_j A^j_k \right) \frac{\partial}{\partial X^i_k}
\]

\[
= \left( X^i_j A^j_k B^k_r - X^i_j B^k_r A^j_k \right) \partial \frac{\partial}{\partial X^i_r} |_{X},
\]

(2.6)

where we have used the fact that \( \frac{\partial X^p_i}{\partial X^i_k} \) is equal to 1 if \( p = i \) and \( q = k \), and 0 otherwise, as well as the fact that the mixed partial derivatives of a smooth function can be taken in any order. Evaluating equation 2.6 at the identity map \( Id \in GL(V) \) results

\[ [A, B]_{Id} = \left( A^i_k B^k_r - B^i_k A^k_r \right) \frac{\partial}{\partial X^i_r} |_{Id}. \]

Since the matrix representation of \( [A, B] \in gl(V) \) is just \( (A^i_k B^k_r - B^i_k A^k_r) \), thus according to the natural isomorphism we get

\[ [\tilde{A}, \tilde{B}] = [\tilde{A}, \tilde{B}]. \]

This proves that the natural isomorphism between \( \text{Lie}(GL(V)) \) and \( gl(V) \) is a Lie algebra isomorphism. Thus we can simply identify \( \text{Lie}(GL(V)) \) with \( gl(V) \). □

Combining Theorem 2.3 and Theorem 2.4, we get the following corollary.

**Corollary 2.7.** If \( \rho : G \to GL(V) \) is a representation of the Lie group \( G \) on a finite-dimensional vector space \( V \), then the induced homomorphism \( \rho_* : g \to gl(V) \) is a representation of the Lie algebra \( g \) of the Lie group \( G \) on the vector space \( V \).

### 3. From Representations of Lie Algebras to Lie Groups

In the previous section, we saw that given a representation \( \rho : G \to GL(V) \) of the Lie group \( G \) on a finite-dimensional vector space \( V \), we obtained an corresponding representation \( \rho_* : g \to gl(V) \) of its Lie algebra on the same vector space \( V \). However, given a representation of the Lie algebra \( g \) of the Lie group \( G \) on a finite-dimensional vector space \( V \), can we then find an corresponding representation of the Lie group \( G \) on the vector space \( V \)? The answer is yes if we impose the additional condition that the Lie group \( G \) is simply-connected.

In order to reach our ultimate goal, we shall use an important fact that every Lie subalgebra corresponds to some Lie subgroup. The proof of this fact involves a lot of new machinery which we have not introduced in this paper. Therefore, we will only state the theorem here without offering the proof.

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9 A Lie subalgebra of the Lie algebra \( g \) is a linear sub-space \( h \subseteq g \) which is also closed under brackets. A Lie subgroup of a Lie group \( G \) is a subgroup \( H \) of \( G \) which also has a Lie group structure and is an immersed submanifold of \( G \).

10 For details of the proof, please refer to either chapter 20 of Lee’s Introduction to Smooth Manifolds or Lecture 8 of Representation Theory: A First Course by Fulton and Harris.
Theorem 3.1. Suppose $G$ is a Lie group with Lie algebra $\mathfrak{g}$. If $\mathfrak{h}$ is any Lie subalgebra of $\mathfrak{g}$, then there exists a unique connected Lie subgroup of $G$ whose Lie algebra is $\mathfrak{h}$.

The most important application of Theorem 3.1 is to prove the following theorem.

Theorem 3.2. Suppose $G$ and $H$ are Lie groups with $G$ simply connected, and let $\mathfrak{g}$ and $\mathfrak{h}$ be their Lie algebras. For any Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$, there exists a Lie group homomorphism $\Phi : G \to H$ such that $\Phi_* = \phi$.

In order to prove Theorem 3.2, we shall first prove the following lemmas.

Lemma 3.3. Suppose $G$ and $H$ are Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Then $\mathfrak{g} \oplus \mathfrak{h}$ is a Lie algebra with the bracket defined by

$[[X, Y], (X', Y')] = ([X, X'], [Y, Y'])$.

The Lie algebra $\mathfrak{g} \oplus \mathfrak{h}$ defined above is indeed isomorphic to $\operatorname{Lie}(G \times H)$, the Lie algebra of the Lie group $G \times H$.

Proof. The verification for the vector space $\mathfrak{g} \oplus \mathfrak{h}$ being a Lie algebra with the bracket defined in the lemma is an immediate consequence of the properties satisfied by the brackets for $\mathfrak{g}$ and $\mathfrak{h}$.

What remains to show is that $\operatorname{Lie}(G \times H)$ is isomorphic to $\mathfrak{g} \oplus \mathfrak{h}$. Let $(U, x^i)$ and $(V, y^j)$ be charts which respectively contain $e_G$ and $e_H$. Then, it follows that $(U \times V, x^i \times y^j)$ is a chart which contains $e_G \times e_H$. So for any tangent vector $Z \in T_e(G \times H)$, we have

$Z = z^{i} \frac{\partial}{\partial x^i}_e + z^{j} \frac{\partial}{\partial y^j}_e$.

By Theorem 1.21, we know that every left-invariant vector field $W$ on a Lie group is uniquely determined by its value at $e$, namely $W_e$. Thus, we get a vector space isomorphism $\psi : \operatorname{Lie}(G \times H) \to \mathfrak{g} \oplus \mathfrak{h}$ by considering the composition of maps

$\tilde{Z} \leftrightarrow Z_e \leftrightarrow \left( z^{i} \frac{\partial}{\partial x^i}_e, z^{j} \frac{\partial}{\partial y^j}_e \right) \leftrightarrow \left( \tilde{Z}_G, \tilde{Z}_H \right)$,

where $\tilde{Z}_G$ and $\tilde{Z}_H$ are the unique left-invariant vector fields determined respectively by $z^{i} \partial / \partial x^i \in T_e G$ and $z^{j} \partial / \partial y^j \in T_e H$. To complete the proof, we need to show that this isomorphism is indeed a Lie algebra isomorphism. For any $W, V \in \operatorname{Lie}(G \times H)$,

$[\tilde{W}, \tilde{V}]_e = \left( w^{x_i} \frac{\partial v^{x_k}}{\partial x^i} - v^{x_i} \frac{\partial w^{x_k}}{\partial x^i} \right) \frac{\partial}{\partial x^k}_e + \left( v^{y_j} \frac{\partial w^{y_j}}{\partial y^j} - w^{y_j} \frac{\partial v^{y_j}}{\partial y^j} \right) \frac{\partial}{\partial y^j}_e$,

where we have used the fact that $\partial w^{x_i} / \partial y^j = 0$ and $\partial w^{y_j} / \partial x^i = 0$ (analogously for $w^{x_i}$ and $v^{y_j}$).11 So we have

$[\tilde{W}, \tilde{V}]_e = ([\tilde{W}_G, \tilde{V}_G], [\tilde{W}_H, \tilde{V}_H])$.

Then it follows that $\psi : \operatorname{Lie}(G \times H) \to \mathfrak{g} \oplus \mathfrak{h}$ is a Lie algebra isomorphism. \hfill $\square$

11This is because for $Z = \frac{\partial}{\partial x^i}, \tilde{Z}_{g \times k} = (L_{g \times k})_* Z = (L_{g \times k})_* \left( \frac{\partial}{\partial x^i} \right)_e = \frac{\partial z^{x_j}}{\partial x^i} \frac{\partial}{\partial x^j}_{g \times k} + \frac{\partial z^{y_j}}{\partial y^j} \frac{\partial}{\partial x^i}_{g \times k}$, where $L$ is the coordinate representation of $L_{g \times k}$. Since $L_{g \times k} = L_g \times L_h$, then clearly $\frac{\partial z^{x_j}}{\partial x^i} = 0$. From this we can see that $w^{x_i}$ and $w^{y_j}$ can only depend on $x^i$ and $y^j$ respectively.
Lemma 3.4. Suppose $G$ and $H$ are connected Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Let $F : G \to H$ be a Lie group homomorphism. Then $F$ is a smooth covering map if and only if the induced homomorphism $F_* : \mathfrak{g} \to \mathfrak{h}$ is an isomorphism.

The proof for Lemma 3.4 requires some extra machinery which we have not introduced in this paper. So for this reason, we shall not provide the proof here.\(^{12}\)

Now, we are ready to prove Theorem 3.2.

Proof. By Lemma 3.3, the Lie algebra of $G \times H$ is isomorphic to $\mathfrak{g} \oplus \mathfrak{h}$. Let $r \subseteq \mathfrak{g} \oplus \mathfrak{h}$ be the graph of $\phi$, namely

$$r = \{(X, \phi X) : X \in \mathfrak{g}\}.$$ 

Clearly, $r$ is a vector sub-space of $\mathfrak{g} \oplus \mathfrak{h}$. Because $\phi$ is a Lie algebra homomorphism, we then have

$$[(X, \phi X), (Y, \phi Y)] = ([X, Y], [\phi X, \phi Y]) = ([X, Y], \phi [X, Y]) \in r.$$ 

So $r$ is in fact a Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{h}$. Then by Theorem 3.1, there exists a unique connected Lie subgroup $R \subseteq G \times H$ whose Lie algebra is $r$.

Let $\pi_1 : G \times H \to G$ and $\pi_2 : G \times H \to H$ be the projection maps to $G$ and $H$ respectively. Obviously, $\pi_1$ and $\pi_2$ are Lie group homomorphisms, so it follows that $\Pi = \pi_1|_R : R \to G$ is also a Lie group homomorphism. We shall show that $\Pi$ is a smooth covering map in the next step. Since $G$ is simply connected, then it will follow that $\Pi$ is bijective, and therefore is a Lie group isomorphism.\(^{13}\)

In order to show that $\Pi$ is a smooth covering map, it is sufficient to show that its induced Lie algebra homomorphism $\Pi_* \mathfrak{g}$ is an isomorphism according to Lemma 3.4. Consider the following sequence of maps

$$R \hookrightarrow G \times H \xrightarrow{\pi_1} G.$$ 

The composition of these maps is exactly $\Pi$. Thus by Lemma 1.5, it follows that the induced Lie algebra homomorphism $\Pi_*$ is just the composition of the maps

$$r \hookrightarrow \mathfrak{g} \oplus \mathfrak{h} \xrightarrow{\varpi_1} \mathfrak{g},$$

where $\varpi_1$ denotes the projection $\mathfrak{g} \oplus \mathfrak{h} \to \mathfrak{g}$. This implies that $\Pi_*$ is just the projection $\varpi_1 : \mathfrak{g} \oplus \mathfrak{h} \to \mathfrak{g}$ restricted to $r$. Since $r$ is the graph of $\phi$, then $r \cap \mathfrak{h} = \{0\}$. So $\Pi_* = \varpi_1|_r \circ \mathfrak{g}$ is an isomorphism. Therefore, by Lemma 3.4, $\Pi$ is a smooth covering map of the simply connected Lie group $G$ and thus a Lie group isomorphism.

Now define a Lie group homomorphism $\Phi : G \to H$ by $\Phi = \pi_2|_R \circ \Pi^{-1}$. Since $\Pi = \pi_1|_R$, then according to the definition of $\Phi$, we have

$$\pi_2|_R \circ \Pi^{-1} = \Phi \circ \varpi_1|_r.$$ 

Since the Lie algebra homomorphism induced by the projection $\pi_2 : G \times H \to H$ is just the projection $\varpi_2 : \mathfrak{g} \oplus \mathfrak{h} \to \mathfrak{h}$, then by Lemma 1.5, we have

$$\varpi_2|_r = \Phi_* \circ \varpi_1|_r : \mathfrak{r} \to \mathfrak{h}.$$ 

\(^{12}\)For details of the proof, please refer to chapter 9 of Lee’s Introduction to Smooth Manifolds.

\(^{13}\)This follows from the property of covering maps of simply connected spaces: If $X$ is a simply connected space, then any covering map $\pi : \tilde{X} \to X$ is a homeomorphism. For proofs, please refer to Lee’s Introduction To Topological Manifolds.
Therefore, for any $X \in \mathfrak{g}$, we have
\[
\phi X = \omega_2|_r(X, \phi X) = \Phi_* \circ \omega_1|_r(X, \phi X) = \Phi_* X.
\]
This shows that $\Phi_* = \phi$. \hfill \Box

By invoking Theorem 3.2, we shall finally reach the ultimate goal of this section.

**Corollary 3.5.** Let $G$ be a simply connected Lie group and $\mathfrak{g}$ be its Lie algebra. If $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of the Lie algebra $\mathfrak{g}$ on a finite-dimensional vector space $V$, then there exists a representation $\Phi : G \to GL(V)$ of the Lie group $G$ on the same vector space $V$ such that the corresponding representation $\Phi_*$ of $G$ on $V$ is exactly $\phi$.

**Proof.** By Theorem 2.4, we know that the Lie algebra of $GL(V)$ is exactly $\mathfrak{gl}(V)$. Since the representation $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra homomorphism, then by Theorem 3.2, there exists a Lie group homomorphism $\Phi : G \to GL(V)$ such that $\Phi_* = \phi$. Then the map $\Phi$ is the desired representation of Lie group $G$ on the vector space $V$. \hfill \Box

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**References**