KHOVANOV HOMOLOGY AS AN INVARIANT

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Abstract. This paper will give a brief overview of the Jones polynomial as a link invariant before introducing Khovanov Homology. Khovanov homology categorizes the Jones polynomial by generalizing the Kauffman bracket upon which the Jones Polynomial is based to an exact chain complex of graded vector spaces whose homology forms the Khovanov invariant. The calculation of the chain complex formed and the calculation of the Khovanov homology derived from the chain complex will be explained. Finally this paper will prove that the Khovanov homology is a link invariant under Reidemeister moves.

This paper relies heavily on the work of Paul Turner and the work of Dror Bar-Natan on Khovanov homology and will use some of their diagrams as well as much of their terminology.

Contents

1. The Jones Polynomial

Any knot is an embedding of S^1 into S^3 that can be represented by its projection to the plane. A link L is an assembly of knots with mutual entanglements in three space. The planar projection of L has a set of crossings, $\mathcal{X}$, where $n \in \mathbb{N}$ such that $n = |\mathcal{X}|$, the number of crossings in L. When resolving a knot diagram, an orientation must first be chosen. Based on the arbitrary orientation, the right-handed crossings are assigned n_+ and the left-handed crossings are assigned n_- (as shown below).

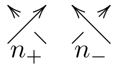


Diagram 1.1

So $n_- + n_+ = n = |\mathcal{X}|$. Further by convention, we define a 0 smoothing and a 1 smoothing as follows:

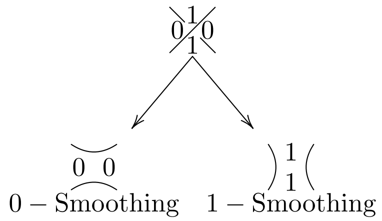


Diagram 1.2

The Kauffman bracket, from which the Jones polynomial is derived recursively, follows from three axioms:

(1) $\langle \emptyset \rangle = 1$

(2) $\langle \bigcirc \coprod L \rangle = (q+q^{-1})\langle L \rangle = \langle k \text{ circles in the plane} \rangle = (q+q^{-1})^k$

(3) $\langle \times \rangle = \langle \asymp \rangle - q\langle)\,(\rangle$

From these axioms, the unormalized Jones Polynomial is

$$\hat{J} = (-1)^{n_-} q^{n_+ - 2n_-} \tag{1.1}$$

and the Jones polynomials is

$$J(L) := \frac{\hat{J}(L)}{q+q^{-1}}. \tag{1.2}$$

To calculate the unormalized Jones Polynomial of a link L we use the above naming conventions to smooth each crossing, $\alpha \in \{0,1\}^{\mathcal{X}}$ in an n-dimension cube $\{0,1\}^{\mathcal{X}}$ to create a set, S_α which is the *complete smoothing* of L (the union of planar cycles). Each union S_α of k cycles is replaced by a term of the form

$$(-1)^{r_\alpha} q^{r_\alpha} (q+q^{-1})^k.$$

Where r_α is the height of the smoothing (or the number of 1 smoothings used). The sum of the resulting terms over all verticies of the cube is then multiplied by the normalization term of $(-1)^{n_-} q^{n_+ - 2n_-}$. However useful the Jones polynomial has been it has some defects, including failure to detect the unknot, that Khovanov attempts to rectify.

2. KHOVANOV HOMOLOGY

In search of a stronger knot invariant than the Jones polynomial, Khovanov develops a way to take an oriented link diagram, L, and transform the diagram into a *bi-graded chain complex*, $\mathcal{C}^{*,*}(L)$, the homology of which is the Khovanov homology of L, $\mathcal{H}(L)$. In the form of a diagram:

$$L \xrightarrow{Khovanov} \mathcal{C}^{*,*}(L) \xrightarrow{homology} \mathcal{H}(L)$$

In essence, by replacing the variable q with this chain complex, an algebraic object, he is *categorifying* the unnormalized Jones polynomial. Khovanov homology enjoys the following properties which will be proven through this paper.

1. The graded Euler characteristic is the unnormalised Jones polynomial.
2. If L is related to another diagram, L' by Reidemeister moves, then there exists an isomorphism
$$\mathcal{H}(L) \cong \mathcal{H}(L').$$

Therefore Khovanov homology is a knot invariant.

The following definitions will be useful in understanding the process of categorification that leads to the Khovanov invariant.

Definition 2.1. A *chain complex* is a sequence of homomorphisms of abelian groups with *differential maps* ∂_n

$$\cdots \to \mathcal{C}_{n+1} \xrightarrow{\partial_{n+1}} \mathcal{C}_n \xrightarrow{\partial_n} \mathcal{C}_{n-1} \to \cdots \to \mathcal{C}_1 \xrightarrow{\partial_1} \mathcal{C}_0 \xrightarrow{\partial_0}\to 0$$

where $\partial_n \partial_{n+1} = 0$ for each n. Notice this gives that the image of ∂_{n+1} is a subset of the Kernel of ∂_n; $\operatorname{Im}\partial_{n+1} \subset \operatorname{Ker}\partial_n$.

Definition 2.2. The *homology* of a chain complex is the set of modules, $H^n(\mathcal{C})$ given by

$$H^n(\mathcal{C}) = \frac{\operatorname{Ker}\partial_n}{\operatorname{Im}\partial_{n+1}}.$$

Definition 2.3. The elements of $\operatorname{Ker}\partial_n$ are called cycles; in other words they are the elements of the complex that go to zero.

Definition 2.4. The elements of $\operatorname{Im}\partial_{n+1}$ are boundaries.

Khovanov's method is to replace the Kauffman bracket of a link projection, $\langle L \rangle$, with the Khovanov bracket $[\![L]\!]$. This bracket generalizes the Kauffman bracket of L to an exact chain complex for graded vector spaces where graded Euler characteristic $\langle L \rangle$ is the Jones polynomial and follows from the axioms:

(1) $[\![\emptyset]\!] = 0 \to \mathbb{Z} \to 0$

(2) $\llbracket \bigcirc \coprod L \rrbracket = V \otimes \llbracket L \rrbracket$
(3) $\llbracket \times \rrbracket = \mathcal{F}(0 \to \llbracket \asymp \rrbracket \xrightarrow{d} \llbracket)\ (\rrbracket \{1\} \to 0)$.

So, where V is a vector space with graded dimension $q + q^{-1}$, $\{1\}$ is a degree shift by 1 operation parallel to the multiplication by q. $\mathcal{F}$ is the flatten operation defined as taking a double complex to a single complex through direct sums on the diagonals of the cube $\{0,1\}^{\mathcal{X}}$. This cube is formed by the 2^n *smoothings*, a smoothing being a resolution by either a 0 or 1 smoothing as defined by the above convention. Each of the smoothings can be indexed by a word of n zeros and ones such that an element of the hyper-cube, $\alpha \in \{0,1\}^n$ is associated with a smoothing and one of these words. So, the Khovanov invariant $\mathcal{H}(L)$ is the homology of a similar renormalization $\llbracket L \rrbracket[-n_-]\{n_+ - 2n_-\}$ of the Khovanov Bracket where $[\cdot]$ and $\{\cdot\}$ are height and degree shifts as defined later in the paper.

Khovanov homology categorifies the Jones polynomial for the purpose of the benefits of a homology. Some of these benefits include

- homology is a stronger invariant than the Euler characteristic
- as will soon be shown, Khovanov homology is a stronger invariant than even the Jones polynomial
- homology reveals richer information (contains more information specific to the knot in the definition of the homology).
- homology is a *functor*

3. CALCULATION

The Khovanov invariant results from a table summarizing the homology induced by Khovanov's categorification of the unormalized Jones polynomial. This homology can be determined once the spaces are determined as well as the differential maps between the spaces, which together form the chain complex.

3.1. **Khovanov Complex.** Khovanov's categorification replaces the polynomials of most link invariants by graded vector spaces to form a homological object. In order to understand how these spaces are formed there are some basic definitions to be understood which determine the categorification of the Jones polynomial.

Definition 3.1. Let $W = \bigoplus_m W_m$ be a graded vector space with homogeneous components $\{W_m\}$. The graded dimension of W is the power series $q\dim W := \sum_m q^m \dim W_m$.

Definition 3.2. Let $\cdot\{l\}$ be the "degree shift" operation on graded vecor spaces. That is, if $W = \bigoplus_m W_m$ is a graded vector space, we set $W\{l\}_m := W_{m-l}$, so that $q\dim W\{l\} = q^l q\dim W$.

Definition 3.3. Likewise, let $\cdot[s]$ be the "height shift" operation on chain complexes. That is, if $\overline{\mathcal{C}}$ is a chain complex $\cdots \to \overline{\mathcal{C}}^{r_\alpha} \xrightarrow{d^r_\alpha} \overline{\mathcal{C}}^{r_\alpha+1} \dots$ of (possibly graded) vector spaces (we call r_α the "height" of a piece $\overline{\mathcal{C}}^{r_\alpha}$ of that complex), and if $\mathcal{C} = \overline{\mathcal{C}}[s]$, then $\mathcal{C}^{r_\alpha} = \overline{\mathcal{C}}^{r_\alpha - s}$ (with all differentials shifted accordingly).

Proposition 3.4. *The graded dimension satisfies the following:*

$$q\dim(W \otimes W') = q\dim(W) q\dim(W'),$$
$$q\dim(W \oplus W') = q\dim(W) + q\dim(W').$$

Definition 3.5. *Khovanov homology* follows from this procedure to calculate the Khovanov Complex of a link.

(1) Let $\mathbb{Q}[1,x]$, which we will call V, be a graded vector space with two basis elements, 1 with degree 1 and x with degree -1. Thus $q\dim V = q + q^{-1}$.
(2) A cube is formed from the complete smoothing of the link where each vertex is associated with a smoothing.
(3) We then define the cube $\{0,1\}^n$ with each vertex α associated with a group in the chain complex as describe below.

$$V_\alpha := V^{\otimes k_\alpha}\{r_\alpha + n_+ - 2n_-\} \quad (3.6)$$

where r_α is the number of 1's in the word denoting the smoothing associated with the vertex α (the "height" of α). And k_α is the number of circles in the plane of the smoothing at α (these circles represent the cycles in the smoothing).

(4) The chain group at the height of r_αth where $i = -n_-, \ldots, n_+$ is the r_αth chain group $[\![L]\!]^{r_\alpha}$ with $i = -n_-, \ldots, n_+$ is the direct sum of all vector spaces at height r_α:

$$\mathcal{C}(L)^{i,*} := \bigoplus_{r_\alpha = i+n_-} V_\alpha. \tag{3.7}$$

So the graded Euler characteristic $\chi_q(\mathcal{C})$ of chain complex $\mathcal{C}$ is the alternating sum of graded dimensions of the chain groups. Later it will be proven that $\mathcal{C}$ has a differential, degree 0, so in the following prove that will be granted.

Definition 3.8. Given a bi-graded chain complex $\mathcal{C}^{*,*}$, the graded euler characteristic of the chain complex is

$$\chi_q(\mathcal{C}(L)) := \sum (-1)^i q^j \dim(\mathcal{C}^{i,j}).$$

Theorem 3.9. *The graded Euler characteristic of $\mathcal{C}(L)$ is the unnormalized Jones polynomial of L:*

$$\chi_q(\mathcal{C}(L)) = \hat{J}(L).$$

Proof. This statement is trivial by design. Homology *categorifies* the Euler characteristic; in other words, the graded Euler characteristic replaces a number with a graded vector space whose dimension is the original number. It is clear that the degree and height shift operations used in the categorification of the Jones polynomial are exactly designed so this theorem holds. ☐

3.2. **Maps.** A few definitions are needed preceding a general definition of the maps between graded vector spaces the result of which brings a final summary of the homology of link L.

Definition 3.10. An element of $\mathcal{C}^{i,j}(L)$ has *homological grading* i and *q-grading* j. So, if $v \in V_\alpha \subset \mathcal{C}^{*,*}(L) = C(L)$ with homological grading i and q-grading j then

$$\begin{aligned} i &= r_\alpha - n_- \\ j &= \deg(v) + i + n_+ - n_- \end{aligned}$$

Basically, since $\mathcal{C}^{*,*}$ is a bi-graded chain complex and the formal sum of all V_α, any element $v \in V_\alpha \subset \mathcal{C}^{*,*}(L)$ has two gradings. The i grading is a calculated constant at height r_α ($i = r_\alpha - n_-$). The j grading is associated with the degree of v, so if $v \in \mathbb{Q}[1, x]$ then v has a computable degree upon which the j degree depends such that $j = \deg(v) + i + n_+ - n_-$.

Definition 3.11. A *cobordism* is an orientable surface whose boundary is a union of the circles in the smoothings at either end

Notice for each edge of the cube there exists a cobordism as each vertex $\alpha \in \{0,1\}^n$ is associated with a smoothing (basically a collection of circles). The edges of the cube are labeled by a string of $n-1$ zeros and ones with a $(*)$ in the position that changes from the tail of the edge to the head. By convention, the $* = 0$ in the word associated with the vertex α at the tail and $* = 1$ in the word at the vertex of the head.

Remark 3.12. By convention pictures of cobordisms go *down* the page.

From the edge denoted ξ connecting $\alpha \xrightarrow{\xi} \alpha'$ a linear map $d_\xi : V_\alpha \to V_{\alpha'}$ is formed. To define d^i (a sum of all the d_ξ with tail α of the same height) two linear maps are needed since each circle in the smoothing requires a copy of vector space V and there exist two smoothings. The map $m : V \otimes V \to V$ merges to cycles and $\Delta : V \to V \otimes V$ splits two cycles. Define d_ξ to be the identity for circles that do not change and otherwise perform m and Δ appropriately.

Define $m : V \otimes V \to V$ by

$$\begin{aligned} &1 \otimes 1 = 1 \\ &1 \otimes x = x \otimes 1 = x \\ &x \otimes x = 0 \end{aligned}$$

and $\Delta : V \to V \otimes V$ by

$$\Delta(1) = 1 \otimes x + x \otimes 1$$
$$\Delta(x) = x \otimes x.$$

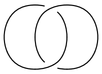
Now, to define $d^i : \mathcal{C}^{i,*}(L) \to \mathcal{C}^{i+1,*}(L)$ for $v \in V_\alpha \subset \mathcal{C}^{i,*}(L)$

$$d^i := \sum_{\xi : \text{Tail}\,(\xi) = \alpha} \text{sign}\,(\xi) d_\xi(v) \tag{3.13}$$

where $\text{sign}\,(\xi) = (-1)^{\text{ number of 1s to the left of } (*) \text{ in } \xi}$.

4. Example

To demonstrate the calculation of Khovanov homology, consider the Hopf link–its projection follows.



First: Orient the diagram, which we will call L, arbitrarily by choosing a direction that remains constant throughout the diagram. This is shown in Diagram 4.1 by the arrows.

Second: Label the crossings n_- and n_+ with regards to Diagram 1.1 to get the following labeled and oriented link diagram.

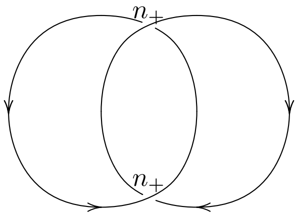
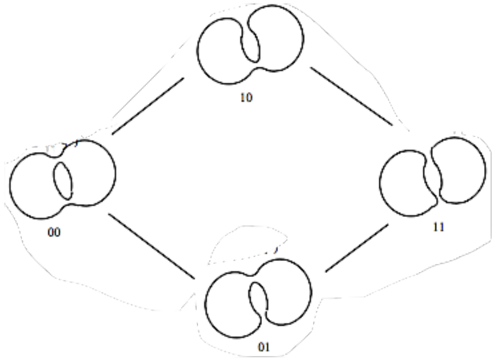


Diagram 4.1

Third: We define, by convention, the smoothings based on Diagram 1.2

Fourth: Construct the cube of smoothings as follows

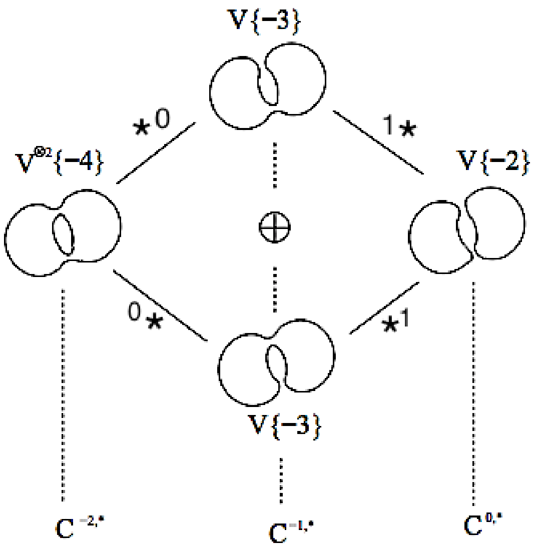


[3]

Diagram 4.2

Consider $V = \mathbb{Q}[1, x]$ where $\deg(1) = 1$ and $\deg(x) = -1$. Define the map $m : V \otimes V \to V$ such that $1 \otimes 1 = 1$, $1 \otimes x = x \otimes 1 = x$ and $x \otimes = 0$ and the map $\Delta : V \to V \otimes V$ such that $\Delta(1) = 1 \otimes x + x \otimes 1$ and $\Delta(x) = x \otimes x$. In this example $n_+ = 0$ and $n_- = 2$ with a total $n = 2$ clearly.

Next we can calculate the V_α using Equation 3.5; at the first smoothing associated with the word 00 we can see $k_\alpha = 2$, there are 2 circles in the smoothing, and $r_\alpha = 0$, there are zero 1s in the word associated with the smoothing. Therefore $V_0 = V^{\otimes 2}\{0 + 0 - 2(2)\} = V^{\otimes 2}\{-4\}$. We can similarly calculate the other spaces and create the following diagram.



[3]

Diagram 4.3

Using Equation 3.12 we can then define the maps that take each cube to the next cube such that $d^i = \mathcal{C}^{i,*}(L) \rightarrow \mathcal{C}^{i+1,*}(L) \ldots$. Then we have the exact chain complex: $0 \rightarrow \mathcal{C}^{-2,*}(L) \rightarrow \mathcal{C}^{-1,*}(L) \rightarrow \mathcal{C}^{0,*}(L) \rightarrow 0$ or more precisely summarized in the diagram.

$$\begin{tikzcd}
 & V\{-3\} \arrow[dr, "-\Delta"] & \\
(V \otimes V)\{-4\} \arrow[ur, "m"] \arrow[dr, "m"'] & \oplus & (V \otimes V)\{-2\} \\
 & V\{-3\} \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

[3]

Diagram 4.4

To find the homology, and finally summarize the homology by the homology degree and q-degree table, we must examine the complexes at the three different heights. We will issolate the cycles which go to zero, the basis for their boundaries, and determine the homology which is the the cycles mod the boundaries. Basically the homology at each height preserves the cycles that are not “killed” by the boundaries.

In this example, at the first height ($r_\alpha = 0$) $\mathcal{C}^{-2,*}(L)$ lives in $\mathbb{Q}[1,x] \otimes \mathbb{Q}[1,y]$, so we use the definition of m to find the cycles that go to zero. We can see that $x \otimes x$ goes to zero by definition of m since $x \otimes x = 0$, and $1 \otimes x - x \otimes 1 = x - x = 0$. These are clearly the only cycles that go to zero as all cycles must live in $\mathbb{Q}[1,x] \otimes \mathbb{Q}[1,y]$ and no other permutation of its elements go to zero by definition of m. To find the boundary basis at height $r_\alpha = 0$ we consider the complex before $\mathcal{C}^{-2,*}(L)$ in the chain since we are considering $\mathcal{C}^{-2,*}(L) \rightarrow 0$ when we consider cycles. However, $0 \rightarrow \mathcal{C}^{-2,*}(L) \rightarrow 0$ at this height, so the boundaries are empty. Therefore the homology is just $\mathbb{Q}[1,x] \oplus \mathbb{Q}[1,x]$.

At the second height ($r_\alpha = 1$) the complex lives in $\mathbb{Q}[1,x] \oplus \mathbb{Q}[1,x]$, so based on the definition of Δ we can see (x,x) is a cycle because

$$\begin{aligned}
(x,x) &= \Delta(x) - \Delta(x) \text{ by the alternating sum definition of the complex} \\
&= x \otimes x - x \otimes x \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

and $(1,1)$ is a cycle because

$$\begin{aligned}(1,1) &= \Delta(1) - \Delta(1)\\ &= 1\otimes x + x\otimes 1 - (1\otimes x + x\otimes 1)\\ &= 0.\end{aligned}$$

We consider the boundaries to be the results of the cycles of $\mathcal{C}^{-2,*}(L)$ which are $(1,1)$ and (x,x) making the homology clearly empty.

At the third height ($r_\alpha = 2$) the complex lives in $\mathbb{Q}[1,x]\otimes\mathbb{Q}[1,x]$ and we can see everything in $\mathcal{C}^{0,*}(L)$ goes to zero because $\mathcal{C}^{0,*}(L)\to 0$ by the definition of the chain complex. Therefore the cycles include $\{1\otimes 1, 1\otimes x, x\otimes 1, x\otimes x\}$. The boundaries can be found as before to be $\{1\otimes x + x\otimes 1, x\otimes x\}$. To find the homology though we can consider another boundary basis that also works, namely $\{1\otimes 1, 1\otimes x + x\otimes 1, x\otimes 1, x\otimes x\}$. We can arbitrarily choose to include $x\otimes 1$ instead of $1\otimes x$ and it will be shown that this choice does not affect the final answer as the results only rely on the i and j components rather than the differences in elements of the homology. Therefore we can see that both $x\otimes x$ and $x\otimes 1$ are killed leaving the homology to be $\{1\otimes 1, 1\otimes x\}$.

We can summarize the results in the following tables.

Homological degree	-2	-1	0
Cycles	$\{1\otimes x - x\otimes 1, x\otimes x\}$	$\{(1,1),(x,x)\}$	$\{1\otimes 1, 1\otimes x, x\otimes 1, x\otimes x\}$
Boundaries	-	$\{(1,1),(x,x)\}$	$\{1\otimes x + x\otimes 1, x\otimes x\}$
Homology	$\{1\otimes x - x\otimes 1, x\otimes x\}$	-	$\{1\otimes 1, 1\otimes x\}$
q-degrees	$-4, \quad -6$		$0, \quad -2$

i j	-2	-1	0
0			$\mathbb{Q}$
-1			
-2			$\mathbb{Q}$
-3			
-4	$\mathbb{Q}$		
-5			
-6	$\mathbb{Q}$		

5. THE MAIN THEOREM

Proposition 5.1. *The sequences* $[\![L]\!]$ *and* $\mathcal{C}(L)$ *are chain complexes.*

Proof. We can see that the cube commutes taking all maps with no signs. So we will prove that $d^{i+1}\circ d^i = 0$ for each differential, proving the proposition. This can be done by evaluating the combinations of functions and the definitions of those functions.

Define f to be $m\circ\Delta = f$. Then either of the following two possibilities arise:

$$\begin{aligned}f(1) &= m(1\otimes x) + m(x\otimes 1)\\ &= m(1\otimes x) + m(1\otimes x)\\ &= x + x\\ &= 0 \text{ because } x^2 = 0,\ \text{therefore } x = 0 = x + x\end{aligned}$$

or

$$\begin{aligned}f(x) &= m(x\otimes x)\\ &= 0.\end{aligned}$$

So, for the composition function f, all possible combinations of inputs result in 0.

Define g to be the composition $\Delta \circ m = g$. Then the following four possibilities arise:

$$\begin{aligned} g(1 \otimes 1) &= \Delta(1) \\ &= 1 \otimes x + x \otimes 1 \\ &= x + x \\ &= 0 \end{aligned}$$

or

$$\begin{aligned} g(x \otimes 1) = g(1 \otimes x) &= \Delta(x) \\ &= x \otimes x \\ &= 0 \end{aligned}$$

or, finally,

$$\begin{aligned} g(x \otimes x) &= \Delta(0) \\ &= 0. \end{aligned}$$

So, for the composition function g, all possible combinations of inputs result in 0. Therefore, for the two possible composition functions (disregarding sign), all outputs are 0 and $d^{i+1} \circ d^i = 0$. This completes the verification that a chain complex is formed by this commutative cube. ☐

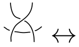
Define $Kh(L)$ as the Poincar polynomial of the complex $\mathcal{C}(L)$ in the variable t where $\mathcal{H}^r(L)$ is the rth *cohomology* of the graded vector space $\mathcal{C}(L)$.

$$Kh(L) := \sum_r t^r q \dim \mathcal{H}^r(L). \tag{5.2}$$

Theorem 5.3 (Khovanov)**.** *The graded dimensions of the homology groups $\mathcal{H}^r(L)$ are link invariants, and hence $Kh(L)$, a polynomial in the variables t and q, is a link invariant that specializes the unnormalised Jones polynomial at $t = -1$.*

Remark 5.4. The below diagrams are from Bar-Natan so all will use V as a vector space with elements v_+ and v_- as the more general form of what this paper uses $V = \mathbb{Q}[1, x]$. In both one element has degree 1, in the former v_+ and in the latter 1, and another element has degree -1, v_- and x.

5.1. **Proof.** The main theorem holds if H^r is invariant under Reidemeister moves so that $H^r(L) = H^r(L')$ for L and L' are related by a series of the three Reidemeister moves that follow.

$\Omega 1$  $\leftrightarrow$ 
$\Omega 2$  $\leftrightarrow$ 
$\Omega 3$  $\leftrightarrow$ .

We approach proving invariance in the way invariance is approached for the Kauffman bracket, reducing the "complicated case," but since we are dealing with complicies and homologies instead of polynomials the language must be different, so we will use the following "cancellation principle".

Lemma 5.5. *Let $\mathcal{C}$ be a chain complex and let $\mathcal{C}' \subset \mathcal{C}$ be a subchain complex.*

- *If $\mathcal{C}'$ is acyclic (has no homology), then it can be "canceled." That is, in that case the homology $H(\mathcal{C})$ of $\mathcal{C}$ is equal to the homology $H(\mathcal{C}/\mathcal{C}')$ of $\mathcal{C}/\mathcal{C}'$.*
- *Likewise, if $\mathcal{C}/\mathcal{C}'$ is acyclic then $H(\mathcal{C}) = H(\mathcal{C}')$.*

Proof. Both assertions follow easily from the long exact sequence of homology groups

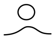
$$\ldots \longrightarrow H^r(\mathcal{C}') \longrightarrow H^r(\mathcal{C}) \longrightarrow H^r(\mathcal{C}/\mathcal{C}') \longrightarrow H^{r+1}(\mathcal{C}') \longrightarrow \ldots$$

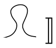
associated with the short exact sequence $0 \longrightarrow \mathcal{C}' \longrightarrow \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{C}' \longrightarrow 0$. Since this is a short exact sequence of chain complexes and these chain complexes are modules, we can apply the Snake Lemma [4] which gives rise to the former long exact sequence of homology groups. ☐

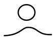
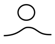
5.1.1. *Invariance Under* $\Omega 1$.

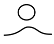
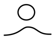
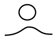
Proof. In our vector space $\mathbb{Q}[1, x]$, which we will call V, a twist that we would like to be isomorphic to the untwist gives raise to the following complex

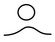
$\mathcal{C} = [\![$  $]\!] = ([\![$  $]\!] \xrightarrow{m} [\![$  $]\!]\{1\}).$

The two smoothed diagrams  and  each represent $(n-1)$-dimensional cubes as part of a larger n-dimensional cube. By construction m represents the differentials between the two cubes. Consider the subcomplex of $\mathcal{C}$:

$\mathcal{C}' = ([\![$  $]\!]_1 \xrightarrow{m} [\![$  $]\!]\{1\}).$

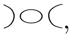
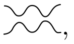
Where the subscript "1" means the subspace of $[\![$  $]\!]$ in which the small circle, or cycle, of  is always marked 1. Each cycle in a smoothing must be assigned an element of V as those elements are defined, so this case considers the special cycle (the small circle) as assigned 1.

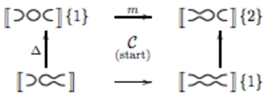
The goal is to find cycles in  that essentially "kill" chains in the boundary, . Since 1 is by definition designated the role of the multiplicative unit element in m, 1 "kills" the cycles in . So, the only homologies to be found lie in $[\![$  $]\!]$ which the small circle denoted x (else  would be acyclic). So we can consider

$\mathcal{C}/\mathcal{C}' = ([\![$  $]\!]_{/1=0} \to 0).$

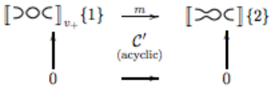
$V/(1=0)$ is generated only by x (making it one-dimensional), so with a simple shift in degrees we can say that by Lemma 5.5, since $\mathcal{C}/\mathcal{C}'$ is acyclic, $[\![$  $]\!]_{/1=0}$ is isomorphic to $[\![$  $]\!]$. This shift is made irrelevant by the shift $[-n_-]\{n_+ - 2n_-\}$ in the definition of the complex. Therefore, we have invariance under the first Reidemeister move because the move does not twist is isomorphic to the untwist developed by the first Reidemeister move. ☐

5.1.2. *Invariance Under* $\Omega 2$.

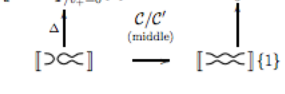
Proof. The second Reidemeister move can be represented in the following complex, $\mathcal{C}$, with the smoothing , , , .

$\mathcal{C} =$  [1]

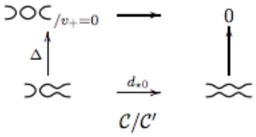
This depicts four cubes forming a larger cube, codimension 2. We intend to determine the cohomology elements using what we know about the differentials between these smaller cubes to check that they represent cohomologies of the larger cube. We can see a natural subcomplex of $\mathcal{C}$ as follows:

$\mathcal{C}' =$  [1]

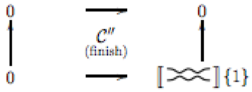
$\mathcal{C}'$ is clearly acyclic so we take the quotient

$\mathcal{C}/\mathcal{C}' =$  [1]

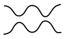
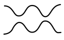
We will, though, for the purpose of simplicity and clarity, however, suppress the height shifts and brackets so

$\mathcal{C}/\mathcal{C}' =$  [1].

Remark 5.6. We can see a subcomplex of $\mathcal{C}/\mathcal{C}'$ is

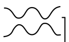
$\mathcal{C}_0''' =$  [1]

which we will want to prove is isomorphic to $(\mathcal{C}/\mathcal{C}')/\mathcal{C}'''$ as we will later define $\mathcal{C}'''$.

The map Δ is a bijection so we can define a new map $\tau :$  $_{v_+} \to$  in terms of its preimage composed with d_{*0} so that $\tau = d_{*0} \circ \Delta^{-1}$. Then $\mathcal{C}'''$ is a subcomplex of $\mathcal{C}/\mathcal{C}'$ containing all $\alpha \in$  and all ordered pairs $(\beta, \tau\beta) \in$  $_{/1=0} \oplus$ .

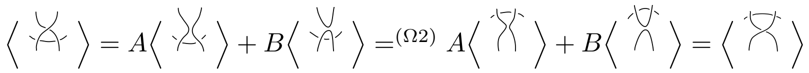
$$\begin{tikzcd} \beta \arrow[r] \arrow[dr, "\tau = d_{*0}\Delta^{-1}"] & 0 \\ \alpha \arrow[u, "\Delta"] \arrow[r, "d_{*0}", "\cdots"'] & \tau\beta \arrow[u] \end{tikzcd}$$

$\mathcal{C}''' =$ [1]

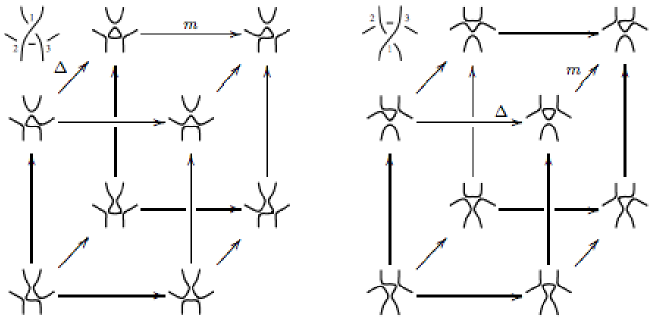
In this complex, since α was arbitrarily chosen, we killed the lower left corner of $\mathcal{C}/\mathcal{C}'$. We have similar freedom to choose β so this identifies all members in the upper left corner of the complex and some members in the lower right (according to the relation $\beta = \tau\beta$ [a relation meaning $(\beta, 0) = (0, \tau\beta)$ in  $_{/1=0} \oplus$ ]). Further we have choice of γ in the lower right corner so $(\mathcal{C}/\mathcal{C}')/\mathcal{C}'''$ is isomorphic to $\mathcal{C}_0'''$ as we first defined it. ☐

5.1.3. *Invariance Under* $\Omega 3$.

Proof. Invariance under the third Reidemeister move follows from invariance under $\Omega 2$ in the Kauffman bracket since



However, this proof does not work as smoothly for the Khovanov bracket, but the idea of using invariance under $\Omega 2$ is the starting point for this proof. First the smoothings of $\Omega 3$ are depicted by the following 3-dimensional cubes (again for convenience the bracket and height shift notation is suppressed).

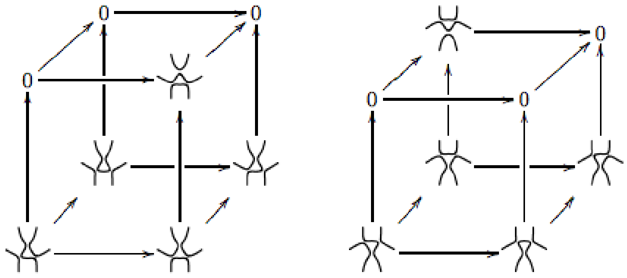


[1]

Figure 5.1

It is clear that the bottom layers of these cubes are partial smoothings  and , so these layers are isomorphic. The top layers seem to correspond accordingly to  and .

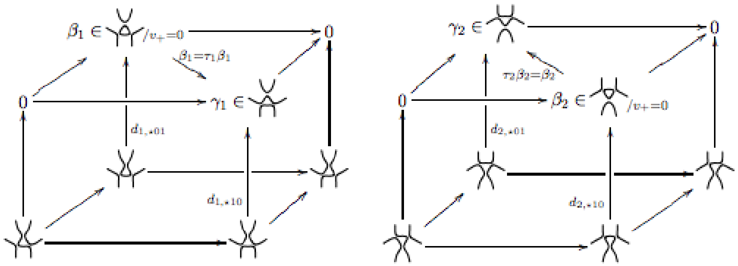
Warning 5.7. The layers do in fact correspond to the $\Omega 2$ smoothing, however this is useless. The cubes would reduce to the following cubes were the smoothing to be used.



[1]

These cubes are not isomorphic though. Even though individually their bottom layers are isomorphic and their top layers are isomorphic, the maps between the verticies are different making the cubes not isomorphic. Further the $\Omega 2$ reduction is invalid because, as a result of the existence of the bottom layers, the analog of $\mathcal{C}'''$ from the proof of $\Omega 2$ invariance (a necessary complex) is not a subcomplex. Instead new maps will be created as were created in the $\Omega 2$ invariance proof (where they were not necessary, but sufficient).

Now define $\mathcal{C}'$ and $\mathcal{C}'''$ as these subcomplexes were defined in the proof of $\Omega 2$ invariance, but within the top layers of each of the cubes of Figure 5.1 reference. Next mod out each cube by its $\mathcal{C}'$ and $\mathcal{C}'''$. This does not change the homology by Lemma 5.4. This results in the following cubes.



[1]

These complecies are now truly isomorphic via the map Υ which keeps the bottom layers in place while transposing the top layers with the map (β_1, γ_1) to (β_2, γ_2). Υ is an isomorphism on spaces level so we have what we wanted. ☐

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