SERRE DUALITY AND APPLICATIONS

JUN HOU FUNG

Abstract. We carefully develop the theory of Serre duality and dualizing sheaves. We differ from the approach in [12] in that the use of spectral sequences and the Yoneda pairing are emphasized to put the proofs in a more systematic framework. As applications of the theory, we discuss the Riemann-Roch theorem for curves and Bott's theorem in representation theory (following [8]) using the algebraic-geometric machinery presented.

Contents

1. Prerequisites

Studying algebraic geometry from the modern perspective now requires a somewhat substantial background in commutative and homological algebra, and it would be impractical to go through the many definitions and theorems in a short paper. In any case, I can do no better than the usual treatments of the various subjects, for which references are provided in the bibliography. To a first approximation, this paper can be read in conjunction with chapter III of Hartshorne [12]. Also in the bibliography are a couple of texts on algebraic groups and Lie theory which may be beneficial to understanding the latter parts of this paper.

Date: September 15, 2013.

A note on notation: we have chosen to adopt most of the notation from the source material. Thus the reader should consult the references, especially [12], if any of the notation is unfamiliar. Fortunately, there is a index of notation available.

1.1. **A crash course in sheaves and schemes.** In this section we review some of the terminology and notation used in this paper; the material is taken from chapter II of [12].

1.1.1. *Sheaves.* Let X be a topological space and $\mathcal{U}(X)$ the category of open sets in X whose morphisms are inclusions. Fix a concrete category $\mathcal{C}$. A $\mathcal{C}$-valued *presheaf* $\mathscr{F}$ is a contravariant functor from $\mathcal{U}(X)$ to $\mathcal{C}$. Usually $\mathcal{C}$ is **Ab** or **Ring**. If $\mathscr{F}$ is a presheaf and $P \in X$ is a point, the *stalk* $\mathscr{F}_P$ is defined to be the direct limit of the groups $\mathscr{F}(U)$ over all open sets containing P via the restriction maps.

A *sheaf* is a presheaf that satisfies the identity and gluing axioms. Given any presheaf $\mathscr{F}$, there is a construction called *sheafification* that gives a sheaf $\mathscr{F}^+$ satisfying the universal property that every morphism from $\mathscr{F}$ to a sheaf $\mathscr{G}$ factors uniquely through $\mathscr{F}^+$. We call $\mathscr{F}^+$ the *sheaf associated* to $\mathscr{F}$.

Let $(X, \mathscr{O}_X)$ be a ringed space. The main kind of sheaves we will work with are *sheaves of $\mathscr{O}_X$-modules*, or *$\mathscr{O}_X$-modules* for short. These are sheaves of abelian groups such that for each open set $U \subseteq X$, the group $\mathscr{F}(U)$ is an $\mathscr{O}_X(U)$-module, and such that the restriction morphisms $\mathscr{F}(U) \to \mathscr{F}(V)$ are compatible with the module structure induced by the ring homomorphism $\mathscr{O}_X(U) \to \mathscr{O}_X(V)$. The category $\mathbf{Mod}(X)$ of $\mathscr{O}_X$-modules is *abelian*, i.e., $\mathrm{Hom}(\mathscr{F}, \mathscr{G})$ has the structure of an abelian group for any two $\mathscr{O}_X$-modules $\mathscr{F}, \mathscr{G}$, and the composition is linear; finite direct sums, kernels, and cokernels exist; and so on. We also have tensor products.

An important example of an $\mathscr{O}_X$-module is the *sheaf* $\mathscr{H}om$, which takes

$$U \mapsto \mathrm{Hom}_{\mathscr{O}_X|_U}(\mathscr{F}|_U, \mathscr{G}|_U).$$

As with Ext, we define the functors $\mathscr{E}xt^i(\mathscr{F}, -)$ as the right derived functors of $\mathscr{H}om(\mathscr{F}, -)$, and we call these the *sheaf* $\mathscr{E}xt$. (But we cannot define $\mathscr{E}xt$ using $\mathscr{H}om(-, \mathscr{G})$, because we may not have enough projectives in the category.)

An $\mathscr{O}_X$-module is *free* if it is isomorphic to a direct sum of copies of $\mathscr{O}_X$, and it is *locally free* if X can be covered by open sets U for which $\mathscr{F}|_U$ is a free $\mathscr{O}_X|_U$-module. In that case, the *rank* of $\mathscr{F}$ on such an open set is the number of copies of the structure sheaf needed; if X is connected, the rank of a locally free sheaf is the same everywhere. An *invertible sheaf* is a locally free sheaf of rank one.

A *sheaf of ideals* on X is a quasicoherent $\mathscr{O}_X$-module $\mathscr{I}$ that is a subsheaf of $\mathscr{O}_X$. If Y is a closed subscheme (discussed below) of a scheme X, and $i : Y \to X$ is the inclusion morphism, the *ideal sheaf* of Y is $\mathscr{I}_Y = \ker(i^\sharp : \mathscr{O}_X \to i_*\mathscr{O}_Y)$.

Two important operations on sheaves are the direct image and inverse image constructions. Let $\mathscr{F}$ be a sheaf and $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ a map of ringed spaces. The *direct image* sheaf $f_*\mathscr{F}$ is defined by $(f_*\mathscr{F})(V) = \mathscr{F}(f^{-1}(V))$ for any open set $V \subseteq Y$. Moreover, if $\mathscr{F}$ is a sheaf of $\mathscr{O}_X$-modules, then $f_*\mathscr{F}$ is a $f_*\mathscr{O}_X$-module. The morphism $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_X$ gives $f_*\mathscr{F}$ the structure of an $\mathscr{O}_Y$-module.

If $\mathscr{G}$ is a sheaf on Y, let $f^{-1}(\mathscr{G})$ be the sheaf associated to the presheaf

$$U \mapsto \lim_{V \supseteq f(U)} \mathscr{G}(V).$$

Furthermore, if $\mathscr{G}$ is a sheaf of $\mathscr{O}_Y$-modules, then $f^{-1}\mathscr{G}$ is a $f^{-1}\mathscr{O}_Y$-module. Because of the adjoint property of f^{-1}, we have a morphism$f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$. We define the *inverse image* of $\mathscr{G}$ to be the tensor product $f^*\mathscr{G} = f^{-1}\mathscr{G} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$, which is an $\mathscr{O}_X$-module. The functors f^* and f_* are adjoint.

1.1.2. *Schemes.* An *affine scheme* $(X, \mathscr{O}_X)$ is a locally ringed space that is isomorphic to the spectrum of some ring. The *spectrum* of a ring A is the pair $(\operatorname{Spec} A, \mathscr{O})$ where the structure sheaf $\mathscr{O}$ is defined as follows. For an open set $U \subseteq \operatorname{Spec} A$, let $\mathscr{O}(U)$ be the ring of functions $s : U \to \bigsqcup_{\mathfrak{p} \in U} A_{\mathfrak{p}}$ such that $s(\mathfrak{p}) \in A_{\mathfrak{p}}$ and such that s is locally a quotient of elements of A. That is, for each $\mathfrak{p} \in U$, there is a neighborhood $V \subseteq U$ of $\mathfrak{p}$ and elements $a, f \in A$ such that for each $\mathfrak{q} \in V$ we have $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = \frac{a}{f}$.

A *scheme* is a locally ringed space $(X, \mathscr{O}_X)$ where each point has an open neighborhood U such that $(U, \mathscr{O}_X|_U)$ is an affine scheme. A scheme may have many interesting properties, in addition to notions such as connectedness, irreducibility, and quasicompactness on the underlying space. For example, a scheme is *integral* if the ring $\mathscr{O}_X(U)$ is an integral domain for every open set U. It is *locally noetherian* if it can be covered by affine open subschemes $\operatorname{Spec} A_i$ where each A_i is noetherian, and it is *noetherian* if it is locally noetherian and only finitely many $\operatorname{Spec} A_i$ is needed to cover X. When we come to Weil divisors, we will treat only schemes that are *regular in codimension one*, which means that every local ring $\mathscr{O}_{X,x}$ of X of dimension one is regular. In general, a scheme is *regular* if all local rings are regular. A scheme is *Cohen-Macaulay* if all local rings are Cohen-Macaulay.

Warning: while an open subscheme of $(X, \mathscr{O}_X)$ is simply a scheme $(U, \mathscr{O}_U)$ whose underlying space is an open subset of X and whose structure sheaf is isomorphic to $\mathscr{O}_X|_U$, the same definition does not work for closed subschemes. A *closed immersion* is a morphism $f : Y \to X$ of schemes such that f induces a homeomorphism of the underlying space of Y onto a closed subset of the underlying space of X and furthermore the induced map $f^\sharp : \mathscr{O}_X \to f_*\mathscr{O}_Y$ of sheaves on X is surjective. A *closed subscheme* of X is an equivalence class of closed immersions where $f : Y \to X$ and $f' : Y' \to X$ are equivalent if there is an isomorphism $\varphi : Y' \to Y$ such that $f' = f \circ \varphi$.

There are also many properties that morphisms of schemes may possess; we mention only a few. Let $f : X \to Y$ be a morphism of schemes. Then f is *locally of finite type* if there exists a covering of Y by open affine subsets $V_i = \operatorname{Spec} B_i$ such that for each i we have an open affine cover $\{U_{i,j} = \operatorname{Spec} A_{i,j}\}$ of $f^{-1}(V_i)$ with each $A_{i,j}$ finitely generated as a B_i-algebra, and f is of *finite type* if it is locally of finite type and for each i the cover $\{U_{i,j}\}$ can be chosen to be finite.

The morphism f is *separated* if the diagonal morphism $X \to X \times_Y X$ is closed, and it is *universally closed* if for all morphisms $Z \to Y$, the morphism $X \times_Y Z \to Z$ is closed. The map f is *proper* if it is of finite type, separated, and universally closed.

The morphism f is *projective* if it factors as a closed immersion into $\mathbb{P}^n_Y$, followed by the projection map $\mathbb{P}^n_Y \to Y$ for some n. It is *quasiprojective* if it factors as an open immersion followed by a projective morphism.

The morphism f is *flat at a point* $x \in X$ if the stalk $\mathscr{O}_{X,x}$ is flat as an $\mathscr{O}_{Y,f(x)}$-module viewed via $f^\sharp$. It is *flat* if it is flat at every point $x \in X$. If $\mathscr{F}$ is an $\mathscr{O}_X$-module, then $\mathscr{F}$ is *flat over* Y *at a point* $x \in X$ if the stalk $\mathscr{F}_x$ is a flat $\mathscr{O}_{Y,f(x)}$-module viewed via $f^\sharp$. The sheaf $\mathscr{F}$ is *flat* if it is flat at every point of X, and X is *flat* if $\mathscr{O}_X$ is so.

1.1.3. Proj *and twisted sheaves.* Let S be a graded ring. We construct a projective scheme $\operatorname{Proj} S$ as follows. As a set, $\operatorname{Proj} S$ consists of all homogeneous prime ideals in S that do not contain S_+, the ideal of all positively-graded elements. If $\mathfrak{a}$ is a homogeneous ideal, let $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Proj} S : \mathfrak{p} \supseteq \mathfrak{a}\}$. The sets $\{V(\mathfrak{a})\}$ form the closed sets of the Zariski topology on $\operatorname{Proj} S$. Define a sheaf of rings on $\operatorname{Proj} S$ by letting $\mathscr{O}(U)$ be the set of functions $s : U \to \bigsqcup_{\mathfrak{p}\in U} S_{(\mathfrak{p})}$ for each open set U, where $S_{(\mathfrak{p})}$ is the ring of degree zero elements in the localized ring $T^{-1}S$ where T is the multiplicative set consisting of all homogeneous elements of S not in $\mathfrak{p}$. The functions s should be locally a quotient of elements in S and satisfy $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$.

Next, let A be a ring and M an A-module. We can in the same vein construct the *sheaf associated to* M on $\operatorname{Spec} A$, denoted $\widetilde{M}$, as follows. For any open set $U \subseteq \operatorname{Spec} A$, let $\widetilde{M}(U)$ be the set of functions $s : U \to \bigsqcup_{\mathfrak{p}\in U} M_{\mathfrak{p}}$ such that $s(\mathfrak{p}) \in M_{\mathfrak{p}}$ for each $\mathfrak{p} \in U$ and such that s is locally a fraction. The sheaf $\widetilde{M}$ is a sheaf of $\mathscr{O}_X$-modules. Generally we say a sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$ is *quasicoherent* if X is covered by open affine subsets $U_i = \operatorname{Spec} A_i$ such that for each i there is an A_i-module M_i with $\mathscr{F}|_{U_i} \cong \widetilde{M}_i$, and $\mathscr{F}$ is *coherent* if it is quasicoherent and furthermore each M_i is finitely generated.

Now suppose S is a graded ring and $X = \operatorname{Proj} S$. For any $m \in \mathbb{Z}$, we define the twisted ring $S(m)$ to be S shifted m places to the left, i.e., $S(m)_d = S_{m+d}$. Then we define the sheaf $\mathscr{O}_X(m)$ to be $\widetilde{S(m)}$. The sheaf $\mathscr{O}_X(1)$ is called the *twisting sheaf of Serre* and generates an infinite cyclic group of invertible sheaves via the operation $\mathscr{O}_X(m) \otimes \mathscr{O}_X(n) \cong \mathscr{O}_X(m+n)$. In general, if $\mathscr{F}$ is any $\mathscr{O}_X$-module, we denote by $\mathscr{F}(m)$ the sheaf $\mathscr{F} \otimes_X \mathscr{O}_X(m)$.

1.1.4. *Divisors and the Picard group.* We introduce some terminology on divisors. Let X be a noetherian integral separated scheme that is regular in codimension one. A *prime divisor* on X is a closed integral subscheme of codimension one, and a *Weil divisor* is an element of the free abelian group $\operatorname{Div} X$ generated by the prime divisors. In the case where X is a curve, prime divisors are simply points Y on the curve, so a Weil divisor is a formal sum of points on X, e.g., $D = \sum n_i Y_i$, where $n_i \in \mathbb{Z}$. If all $n_i \geq 0$, we say the divisor D is *effective.*

If Y is a prime divisor, let η be its generic point. The local ring $\mathscr{O}_{X,\eta}$ is a DVR with quotient field K, the function field of X. Call the discrete valuation ν_Y the *valuation of* Y. If $f \in K^*$, then $\nu_Y(f)$ is an integer; if it is positive, we say f has a *zero* along Y and if it is negative, we say f has a *pole* along Y. We define the *divisor of* f to be $(f) = \sum \nu_Y(f) \cdot Y$, where the sum is taken over all prime divisors of X. This is a finite sum. Any divisor equal to the divisor of some function is called a *principal divisor*. Two divisors D and D' are *linearly equivalent*, written $D \sim D'$, if $D - D'$ is principal. A *complete linear system* on a nonsingular projective variety is the (possibly empty) set of all effective divisors linearly equivalent to some given divisor D_0, and it is denoted by $|D_0|$.

The group of divisors modulo principal equivalence is the *divisor class group* of X, denoted $\operatorname{Cl} X$. Next, for any ringed space X, we can define the *Picard group* of X, denoted $\operatorname{Pic} X$, to be the group of isomorphism classes of invertible sheaves on X with the operation $\otimes$. If X is a noetherian, integral, separated, locally factorial scheme, then $\operatorname{Cl} X$ and $\operatorname{Pic} X$ are naturally isomorphic.

1.1.5. *Sheaves of differentials and smooth varieties.* Let $f : X \to Y$ be a morphism of schemes, and let $\Delta : X \to X \times_Y X$ be the diagonal morphism. The image $\Delta(X)$ is a locally closed subscheme of $X \times_Y X$, i.e., a closed subscheme of an open subset W of $X \times_Y X$. Let $\mathscr{I}$ be the sheaf of ideals of $\Delta(X)$ in W. Define the *sheaf of relative differentials* of X over Y to be the sheaf $\Omega_{X/Y} = \Delta^*(\mathscr{I}/\mathscr{I}^2)$ on X. For example, if $X = \mathbb{A}^n_Y$, then $\Omega_{X/Y}$ is a free $\mathscr{O}_X$-module of rank n, generated by the global sections $dx_1, \ldots, dx_n$, where $x_1, \ldots, x_n$ are the affine coordinates for $\mathbb{A}^n_Y$.

If X is a variety over an algebraically closed field, then we say that X is *nonsingular* or *smooth* if all its local rings are regular local rings. There is a result that asserts that $\Omega_{X/k}$ is locally free of rank $\dim X$ iff X is nonsingular over k. (Compare with the case of smooth manifolds.) If X is a nonsingular variety over k, we define the *canonical sheaf* of X to be $\omega_X = \wedge^n \Omega_{X/k}$, where $n = \dim X$.

1.1.6. *Cohomology of sheaves.* Let $\mathscr{F}$ be a sheaf on a space X and $U \subseteq X$ an open subset. Define $\Gamma(U, \mathscr{F}) = \mathscr{F}(U)$. One way to approach the cohomology of sheaves is to examine the exactness properties of the *global section functor* $\Gamma(X, -) : \mathbf{Sh}(X) \to \mathbf{Ab}$. Briefly, $\Gamma(X, -)$ is left-exact, so we define the *cohomology functors* $H^i(X, -)$ to be the right derived functors of $\Gamma(X, -)$. For any sheaf $\mathscr{F}$, the groups $H^i(X, \mathscr{F})$ are the *cohomology groups* of $\mathscr{F}$.

While this definition via derived functors enjoys very nice theoretical properties, in practice one computes sheaf cohomology using the *Čech complex*. Let X be a topological space, and let $\mathfrak{U} = (U_i)_{i \in I}$ be an open covering of X. Fix a well-ordering of the index set I, and for any finite set of indices $i_0, \ldots, i_p$, denote the intersection $U_{i_0} \cap \cdots \cap U_{i_p}$ by $U_{i_0, \ldots, i_p}$. Let $\mathscr{F}$ be a sheaf on X. We define the complex $C^\bullet(\mathfrak{U}, \mathscr{F})$ as follows. For each $p \geq 0$, let $C^p(\mathfrak{U}, \mathscr{F}) = \prod_{i_0 < \cdots < i_p} \mathscr{F}(U_{i_0, \ldots, i_p})$. Then an element $\alpha \in C^p(\mathfrak{U}, \mathscr{F})$ is given by an element $\alpha_{i_0, \ldots, i_p} \in \mathscr{F}(U_{i_0, \ldots, i_p})$ for every $(p+1)$-tuple $i_0 < \cdots < i_p$ in I. We define the coboundary map $d : C^p \to C^{p+1}$ by setting

$$(d\alpha)_{i_0, \ldots, i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0, \ldots, \hat{i_k}, \ldots, i_{p+1}} |_{U_{i_0, \ldots, i_{p+1}}}.$$

It is easy to see that $d^2 = 0$. We use this cochain complex to define the pth *Čech cohomology group of $\mathscr{F}$ with respect to the covering* $\mathfrak{U}$, denoted $\check{H}^p(\mathscr{U}, \mathscr{F})$.

The investigation of sheaf cohomology is what we will concern ourselves in the next section, where we develop the some of the basic theory. For now, we give only a simple definition. A sheaf $\mathscr{F}$ is *flasque* if for every inclusion of open sets, the restriction map is surjective. The important thing about flasque sheaves is that if $\mathscr{F}$ is flasque, then $H^i(X, \mathscr{F}) = 0$ for all $i > 0$, i.e., $\mathscr{F}$ is *acyclic* for $\Gamma(X, -)$.

2. Serre duality theory

Duality is an indispensable tool both computationally and conceptually. In this section we will prove the following theorem about duality for coherent sheaves and discuss some of its generalizations.

Theorem 2.1 (Serre duality for $\mathbb{P}^n_k$)**.** *Let k be a field and $P = \mathbb{P}^n_k$ be projective n-space over k. Let ω_P be the sheaf $\mathscr{O}_P(-n-1)$ and let $\mathscr{F}$ be a coherent sheaf. Then for $0 \leq r \leq n$, the Yoneda pairing $H^r(P, \mathscr{F}) \times \operatorname{Ext}^{n-r}_P(\mathscr{F}, \omega_P) \to H^n(P, \omega_P)$ is perfect.*

As we will see, $H^n(P, \omega_P) \cong k$, so this gives a natural functorial isomorphism $\operatorname{Ext}_P^{n-r}(\mathscr{F}, \omega_P) \xrightarrow{\sim} H^r(P, \mathscr{F})^*$.

The proof of this theorem from scratch following the outline in [1] and [12] is rather long, and the individual components of the proof are not without independent interest, so we will explore each of them briefly in turn.

2.1. **The cohomology of projective space.** In this section we perform the explicit calculations of the cohomology of the line bundles $\mathscr{O}(m)$ on projective space, which will form the basis for all future computations. First, we need a result about the cohomology of quasicoherent sheaves on affine noetherian schemes.

Theorem 2.2. *Let X be an affine noetherian scheme and $\mathscr{F}$ a quasicoherent sheaf. Then for all $i > 0$, we have $H^i(X, \mathscr{F}) = 0$.*

So there is nothing much to say about cohomology on affine noetherian schemes. The case with projective schemes is a different story, however. First, there is a general result about the vanishing of higher cohomology groups given by Grothendieck.

Theorem 2.3 (Grothendieck vanishing)**.** *Let X be a noetherian topological space. For all $i > \dim X$ and all sheaves of abelian groups $\mathscr{F}$ on X, we have $H^i(X, \mathscr{F}) = 0$.*

Next, instead of doing the calculation of $H^i(\mathbb{P}^n, \mathscr{O}_X(m))$ for each sheaf individually, we consider the combined sheaf $\mathscr{F} = \bigoplus_{m \in \mathbb{Z}} \mathscr{O}_X(m)$. By keeping careful track of the grading, we will obtain the cohomology groups for each individual sheaf as wanted. Now, without further ado, we have the following description of the cohomology:

Theorem 2.4. *Let A be a noetherian ring, and S be the graded ring $A[x_0, \dots, x_n]$, graded by polynomial degree. Let $P = \operatorname{Proj} S = \mathbb{P}^n_A$ with $n \geq 1$. Then*
(a) The natural map $S \to \Gamma_*(\mathscr{O}_P) := \bigoplus_{m \in \mathbb{Z}} H^0(P, \mathscr{O}_P(m))$ is an isomorphism of graded S-modules, where the grading on the target is given by m.
(b) $H^n(P, \mathscr{O}_P(-n-1)) \cong A$.
(c) $H^i(P, \mathscr{O}_P(m)) = 0$ for $0 < i < n$ and all $m \in \mathbb{Z}$.

Notation 2.5. For the following proof and elsewhere in this paper, we will want to work locally. If S is a graded ring, there is a nice basis for the Zariski topology of $\operatorname{Proj} S$. Let S_+ be the ideal consisting of elements with positive degree, and let $f \in S_+$. Denote $D_+(f) = \{\mathfrak{p} \in \operatorname{Proj} S : f \notin \mathfrak{p}\}$. Then $D_+(f)$ is open in $\operatorname{Proj} S$, and furthermore these open sets cover $\operatorname{Proj} S$ and for each such open set there is a isomorphism $(D_+(f), \mathscr{O}|_{D_+(f)}) \cong \operatorname{Spec} S_{(f)}$ of locally ringed spaces where $S_{(f)}$ is the subring of degree 0 elements in S_f.

Proof of (a). Cover P with the open sets $\{D_+(x_i) : 0 \leq i \leq n\}$. Then, a global section $t \in \Gamma(P, \mathscr{O}_P(m))$ is specified by giving sections $t_i \in \Gamma(D_+(x_i), \mathscr{O}_P(m))$ for $0 \leq i \leq n$, which agree on the pairwise intersections $D_+(x_i x_j)$ for $0 \leq i, j \leq n$. The section t_i is just a homogeneous polynomial of degree m in the localization S_{x_i}, and its restriction to $D_+(x_i x_j)$ is the image of t_i in $S_{x_i x_j}$. So summing over all m, we see that $\Gamma_*(\mathscr{O}_P)$ can be identified with the set of $(n+1)$-tuples $(t_0, \dots, t_n)$, where $t_i \in S_{x_i}$ for each i, such that the images of t_i and t_j in $S_{x_i x_j}$ coincide.

Since the x_i are not zero divisors in S, the localization maps $S \to S_{x_i}$ and $S_{x_i} \to S_{x_i x_j}$ are all injective, and these rings are all subrings of $S_{x_0 \cdots x_n}$. Thus $\Gamma_*(\mathscr{O}_P)$ is the intersection $\bigcap_{i=0}^n S_{x_i}$ taken inside $S_{x_0 \cdots x_n}$, which is exactly S. In conclusion, we have proven that $H^0(P, \mathscr{O}_P(m))$ is the group of degree m homogeneous polynomials in the variables $x_0, \ldots, x_n$ if $m \geq 0$, and is the trivial group if $m < 0$. $\square$

We perform the remainder of the calculations using Čech cohomology. For $i = 0, \ldots, n$, let U_i be the open set $D_+(x_i)$. Then $\mathfrak{U} = \{U_i\}_{i=0}^n$ forms an open affine cover of P. Let $\mathscr{F}$ be the sheaf $\bigoplus_{m\in\mathbb{Z}} \mathscr{O}_P(m)$. For any set of indices $i_0, \ldots, i_p$, we have $U_{i_0,\ldots,i_p} := U_{i_0} \cap \cdots \cap U_{i_p} = D_+(x_{i_0} \cdots x_{i_p})$, so $\mathscr{F}(U_{i_0,\ldots,i_p}) \cong S_{x_{i_0}\cdots x_{i_p}}$.

Proof of (b). The cohomology group $H^n(P, \mathscr{F})$ is the cokernel of the map

$$d^{n-1} : \prod_{k=0}^{n} S_{x_0\cdots\hat{x_k}\cdots x_n} \to S_{x_0\cdots x_n}$$

in the Čech complex. Think of $S_{x_0\cdots x_n}$ as a free A-module with basis $x_0^{l_0} \cdots x_n^{l_n}$ with $l_i \in \mathbb{Z}$. The image of d^{n-1} is the free submodule generated by those basis elements for which at least one $l_i \geq 0$. Thus $H^n(P, \mathscr{F})$ is a free A-module with basis consisting of the negative monomials $\{x_0^{l_0} \cdots x_n^{l_n} : l_i < 0 \text{ for each } i\}$. This module has a natural grading given by $\sum_{i=0}^n l_i$. The only monomial of degree $-n-1$ in $H^n(P, \mathscr{F})$ is $x_0^{-1} \cdots x_n^{-1}$, so we see that $H^n(P, \mathscr{O}_P(-n-1))$ is a free A-module of rank one.

Indeed, we have proven a stronger result: for $r \geq 0$, $H^n(P, \mathscr{O}_P(-n-r-1))$ is the free A-module generated by $\{x_0^{l_0} \cdots x_n^{l_n} : l_i < 0, \sum_{i=0}^n l_i = -n-r-1\}$, which has rank $\binom{n+r}{n}$. $\square$

Proof of (c). We induct on the dimension n. If $n = 1$, then there is nothing to prove. So suppose $n > 1$ and that the result holds for $n - 1$. If we localize the Čech complex $C^\bullet(\mathfrak{U}, \mathscr{F})$ at x_n, we obtain a Čech complex for the sheaf $\mathscr{F}|_{U_n}$ on the space U_n, with respect to the open affine cover $\{U_i \cap U_n : 0 \leq i \leq n-1\}$. This complex computes the cohomology of $\mathscr{F}|_{U_n}$ on U_n. Since U_n is affine, theorem 2.2 says $H^i(U_n, \mathscr{F}|_{U_n}) = 0$ for all $i > 0$. Since localization is exact (in particular, it commutes with the quotient $H^i(P, \mathscr{F}) = \ker d^i / \operatorname{im} d^{i-1}$), we conclude that $H^i(P, \mathscr{F})_{x_n} \cong H^i(U_n, \mathscr{F}|_{U_n}) = 0$ for all $i > 0$. In other words, every element of $H^i(P, \mathscr{F})$, $i > 0$, is annihilated by some power of x_n. If we can show that multiplication by x_n induces a bijective map of $H^i(P, \mathscr{F})$ into itself for $0 < i < n$, then this will prove that $H^i(P, \mathscr{F}) = 0$.

Consider the exact sequence of graded S-modules

$$0 \to S(-1) \xrightarrow{x_n} S \to S/(x_n) \to 0.$$

The -1 indicates that the grading of the ring S is shifted by -1 (see the previous section), and reflects that multiplication by the degree 1 element x_n preserves the grading, e.g. x_0^2 is a degree 3 element in $S(-1)$, and $x_0^2 x_n$ has degree 3 in S. Taking the associated modules, we obtain an exact sequence of sheaves

$$0 \to \mathscr{O}_P(-1) \to \mathscr{O}_P \to \mathscr{O}_H \to 0$$

on P, where H is the hyperplane defined by the equation $x_n = 0$. Twisting by each $m \in \mathbb{Z}$ and taking the infinite direct sum, we have

$$0 \to \mathscr{F}(-1) \to \mathscr{F} \to \mathscr{F}_H \to 0,$$

where $\mathscr{F}_H = \bigoplus_{m\in\mathbb{Z}} \mathscr{O}_H(m)$. Clearly, we see from the definition that $\mathscr{F}(-1)$ can be identified with $\mathscr{F}$, but we will continue to write $\mathscr{F}(-1)$ for clarity. The long exact sequence in cohomology associated to this short exact sequence is

$$\begin{tikzcd}
0 \arrow[r] & S(-1) \arrow[r] & S \arrow[r] & S/(x_n) \arrow[dll, "\delta"] \\
& H^1(P,\mathscr{F}(-1)) \arrow[r, "x_n"] & H^1(P,\mathscr{F}) \arrow[r] & H^1(P,\mathscr{F}_H) \arrow[dll, "\delta"] \\
& H^2(P,\mathscr{F}(-1)) \arrow[r, "x_n"] & H^2(P,\mathscr{F}) \arrow[r] & H^2(P,\mathscr{F}_H) \arrow[dll, "\cdots"] \\
& H^{n-1}(P,\mathscr{F}(-1)) \arrow[r, "x_n"] & H^{n-1}(P,\mathscr{F}) \arrow[r] & H^{n-1}(P,\mathscr{F}_H) \arrow[dll, "\delta"] \\
& H^n(P,\mathscr{F}(-1)) \arrow[r, "x_n"] & H^n(P,\mathscr{F}) \arrow[r] & 0.
\end{tikzcd}$$

The long exact sequence stops because of Grothendieck's vanishing theorem. By induction, $H^i(P,\mathscr{F}_H) = 0$ for $0 < i < n-1$. So we immediately have isomorphisms $H^i(P,\mathscr{F}(-1)) \xrightarrow{x_n} H^i(P,\mathscr{F})$ for $1 < i < n-1$. It remains to show that they are isomorphisms even for $i = 1$ and $i = n-1$.

First, the map $S \to S/(x_n)$ is surjective, so $S/(x_n) \xrightarrow{\delta} H^1(P,\mathscr{F}(-1))$ is the zero map, and thus $H^1(P,\mathscr{F}(-1)) \xrightarrow{x_n} H^1(P,\mathscr{F})$ is an isomorphism.

Finally, consider the case $i = n-1$. The map $H^n(P,\mathscr{F}(-1)) \xrightarrow{x_n} H^n(P,\mathscr{F})$ is surjective, and we know from part (b) that $H^n(P,\mathscr{F})$ is the space of all negative monomials in the $n+1$ variables. So the kernel of this map is the free A-module generated by $\{x_0^{l_0}\cdots x_{n-1}^{l_{n-1}}x_n^{-1} : l_i < 0\} \subseteq H^n(P,\mathscr{F}(-1))$. Since $H^{n-1}(P,\mathscr{F}_H)$ is free on the generators $\{x_0^{l_0}\cdots x_{n-1}^{l_{n-1}}\}$ from part (b) again, we see that the connecting homomorphism $H^{n-1}(P,\mathscr{F}_H) \xrightarrow{\delta} H^n(P,\mathscr{F}(-1))$ is injective by counting the dimensions of each graded part; indeed, the map δ is given by multiplication by x_n^{-1}. Hence the map $H^{n-1}(P,\mathscr{F}) \to H^{n-1}(P,\mathscr{F}_H)$ is the zero map, and so $H^{n-1}(P,\mathscr{F}(-1)) \xrightarrow{x_n} H^{n-1}(P,\mathscr{F})$ is an isomorphism as wanted. □

We can also prove the first instance of Serre duality using these calculations. Recall that $H^0(P,\mathscr{O}_P(m))$ is a free A-module with basis $\{x_0^{m_0}\cdots x_n^{m_n} : \sum_{i=0}^n m_i = m\}$ and $H^n(P,\mathscr{O}_P(-m-n-1))$ is free with basis $\{x_0^{l_0}\cdots x_n^{l_n} : \sum_{i=0}^n l_i = -m-n-1\}$.

Proposition 2.6. *Let $P = \mathbb{P}^n_A$. The natural map*

$$H^0(P,\mathscr{O}_P(m)) \times H^n(P,\mathscr{O}_P(-m-n-1)) \to H^n(P,\mathscr{O}_P(-n-1)) \cong A$$

is a perfect pairing of finitely generated free A-modules for each $m \in \mathbb{Z}$.

Indeed, the map takes $(x_0^{m_0}\cdots x_n^{m_n}, x_0^{l_0}\cdots x_n^{l_n})$ to $x_0^{m_0+l_0}\cdots x_n^{m_n+l_n}$, which is zero in $H^n(P,\mathscr{O}_P(-n-1))$ unless $m_i + l_i = -1$ for all i.

Remark 2.7. Note that in this section we have used the fact that the cohomology computed using the Čech complex coincides with the derived-functor cohomology, i.e., $\check{H}^*(\mathfrak{U},\mathscr{F}) \cong H^*(X,\mathscr{F})$. One way to see this is to consider the spectral sequence

$$E_2^{p,q} = \check{H}^p(\mathfrak{U}, \underline{H}^q(X,\mathscr{F})) \Rightarrow H^{p+q}(X,\mathscr{F}).$$

This is an example of the Grothendieck spectral sequence we will see later, applied to the functors $F = \check{H}^0(\mathfrak{U}, -) : \mathbf{PSh}(X) \to \mathbf{Ab}$ and $G : \mathbf{Sh}(X) \hookrightarrow \mathbf{PSh}(X)$.

2.2. **Twisted sheaves.** In this section we collect several results about the twisting sheaf $\mathscr{O}(m)$. We first prove a theorem about how any coherent $\mathscr{O}_X$-module is generated by global sections after enough twists. Two main uses of this theorem are the existence of a useful short exact sequence involving coherent sheaves and Serre vanishing.

Lemma 2.8. *Let X be a scheme, let $\mathscr{L}$ be an invertible sheaf on X, let f be a global section of $\mathscr{L}$, let X_f be the open set of points $x \in X$ where $f_x \notin \mathfrak{m}_x \mathscr{L}_x$ (here $\mathfrak{m}_x$ is the maximal ideal in the local ring $\mathscr{O}_{X,x}$), and let $\mathscr{F}$ be a quasicoherent sheaf on X. Suppose that X has a finite covering by open affine subsets U_i such that $\mathscr{L}|_{U_i}$ is free for each i, and such that $U_i \cap U_j$ is quasicompact for each i, j. Then given a section $s \in \Gamma(X_f, \mathscr{F})$, there is some $m \geq 0$ such that the section $f^m s \in \Gamma(X_f, \mathscr{F} \otimes \mathscr{L}^{\otimes m})$ extends to a global section of $\mathscr{F} \otimes \mathscr{L}^{\otimes m}$.*

Theorem 2.9 (Serre)**.** *Let X be a projective scheme over a noetherian ring A and let $\mathscr{O}_X(1)$ be a very ample invertible sheaf on X. Let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. Then there is an integer m_0 such that for all $m \geq m_0$, the sheaf $\mathscr{F}(m)$ can be generated by a finite number of global sections.*

Proof. Let $i : X \to \mathbb{P}^n_A$ be a closed immersion of X into a projective space over A such that $i^*(\mathscr{O}_{\mathbb{P}^n_A}) = \mathscr{O}_X(1)$. Then $i_*\mathscr{F}$ is a coherent sheaf on $\mathbb{P}^n_A$ and $i_*(\mathscr{F}(m)) = (i_*\mathscr{F})(m)$. Moreover, $\mathscr{F}(m)$ is generated by finitely many global sections iff $i_*\mathscr{F}(m)$ is so. So we are reduced to the case $X = \mathbb{P}^n_A = \operatorname{Proj} A[x_0, \ldots, x_n]$.

Cover X by the open sets $D_+(x_i)$ for $0 \leq i \leq n$. Since $\mathscr{F}$ is coherent, for each i there is a finitely generated module M_i over $B_i = A[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]$ such that $\mathscr{F}|_{D_+(x_i)} \cong \widetilde{M_i}$. For each i, take a finite number of elements $s_{ij} \in M_i$ which generate B_i. By the previous lemma with $\mathscr{L} = \mathscr{O}_X(1)$ and $X_f = D_+(x_i)$, there is an integer $m_{ij} \geq 0$ such that $x_i^{m_{ij}} s_{ij}$ extends to a global section t_{ij} of $\mathscr{F}(m)$. Take m large enough to work for all i, j. Then $\mathscr{F}(m)|_{D_+(x_i)}$ corresponds to a B_i-module M_i', and the map $x_i^m : \mathscr{F} \to \mathscr{F}(m)$ induces an isomorphism of M_i with M_i'. So the sections $x_i^m s_{ij}$ generate M_i', and hence the global sections $t_{ij} \in \Gamma(X, \mathscr{F}(m))$ generate the sheaf $\mathscr{F}(m)$ everywhere. □

Corollary 2.10. *Let X be a projective scheme over a noetherian ring. If $\mathscr{F}$ is a coherent $\mathscr{O}_X$-module, then there is a short exact sequence*

$$0 \to \mathscr{G} \to \bigoplus_{i=1}^{N} \mathscr{O}_X(-m) \to \mathscr{F} \to 0$$

of coherent $\mathscr{O}_X$-modules for some large enough m.

Proof. Choose m large enough so that $\mathscr{F}(m)$ is generated by N global sections. So we have a surjection $\bigoplus_{i=1}^N \mathscr{O}_X \to \mathscr{F}(m)$. Twisting by $-m$ gives the surjection $\bigoplus_{i=1}^N \mathscr{O}_X(-m) \to \mathscr{F}$. □

The next result concerns the cohomology of coherent $\mathscr{O}_X$-modules on projective schemes. Unlike the cohomology on affine noetherian schemes, we do not generally have trivial cohomology, but it turns out that they *are* finitely generated, and *do* become zero if the sheaf is twisted enough times, as we have already seen for the case $X = \mathbb{P}^n_A$ and $\mathscr{F} = \mathscr{O}_X(m)$, $m \in \mathbb{Z}$. In fact, the proof essentially reduces to this case using the short exact sequence above.

Theorem 2.11 (Serre finiteness and vanishing)**.** *Let X be a projective scheme over a noetherian ring A, and let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. Then*
(a) The A-modules $H^i(X, \mathscr{F})$ are finitely generated over A.
(b) There is an integer l_0 such that $H^i(X, \mathscr{F}(l)) = 0$ for all $l \geq l_0$ and $i > 0$.

Proof. Since X is a projective scheme, there is a closed immersion $i : X \to \mathbb{P}^n_A$, and we may reduce to the case $X = \mathbb{P}^n_A$.
(a) By corollary 2.10, there is a short exact sequence of coherent sheaves

$$0 \to \mathscr{G} \to \mathscr{O}_X(-m)^{\oplus N} \to \mathscr{F} \to 0$$

on X. Taking the long exact sequence in cohomology, we get

$$\begin{aligned} 0 \longrightarrow H^0(X, \mathscr{G}) &\longrightarrow H^0(X, \mathscr{O}_X(-m)^{\oplus N}) \longrightarrow H^0(X, \mathscr{F}) \\ \longrightarrow H^1(X, \mathscr{G}) &\longrightarrow H^1(X, \mathscr{O}_X(-m)^{\oplus N}) \longrightarrow H^1(X, \mathscr{F}) \\ &\cdots \\ \longrightarrow H^n(X, \mathscr{G}) &\longrightarrow H^n(X, \mathscr{O}_X(-m)^{\oplus N}) \longrightarrow H^n(X, \mathscr{F}) \longrightarrow 0. \end{aligned}$$

By the explicit calculations of the cohomology of line bundles over $X = \mathbb{P}^n_A$ above, we see that $H^i(X, \mathscr{O}(-m)^{\oplus N}) = H^i(X, \mathscr{O}(-m))^{\oplus N}$ is finitely generated for all i. In particular, $H^n(X, \mathscr{O}(-m)^{\oplus N})$ is finitely generated; hence $H^n(X, \mathscr{F})$ is finitely generated. Since $\mathscr{G}$ is also a coherent sheaf, $H^n(X, \mathscr{G})$ is also finitely generated, and the result then follows from descending induction.
(b) Twist the short exact sequence in (2.10) by $\mathscr{O}(l)$, and consider the associated long exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(X, \mathscr{G}(l)) &\longrightarrow H^0(X, \mathscr{O}(l-m)^{\oplus N}) \longrightarrow H^0(X, \mathscr{F}(l)) \\ \longrightarrow H^1(X, \mathscr{G}(l)) &\longrightarrow H^1(X, \mathscr{O}(l-m)^{\oplus N}) \longrightarrow H^1(X, \mathscr{F}(l)) \\ &\cdots \\ \longrightarrow H^n(X, \mathscr{G}(l)) &\longrightarrow H^n(X, \mathscr{O}(l-m)^{\oplus N}) \longrightarrow H^n(X, \mathscr{F}(l)) \longrightarrow 0. \end{aligned}$$

For large enough l and all $i > 0$, we have $H^i(X, \mathscr{O}(l-m)^{\oplus N}) = 0$. In particular, $H^n(X, \mathscr{O}(l-m)^{\oplus N}) = 0$, so $H^n(X, \mathscr{F}(l)) = 0$. Since $\mathscr{G}$ is also coherent, $H^n(X, \mathscr{G}(l)) = 0$ for large enough l, and the result again follows from descending induction.

□

2.3. **The Yoneda pairing.** The statement of theorem 2.1 refers to the Yoneda pairing which we describe here, using the language of abelian categories and derived functors. Recall, an object I in an abelian category $\mathcal{C}$ is *injective* if $\mathrm{Hom}(-, I)$ is exact. The category $\mathcal{C}$ *has enough injectives* if every object is isomorphic to a subobject of an injective object.

Theorem/Definition 2.12 (Yoneda-Cartier)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories and suppose $\mathcal{C}$ has enough injectives. Let $F : \mathcal{C} \to \mathcal{D}$ be an additive, left-exact* *functor. Then for any two objects A, B in $\mathcal{C}$, there exist δ-functorial (i.e., functorial in A and B and compatible with connecting morphisms) pairings*

$$R^pF(A) \times \operatorname{Ext}^q_{\mathcal{C}}(A, B) \to R^{p+q}F(B)$$

for all nonnegative integers p and q.

Examples 2.13.
(i) As we will see, the pairing in (2.6) is an example of the Yoneda pairing, where $F = \Gamma(P, -)$, $A = \mathscr{O}_P(m)$, $B = \mathscr{O}_P(-n-1)$, $p = n$, and $q = 0$.
(ii) Another example is the Yoneda product of long extensions

$$\operatorname{Ext}^n_R(N, P) \otimes \operatorname{Ext}^m_R(M, N) \to \operatorname{Ext}^{m+n}_R(M, P)$$

in the category of R-modules. See [18] for details.

Proof. Choose injective resolutions $0 \to A \to I^\bullet$ and $0 \to B \to J^\bullet$, and define a complex of abelian groups $\operatorname{Hom}^*(I^\bullet, J^\bullet)$ as follows. Let the q-th cochain $\operatorname{Hom}^q(I^\bullet, J^\bullet)$ be the free group generated by all families $u = (u_p)_{p\in\mathbb{Z}}$ of morphisms $u_p : I^p \to J^{p+q}$ (not assumed to be compatible with the boundary). Define the differential $\partial^q : \operatorname{Hom}^q(I^\bullet, J^\bullet) \to \operatorname{Hom}^{q+1}(I^\bullet, J^\bullet)$ by $\partial^q u = du + (-1)^q ud$. That is, we have $\partial^q u \in \operatorname{Hom}^{q+1}(I^\bullet, J^\bullet)$, given by the family of morphisms $((\partial^q u)_p)_{p\in\mathbb{Z}}$, where $(\partial^q u)_p = d^p u_p + (-1)^q u_{p+1} d^p$ as in the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & \cdots \arrow[r] & I^p \arrow[r, "d^p"] \arrow[d, "u_p"'] \arrow[dr, dashed, "(\partial^q u)_p"] & I^{p+1} \arrow[r] \arrow[d, "u_{p+1}"] & \cdots \\
0 \arrow[r] & B \arrow[r] & \cdots \arrow[r] & J^{p+q} \arrow[r, "d^p"'] & J^{p+q+1} \arrow[r] & \cdots
\end{tikzcd}$$

Then:
(i) $\partial^{q+1} \circ \partial^q = 0$.
(ii) If $u \in \ker \partial^q$, then u (anti)commutes with the boundary d.
(iii) If $u \in \operatorname{im} \partial^{q-1}$, then u is chain-homotopic to the zero map.
(iv) The cohomology group $H^q(\operatorname{Hom}^*(I^\bullet, J^\bullet))$ is the group of homotopy classes of morphisms which (anti)commute with the boundary.

Each $u \in \operatorname{Hom}^q(I^\bullet, J^\bullet)$ induces a morphism $Fu : F(I^\bullet) \to F(J^\bullet)$ of degree q. If $u \in \ker \partial^q$, then u (anti)commutes with the boundary, and so preserves cocycles and coboundaries, and Fu induces a morphism $H^p(Fu) : R^pF(A) \to R^{p+q}F(B)$ for each p. If $u \in \operatorname{im} \partial^{q-1}$, then u is homotopic to zero, so $H^p(Fu) = 0$. So $H^p(Fu)$ depends only on the homotopy class $[u]$ of u. Therefore, there exists a bilinear pairing $R^pF(A) \times H^q(\operatorname{Hom}^*(I^\bullet, J^\bullet)) \to R^{p+q}F(B)$, given by $(\alpha, [u]) \mapsto H^p(Fu)(\alpha)$. It remains to see that $H^q(\operatorname{Hom}^*(I^\bullet, J^\bullet)) \cong \operatorname{Ext}^q_{\mathcal{C}}(A, B)$, which is the content of the following lemma. The proof of δ-functoriality is omitted. □

Lemma 2.14. *With assumptions and definitions in the previous theorem, we have $H^q(\operatorname{Hom}^*(I^\bullet, J^\bullet)) \cong \operatorname{Ext}^q_{\mathcal{C}}(A, B)$.*

Proof. We are going to present a slick proof of this fact, which becomes more or less a triviality from our vantage point. For a different proof with more details (in particular, the explicit isomorphism), see [1].

Fix, once and for all, an injective resolution $J^\bullet(B)$ for every object B in $\mathcal{C}$. The method of proof is to show that $H^*(\operatorname{Hom}^*(I^\bullet, J^\bullet(-)))$ and $\operatorname{Ext}^*_{\mathcal{C}}(A, -)$ are both universal covariant δ-functors with the same zeroth level, which will then guarantee uniqueness up to unique isomorphism.

As right-derived functors, $H^*(\mathrm{Hom}^*(I^\bullet, J^\bullet(-)))$ and $\mathrm{Ext}^*_{\mathcal{C}}(A,-)$ are automatically universal δ-functors. By definition, $\mathrm{Ext}^0_{\mathcal{C}}(A,-) = \mathrm{Hom}_{\mathcal{C}}(A,-)$. Unpacking all the definitions, we also see that the assertion $H^0(\mathrm{Hom}^*(I^\bullet, J^\bullet(-))) = \mathrm{Hom}_{\mathcal{C}}(A,-)$ amounts to the fact that any cocycle $A \to B$ on the zeroth level lifts uniquely up to homotopy to a cochain map $I^\bullet \to J^\bullet(B)$ and any coboundary on the zeroth level lifts to a chain-homotopy. $\square$

To be able to apply these results, we need to show that the category of $\mathscr{O}_X$-modules has enough injectives.

Proposition 2.15. *Let $(X, \mathscr{O}_X)$ be a ringed space. Then the category* $\mathbf{Mod}(X)$ *of $\mathscr{O}_X$-modules has enough injectives.*

Proof sketch. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module, and let $\mathscr{Q}$ be the $\mathscr{O}_X$-module defined by $\mathscr{Q}(U) = \prod_{x \in U} \mathscr{Q}_x$, where $\mathscr{Q}_x$ is a fixed injective $\mathscr{O}_{X,x}$-module containing $\mathscr{F}_x$, for any open set U of X. Then $\mathscr{Q}$ is injective and contains $\mathscr{F}$. $\square$

2.4. **Proof of theorem 2.1.** Having set up all the theory, the proof of Serre duality is not too difficult at this point. We have already proved one instance of it (proposition 2.6). From there, it remains to extend the result to an arbitrary coherent sheaf using the short exact sequence (2.10), and to other degrees via induction.

Theorem (Serre duality for $\mathbb{P}^n_k$)**.** *Let k be a field and $P = \mathbb{P}^n_k$ be projective n-space over k. Let $\omega_P = \mathscr{O}_P(-n-1)$ and let $\mathscr{F}$ be a coherent sheaf. Then the Yoneda pairing $H^r(P, \mathscr{F}) \times \mathrm{Ext}^{n-r}_P(\mathscr{F}, \omega_P) \to H^n(P, \omega_P)$ is perfect.*

Proof. First consider the case $\mathscr{F} = \mathscr{O}_P(-m-n-1)$ and $r = n$. Since

$$\begin{aligned}
\mathrm{Ext}^0_P(\mathscr{F}, \omega_P) &= \mathrm{Hom}_P(\mathscr{F}, \omega_P) = \Gamma(P, \mathscr{H}om_P(\mathscr{F}, \omega_P)) \\
&= \Gamma(P, \mathscr{H}om_P(\mathscr{O}_P, \mathscr{O}_P(m))) = \Gamma(P, \mathscr{O}_P(m)) \\
&= H^0(P, \mathscr{O}_P(m)),
\end{aligned}$$

we get the pairing $H^n(P, \mathscr{O}_P(-m-n-1)) \times H^0(P, \mathscr{O}_P(m)) \to H^n(P, \mathscr{O}_P(-n-1))$, which we already showed was perfect. So Serre duality holds for sheaves of the form $\mathscr{F} = \bigoplus_i \mathscr{O}_P(m_i)$.

In general, there is a presentation $\mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0$ of the coherent sheaf $\mathscr{F}$ where the $\mathscr{E}_i$ are of the form $\mathscr{O}_P(-m_i)^{\oplus N_i}$. Fix an isomorphism $\eta : H^n(P, \omega_P) \cong k$, so we get induced maps $y_r(\mathscr{F}) : \mathrm{Ext}^{n-r}_P(\mathscr{F}, \omega_P) \to H^r(P, \mathscr{F})^*$ from the Yoneda pairing. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_P(\mathscr{F}, \omega_P) \arrow[r] \arrow[d, "y_n(\mathscr{F})"] & \mathrm{Hom}_P(\mathscr{E}_0, \omega_P) \arrow[r] \arrow[d, "y_n(\mathscr{E}_0)"] & \mathrm{Hom}_P(\mathscr{E}_1, \omega_P) \arrow[d, "y_n(\mathscr{E}_1)"] \\
0 \arrow[r] & H^n(P, \mathscr{F})^* \arrow[r] & H^n(P, \mathscr{E}_0)^* \arrow[r] & H^n(P, \mathscr{E}_1)^*
\end{tikzcd}$$

The top row is exact since $\mathrm{Hom}_P(-, \omega_P)$ is left-exact, and the bottom row is exact because it comes from the long exact sequence in cohomology and the fact that $H^{n+1}(P, -) = 0$. The diagram commutes by naturality. The maps $y_n(\mathscr{E}_0)$ and $y_n(\mathscr{E}_1)$ are isomorphisms; thus $y_n(\mathscr{F})$ is also an isomorphism.

Now induct downwards on r from n, and consider an exact sequence of the form $0 \to \mathscr{G} \to \mathscr{E} \to \mathscr{F} \to 0$, where $\mathscr{E}$ is $\mathscr{O}_P(-m)^{\oplus N}$ and $\mathscr{G}$ is coherent. Then we have the diagram

$$\begin{tikzcd}
\mathrm{Ext}_P^{n-r-1}(\mathscr{E},\omega_P) \arrow[r] \arrow[d, "y_{r+1}(\mathscr{E})"] & \mathrm{Ext}_P^{n-r-1}(\mathscr{G},\omega_P) \arrow[r] \arrow[d, "y_{r+1}(\mathscr{G})"] & \mathrm{Ext}_P^{n-r}(\mathscr{F},\omega_P) \arrow[r] \arrow[d, "y_r(\mathscr{F})"] & \mathrm{Ext}_P^{n-r}(\mathscr{E},\omega_P) \arrow[d, "y_r(\mathscr{E})"] \\
H^{r+1}(P,\mathscr{E})^* \arrow[r] & H^{r+1}(P,\mathscr{G})^* \arrow[r] & H^r(P,\mathscr{F})^* \arrow[r] & H^r(P,\mathscr{E})^*
\end{tikzcd}$$

Again, the rows are exact and the diagram commutes by naturality. The map $y_{r+1}(\mathscr{E})$ is an isomorphism, and so is $y_{r+1}(\mathscr{G})$ by the induction hypothesis. For large enough m, we have $\mathrm{Ext}_P^{n-r}(\mathscr{E},\omega_P) = H^{n-r}(P,\omega_P(m))^{\oplus N} = 0$ by Serre vanishing. Also, direct calculations on $\mathbb{P}^n$ from theorem 2.4 show that $H^r(P,\mathscr{E}) = 0$ for $r < n$. Hence $y_r(\mathscr{F})$ is an isomorphism as wanted. □

Conceptually, we can explain the outline of this proof in terms of universal δ-functors (see for example the proof of lemma 2.14). Our goal was to show that the map

$$y_{n-i} : \mathrm{Ext}_P^i(-,\omega_P) \to H^{n-i}(P,-)^*$$

we obtain from the Yoneda pairing is a natural functorial isomorphism. First we showed that this was true for $i = 0$. Then, observing that $\mathrm{Ext}_P^i(-,\omega_P)$ and $H^{n-i}(P,-)^*$ are contravariant δ-functors that are isomorphic on the zeroth level, it is enough to show that they are both coeffaceable and thus universal, and we would have uniqueness up to unique isomorphism.

Given any coherent sheaf, we have a surjection $\mathscr{O}_P(-m)^{\oplus N} \to \mathscr{F} \to 0$. The assertions $\mathrm{Ext}_P^i(\mathscr{E},\omega_P) = 0$ and $H^{n-i}(P,\mathscr{E}) = 0$ for $i > 0$ that we proved suffice to prove coeffaceability.

2.5. **The Grothendieck spectral sequence.** First described in Grothendieck's *Tohoku* paper (1957), the Grothedieck spectral sequence is an ubiquitous and powerful tool to compute composites of derived functors. A cute analogy: the spectral sequence does for derived functors what the chain rule does for derivatives in high-school calculus.

Theorem 2.16. *Let $\mathcal{A},\mathcal{B},\mathcal{C}$ be abelian categories such that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives. Let $G : \mathcal{A} \to \mathcal{B}$ and $F : \mathcal{B} \to \mathcal{C}$ be left-exact functors. Suppose G takes injectives in $\mathcal{A}$ to F-acyclic objects in $\mathcal{B}$. Then for each object A in $\mathcal{A}$, there exists a convergent first-quadrant cohomological spectral sequence*

$$E_2^{p,q} = (R^pF)(R^qG(A)) \Rightarrow R^{p+q}(FG)(A).$$

Definition 2.17. Let $\mathcal{A}$ be an abelian category with enough injectives. A (right) *Cartan-Eilenberg resolution* of a cochain complex $A^\bullet$ in $\mathcal{A}$ is an upper half-plane complex $I^{\bullet,\bullet}$ of injective objects in $\mathcal{A}$, with an augmentation map $A^\bullet \to I^{\bullet,0}$. We require that maps on the horizontal coboundaries and cohomologies are injective resolutions of $B^p(A^\bullet)$ and $H^p(A^\bullet)$ respectively.

Proposition 2.18. *Every cochain complex admits a Cartan-Eilenberg resolution.*

Proof sketch. For each p, select injective resolutions $I_B^{p,\bullet}$ of $B^p(A^\bullet)$ and $I_H^{p,\bullet}$ of $H^p(A^\bullet)$ respectively. By the horseshoe lemma (the lemma on simultaneous resolution) applied to the short exact sequence

$$0 \to B^p(A^\bullet) \to Z^p(A^\bullet) \to H^p(A^\bullet) \to 0,$$

we obtain an injective resolution $I_Z^{p,\bullet}$ of $Z^p(A^\bullet)$.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d, dashed] & 0 \arrow[d, dashed] & \\
0 \arrow[r] & B^p(A^\bullet) \arrow[r] \arrow[d] & I_B^{p,1} \arrow[r] \arrow[d, dashed] & I_B^{p,2} \arrow[r] \arrow[d, dashed] & \cdots \\
0 \arrow[r, dashed] & Z^p(A^\bullet) \arrow[r, dashed] \arrow[d] & I_Z^{p,1} \arrow[r, dashed] \arrow[d, dashed] & I_Z^{p,2} \arrow[r, dashed] \arrow[d, dashed] & \cdots \\
0 \arrow[r] & H^p(A^\bullet) \arrow[r] \arrow[d] & I_H^{p,1} \arrow[r] \arrow[d, dashed] & I_H^{p,2} \arrow[r] \arrow[d, dashed] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Using the horseshoe lemma again, this time with

$$0 \to Z^p(A^\bullet) \to A^p \to B^{p+1}(A^\bullet) \to 0,$$

construct an injective resolution $(I_A^{p,\bullet}, \varepsilon^p)$ of A^p. Now define $I^{\bullet,\bullet}$ by $I^{p,\bullet} = I_A^{p,\bullet}$. The vertical differentials are $d^p = (-1)^p \varepsilon^p$. The horizontal differentials are the composition $d^p : I_A^{p,\bullet} \twoheadrightarrow I_B^{p+1,\bullet} \hookrightarrow I_Z^{p+1,\bullet} \hookrightarrow I_A^{p+1,\bullet}$. We leave it to the reader to check that this construction works. $\square$

Proof sketch of theorem 2.16. Choose an injective resolution $A \to I^\bullet$ of A in $\mathcal{A}$. Apply G to get a cochain complex $G(I^\bullet)$ in $\mathcal{B}$. Since $G(I^\bullet)$ is a bounded below cochain complex, the Cartan-Eilenberg resolution of $G(I^\bullet)$ is a first-quadrant double complex. There are two spectral sequences associated to this double complex, both of which converge since only finitely many nonzero differentials enter and exit any given spot (p, q). First,

$$\begin{aligned} {}^I E_2^{p,q} &= H^p_{\text{hor}} H^q_{\text{vert}}(F(G(I^\bullet))) = H^p_{\text{hor}}((R^q F)(G(I^\bullet))) \\ &\Rightarrow H^{p+q}(\operatorname{Tot}(F(G(I^\bullet)))) =: (\mathbb{R}^{p+q}F)(G(I^\bullet)). \end{aligned}$$

(Notation: If C is a double complex, we write $\operatorname{Tot}(C)$ to denote the *total complex* obtained by taking the product of the entries by antidiagonals. If T is a functor, we write $\mathbb{R}^i T$ to denote the *right hyperderived functors of* T.)

By hypothesis, each $G(I^p)$ is F-acyclic, so $(R^q F)(G(I^\bullet)) = 0$ for $q > 0$. The nonzero objects on the second page lie exclusively in the zeroth row. Therefore, this spectral sequence collapses and we get

$$(\mathbb{R}^{p+q}F)(G(I^\bullet)) \cong H^{p+q}_{\text{hor}}(FG(I^\bullet)) = R^{p+q}(FG)(A).$$

The second spectral sequence is

$$ {}^{II}E_2^{p,q} = H^q_{\mathrm{vert}} H^p_{\mathrm{hor}}(F(G(I^\bullet))) = H^q_{\mathrm{vert}}((R^pF)(G(I^\bullet))). $$

By a splitting argument, this last term is $(R^pF)(H^q_{\mathrm{vert}}(G(I^\bullet))) = (R^pF)(R^qG(A))$. This spectral sequence has the same abutment as the first, so

$$ \mathbb{R}^{p+q}(F(G(I^\bullet))) \cong R^{p+q}(FG)(A). $$

Thus we have a spectral sequence

$$ E_2^{p,q} = (R^pF)(R^qG(A)) \Rightarrow R^{p+q}(FG)(A). $$

This is the Grothendieck spectral sequence. □

One particular instance of the Grothendieck spectral sequence is the Leray spectral sequence. We specialize to the case where the abelian categories are $\mathbf{Ab}(X)$ on some ringed space X, F is the global section functor, and $G = f_*$ for some morphism $f : X \to Y$.

Theorem 2.19 (Leray spectral sequence)**.** *Let $f : X \to Y$ be a morphism of ringed spaces. Then the functor f_* is left exact. Furthermore, if $\mathscr{Q}$ is an injective $\mathscr{O}_X$-module, then $\mathscr{Q}$ and $f_*\mathscr{Q}$ are flasque. So for any sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules, there exists a spectral sequence*

$$ E_2^{p,q} = H^p(Y, R^qf_*\mathscr{F}) \Rightarrow H^{p+q}(X, \mathscr{F}). $$

Remark 2.20. Let $\pi : E \to B$ be a fibration in the sense of Serre and $\mathscr{F}$ be the constant sheaf $\underline{A}$. Under certain conditions, we may think of $R^q\pi_*\underline{A}$ roughly as the local coefficient system $\underline{\mathcal{H}}^q(X; A)$. This gives a version of the spectral sequence that was used extensively, but without proof, in the author's previous REU paper to study the cohomology of Lie groups.

We will see the Leray spectral sequence again when we discuss Bott's theorem.

Here are two other examples of the Grothendieck spectral sequence that will be used later.

Proposition 2.21. *Let X be a ringed space and $\mathscr{F}, \mathscr{G}$ two $\mathscr{O}_X$-modules. Then there exists a spectral sequence*

$$ H^p(X, \mathscr{E}\mathrm{xt}^q_X(\mathscr{F}, \mathscr{G})) \Rightarrow \mathrm{Ext}^{p+q}_X(\mathscr{F}, \mathscr{G}). $$

Proof. Let $F = \Gamma(X, -)$ and $G = \mathscr{H}\mathrm{om}_X(\mathscr{F}, -)$. Then

$$ FG = \Gamma(X, \mathscr{H}\mathrm{om}_X(\mathscr{F}, -)) = \mathrm{Hom}_X(\mathscr{F}, -). $$

It remains to check that if $\mathscr{Q}$ is an injective $\mathscr{O}_X$-module, then $\mathscr{H}\mathrm{om}_X(\mathscr{F}, \mathscr{Q})$ is F-acyclic, i.e., flasque. Let U be an open subset of X and $f \in \Gamma(U, \mathscr{H}\mathrm{om}_X(\mathscr{F}, \mathscr{Q}))$. Let $\mathscr{F}_U$ be the extension of $\mathscr{F}|_U$ by zero to all of X. Since $\mathscr{Q}$ is injective, the map $\mathscr{F}_U \to \mathscr{Q}$ induced by f extends to an element $g \in \Gamma(X, \mathscr{H}\mathrm{om}_X(\mathscr{F}, \mathscr{Q}))$. Then $g|_U = f$ as wanted. □

Corollary 2.22. *If $\mathscr{F}$ is locally free of finite type, $\mathrm{Ext}^p_X(\mathscr{F}, \mathscr{G}) \cong H^p(X, \mathscr{G} \otimes \mathscr{F}^\vee)$, where $\mathscr{F}^\vee = \mathscr{H}\mathrm{om}_X(\mathscr{F}, \mathscr{O}_X)$.*

Notation 2.23. Let $j : X \to P$ be a closed immersion. If we write $\operatorname{Ext}_X(\mathscr{F}, \mathscr{G})$ or $\mathscr{E}\mathrm{xt}_X(\mathscr{F}, \mathscr{G})$, we abuse notation and treat $\mathscr{F}$ and $\mathscr{G}$ as sheaves on X, even if they are defined as sheaves on P, via the pullback j^*. Similarly, if we write $\operatorname{Ext}_P(\mathscr{F}, \mathscr{G})$ or $\mathscr{E}\mathrm{xt}_P(\mathscr{F}, \mathscr{G})$, we are treating the sheaves as sheaves on P, even if they are defined as sheaves on X, via the pushforward j_*.

Proposition 2.24. *Let $i : X \to P$ be a closed immersion of ringed spaces, $\mathscr{E}$, $\mathscr{F}$ two $\mathscr{O}_X$-modules, and $\mathscr{G}$ an $\mathscr{O}_P$-module. Suppose $\mathscr{E}$ is locally free of finite type. Then there exist a spectral sequence*

$$\operatorname{Ext}^p_X(\mathscr{F}, \mathscr{E}\mathrm{xt}^q_P(\mathscr{E}, \mathscr{G})) \Rightarrow \operatorname{Ext}^{p+q}_P((\mathscr{E} \otimes \mathscr{F}), \mathscr{G}).$$

Proof. Applying Grothendieck's spectral sequence with $F = \operatorname{Hom}_X(\mathscr{F}, -)$ and $G = \mathscr{H}\mathrm{om}_P(\mathscr{E}, -)$ gives the spectral sequence

$$\begin{aligned} E_2^{p,q} = \operatorname{Ext}^p_X(\mathscr{F}, \mathscr{E}\mathrm{xt}^q_P(\mathscr{E}, \mathscr{G})) \Rightarrow & R^{p+q} \operatorname{Hom}_X(\mathscr{F}, \mathscr{H}\mathrm{om}_P(\mathscr{E}, \mathscr{G})) \\ &= R^{p+q} \operatorname{Hom}_P(\mathscr{E} \otimes \mathscr{F}, \mathscr{G}) \\ &= \operatorname{Ext}^{p+q}_P(\mathscr{E} \otimes \mathscr{F}, \mathscr{G}) \end{aligned}$$

using the tensor-hom adjunction.

It remains to show that G takes injectives to F-acyclics, i.e., if $\mathscr{Q}$ is an injective $\mathscr{O}_P$-module and $\mathscr{J} = G\mathscr{Q} = \mathscr{H}\mathrm{om}_P(\mathscr{E}, \mathscr{Q})$, then $\operatorname{Hom}_X(-, \mathscr{J})$ is right-exact. Let $0 \to \mathscr{F}' \to \mathscr{F}$ be an exact sequence of $\mathscr{O}_X$-modules. Since $\mathscr{E}$ is locally free and $\mathscr{Q}$ is injective, the sequence $\operatorname{Hom}_P(\mathscr{F} \otimes \mathscr{E}, \mathscr{Q}) \to \operatorname{Hom}_P(\mathscr{F}' \otimes \mathscr{E}, \mathscr{Q}) \to 0$ is exact. Thus using the tensor-hom adjunction again, $\operatorname{Hom}_X(\mathscr{F}, \mathscr{J}) \to \operatorname{Hom}_X(\mathscr{F}', \mathscr{J}) \to 0$ is exact as wanted. □

2.6. **Towards Grothendieck duality: dualizing sheaves.** This section is concerned primarily with extending Serre duality to projective schemes. The first difficulty in generalizing Serre duality is the question of which sheaf should play the role of $\omega_P = \mathscr{O}_P(-n-1)$.

Definition 2.25. Let X be a proper scheme of dimension r over a field k. A *dualizing sheaf* for X is a coherent sheaf ω_X° on X together with a *trace* morphism $t : H^r(X, \omega_X^\circ) \to k$ such that for all coherent sheaves $\mathscr{F}$ on X, the natural pairing $\operatorname{Hom}(\mathscr{F}, \omega_X^\circ) \times H^r(X, \mathscr{F}) \to H^r(X, \omega_X^\circ)$ followed by t induces an isomorphism $y_0(\mathscr{F}) : \operatorname{Hom}(\mathscr{F}, \omega_X^\circ) \to H^r(X, \mathscr{F})^*$.

Dualizing sheaves are unique. The proof given in [12] is a particular instance of the more abstract version which we give here. Let $\mathcal{C}$ be a category and $\mathcal{D}$ be a concrete category, and let $F : \mathcal{C} \to \mathcal{D}$ be a contravariant functor. Let $\omega \in \operatorname{Ob}(\mathcal{C})$ and $t \in F\omega$. We say that the pair (ω, t) *represents* the functor F if there is a natural isomorphism $\operatorname{Hom}_{\mathcal{C}}(-, \omega) \cong F(-)$. More precisely, the map $\operatorname{Hom}_{\mathcal{C}}(A, \omega) \xrightarrow{F} \operatorname{Hom}_{\mathcal{D}}(F\omega, FA) \xrightarrow{\mathrm{eval}_t} FA$ is a bijection of sets for any $A \in \operatorname{Ob}(\mathcal{C})$.

Proposition 2.26. *The pair (ω, t) representing F is unique up to unique isomorphism.*

Proof. Let (ω', t') be another pair that represents F. By definition of ω, the map $\operatorname{Hom}(\omega', \omega) \to F\omega'$ is bijective. Since $t' \in F\omega'$, there exists a unique $\varphi \in \operatorname{Hom}(\omega', \omega)$ such that $(F\varphi)(t) = t'$. Similarly, there exists a unique $\psi \in \operatorname{Hom}(\omega, \omega')$ such that $(F\psi)(t') = t$.

We claim that $\psi \circ \varphi = \mathrm{id}_{\omega'} \in \mathrm{Hom}(\omega', \omega')$. Under the map $\mathrm{Hom}(\omega', \omega') \xrightarrow{\sim} F\omega'$, $(\psi \circ \varphi)$ goes to $F(\psi \circ \varphi)(t') = t'$. Since $\mathrm{id}_{\omega'}$ also goes to t' under this bijection, we see that $\psi \circ \varphi = \mathrm{id}_{\omega'}$ as desired. Analogously, $\varphi \circ \psi = \mathrm{id}_{\omega}$. So $(\psi, F\psi)$ is the (unique) isomorphism between (ω, t) and (ω', t'). $\square$

Notice that if the pair (ω_X°, t) is a dualizing sheaf for X, then it represents the contravariant functor $H^r(X, -)^* : \mathbf{Coh}(X) \to \mathbf{Vect}_k$. Thus we have uniqueness of dualizing sheaves:

Corollary 2.27. *Let X be a proper scheme over k. Then a dualizing sheaf for X, if it exists, is unique in the sense that if (ω°, t) and (ω', t') are dualizing sheaves for X, then there is a unique isomorphism $\varphi : \omega^\circ \to \omega'$ such that $t = t' \circ H^r(\varphi)$.*

We now turn to existence of dualizing sheaves, which is a more tricky and more interesting matter. It is true for any proper scheme over k, but we prove it only for projective schemes.

Before we begin the formal proof, let's motivate how one might try to find such a dualizing sheaf. Let $P = \mathbb{P}_k^n$, and let $i : X \hookrightarrow P$ be a closed subscheme of dimension r in P. The idea is that we already know what the dualizing sheaf for P is. Using Serre duality for P, we see that if $\mathscr{F}$ is any coherent sheaf on X, then $H^r(X, \mathscr{F})^* = H^r(P, \mathscr{F})^* \cong \mathrm{Ext}_P^{n-r}(\mathscr{F}, \omega_P)$. It remains to determine a sheaf ω_X° such that there is a functorial isomorphism $\mathrm{Ext}_P^{n-r}(\mathscr{F}, \omega_P) \cong \mathrm{Hom}_X(\mathscr{F}, \omega_X^\circ)$. Hypothetically, if we sheafify and let $\mathscr{F} = \mathscr{O}_X$, we would have the expression $\omega_X^\circ = \mathscr{E}\mathrm{xt}_P^{n-r}(\mathscr{O}_X, \omega_P)$, which turns out to be the correct answer. The trace map can then be recovered from figuring out the element $\mathrm{id}_{\omega_X^\circ} \in \mathrm{Hom}_X(\omega_X^\circ, \omega_X^\circ)$ under the identifications $\mathrm{Hom}_X(\omega_X^\circ, \omega_X^\circ) \cong H^r(X, \omega_X^\circ)^*$, as in the argument for uniqueness of representations for representable functors. We now carry out the verification that $\mathscr{E}\mathrm{xt}_P^{n-r}(\mathscr{O}_X, \omega_P)$ has the required properties of a dualizing sheaf.

Lemma 2.28.

(a) *Let P be a ringed space, and $\mathscr{E}, \mathscr{F}, \mathscr{G}$ three $\mathscr{O}_P$-modules. Suppose $\mathscr{E}$ is locally free of finite rank. Then the canonical homomorphisms*
$$\mathscr{E}\mathrm{xt}_P^q(\mathscr{F}, \mathscr{G}) \otimes_P \mathscr{E}^\vee \to \mathscr{E}\mathrm{xt}_P^q(\mathscr{E} \otimes_P \mathscr{F}, \mathscr{G})$$
are isomorphisms for all $q \geq 0$.

(b) *Let $P = \mathbb{P}_k^n$, and let X be a closed subscheme of dimension r in P. Then, $\mathscr{E}\mathrm{xt}_P^q(\mathscr{O}_X, \omega_P) = 0$ for all $q < n - r$.*

Proof.

(a) The proof is straightforward: since $\mathscr{E}$ is locally free, we may assume $\mathscr{E} = \mathscr{O}_P^N$. Then
$$\mathscr{E}\mathrm{xt}_P^q(\mathscr{F}, \mathscr{G}) \otimes \mathscr{E}^\vee = \mathscr{E}\mathrm{xt}_P^q(\mathscr{F}, \mathscr{G})^{\oplus N}$$
$$\text{and} \quad \mathscr{E}\mathrm{xt}_P^q(\mathscr{E} \otimes \mathscr{F}, \mathscr{G}) = \mathscr{E}\mathrm{xt}_P^q(\mathscr{F}^{\oplus N}, \mathscr{G}).$$
On the level of $\mathscr{H}\mathrm{om}$, i.e. $q = 0$, we have the isomorphism
$$\mathscr{H}\mathrm{om}_X(\mathscr{F}, \mathscr{G})^{\oplus N} \to \mathscr{H}\mathrm{om}_X(\mathscr{F}^{\oplus N}, \mathscr{G}),$$
$$(\varphi_1, \ldots, \varphi_N) \mapsto \varphi = (\varphi_1, \ldots, \varphi_N).$$
Now choose an injective resolution of $\mathscr{G}$, and use the isomorphism on $\mathscr{H}\mathrm{om}$ to define a chain isomorphism, which descends to the desired isomorphism on $\mathscr{E}\mathrm{xt}$.

(b) For any q, the sheaf $\mathscr{F}^q := \mathscr{E}\mathrm{xt}^q_P(\mathscr{O}_X, \omega_P)$ is coherent, so after twisting enough times, it is generated by global sections. Thus to show that $\mathscr{F}^q = 0$, it is enough to show that $\Gamma(P, \mathscr{F}^q(m)) = 0$ for all large m. Using part (a), we have

$$\begin{aligned}\Gamma(P, \mathscr{F}^q(m)) &= \Gamma(P, \mathscr{E}\mathrm{xt}^q_P(\mathscr{O}_X, \omega_P) \otimes \mathscr{O}_P(m)) \\ &= \Gamma(P, \mathscr{E}\mathrm{xt}^q_P(\mathscr{O}_X(-m), \omega_P)) \\ &= \mathrm{Ext}^q_P(\mathscr{O}_X(-m), \omega_P).\end{aligned}$$

By Serre duality, this group is dual to $H^{n-q}(P, \mathscr{O}_X(-m))$. If $q < n - r$, then $n - q > \dim X$, and therefore the group $H^{n-q}(P, \mathscr{O}_X(-m))$ vanishes. Hence, $\Gamma(P, \mathscr{F}^q(m)) = 0$ as wanted.

□

Remark 2.29. Part (b) of the lemma also holds if P is any regular k-scheme.

Theorem 2.30. *Let X be a projective scheme over k of dimension r that embeds in $P = \mathbb{P}^n_k$. Then $\omega^\circ_X = \mathscr{E}\mathrm{xt}^{n-r}_P(\mathscr{O}_X, \omega_P)$ is a dualizing sheaf for X.*

Proof. Given the discussion from above, it remains to show that there is a natural functorial isomorphism $\mathrm{Hom}_X(\mathscr{F}, \omega^\circ_X) \cong \mathrm{Ext}^{n-r}_P(\mathscr{F}, \omega_P)$. Let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module, and consider the spectral sequence in proposition 2.24:

$$E_2^{p,q} = \mathrm{Ext}^p_X(\mathscr{F}, \mathscr{E}\mathrm{xt}^q_P(\mathscr{O}_X, \omega_P)) \Rightarrow \mathrm{Ext}^{p+q}_P(\mathscr{F}, \omega_P).$$

By the previous lemma, $E_2^{p,q} = 0$ for all $q < n - r$. Consider the edge homomorphism $\varepsilon^0(\mathscr{F}) : E_2^{0,n-r} \to E_3^{0,n-r} \to \cdots \to E_\infty^{0,n-r}$. We have

$$\begin{aligned}E_2^{0,n-r} &= \mathrm{Hom}_X(\mathscr{F}, \mathscr{E}\mathrm{xt}^{n-r}_P(\mathscr{O}_X, \omega_P)) = \mathrm{Hom}_X(\mathscr{F}, \omega^\circ_X) \\ E_\infty^{0,n-r} &= \mathrm{Ext}^{n-r}_P(\mathscr{F}, \omega_P),\end{aligned}$$

and moreover, $\varepsilon^0(\mathscr{F})$ is an isomorphism by lacunary considerations.

Hence $\mathrm{Hom}_X(\mathscr{F}, \omega^\circ_X)$ and $\mathrm{Ext}^{n-r}_P(\mathscr{F}, \omega_P)$ are functorially isomorphic. □

For a more hands-on proof of this theorem, see [12].

If a dualizing sheaf exists for a projective scheme X over k, then Serre duality falls out in much the same way as it did for $\mathbb{P}^n_k$ (using universal δ-functors), so we omit the proof.

Theorem 2.31 (Duality for a projective scheme)**.** *Let X be a projective scheme of dimension r over an algebraically closed field k, and let ω°_X be a dualizing sheaf for X. Then for all $0 \le i \le n$ and coherent sheaves $\mathscr{F}$ on X, there exist natural functorial maps*

$$y_i(\mathscr{F}) : \mathrm{Ext}^{r-i}_X(\mathscr{F}, \omega^\circ_X) \to H^i(X, \mathscr{F})^*$$

such that y_0 in the same as in the definition of a dualizing sheaf.

Remark 2.32. The importance of a duality theorem cannot be overstated. The role Serre duality plays in algebraic geometry is akin to the role of Poincaré duality in the algebraic topology of manifolds (and the two duality theorems are *not* wholly unrelated!). Duality reduces the amount of computation needed and offers a way to deduce quite specific information about particular cohomology groups using more easily obtained knowledge; the use of Serre duality in the proof of the Riemann-Roch theorem in the next section is a prime example. Depending on the context, a duality theorem can also be a hint towards a sort of equivalence between different objects or theories.

Serre duality is important also because of its many generalizations. Serre duality has been generalized by Grothendieck (see [14]), and subsequently a even further-reaching generalization was attained by Greenlees-May [11].

In the case where X is smooth, there is a particularly nice description of the dualizing sheaf as the canonical sheaf for X. To prove this, we will introduce the *Koszul complex*, so let us fix some notation. If A is a ring and $x_1, \dots, x_r \in A$, define a complex $K_*(\underline{x}) = K_*(x_1, \dots, x_r)$ by

$$K_p(\underline{x}) = \begin{cases} \wedge^p(\bigoplus_{i=1}^r Ae_i) & \text{for } 0 \le p \le r \\ 0 & \text{otherwise.} \end{cases}$$

The differentials are $d_p(e_{i_1} \wedge \cdots e_{i_p}) = \sum_{j=1}^p (-1)^j x_j e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_p}$. Given an A-module M, write $K_*(\underline{x}; M)$ for $K_*(\underline{x}) \otimes_A M$, and $K^*(\underline{x}; M)$ for $\mathrm{Hom}_A(K_*(\underline{x}, M))$.

Lemma 2.33. *Let $(x_1, \dots, x_r)$ be an A-regular sequence and $I = (x_1, \dots, x_r)$. Let M be an A-module.*
(a) $K_*(\underline{x}; M)$ is a resolution of M/IM.
(b) Let $H^r(\underline{x}; M) = H^r(K^*(\underline{x}; M))$. The map $\varphi'_{\underline{x}} : K^r(\underline{x}; M) \to M$ given by $a \mapsto a(e_1 \wedge \cdots \wedge e_r)$ induces an isomorphism $\varphi_{\underline{x}} : H^r(\underline{x}; M) \xrightarrow{\sim} M/IM$.
In particular, if $M = A$, then the resolution in (a) is free and $H^r(\underline{x}; M)$ computes $\mathrm{Ext}^r_A(A/I, M)$, so we have isomorphisms $\mathrm{Ext}^r_A(A/I, M) \xrightarrow{\varphi_{\underline{x}}} M/IM$.
(c) Let $(y_1, \dots, y_r)$ be another A-regular sequence that generates I. We can write $y_i = \sum_i c_{ij} x_j$ for some $c_{ij} \in A$. Then $\varphi_{\underline{y}} \equiv \det(c_{ij}) \varphi_{\underline{x}}$ as functions from $\mathrm{Ext}^r_A(A/I, M)$ to M/IM.

Proof. For (a), note that $K_*(\underline{x}; M)$ is the single complex associated to the double complex $K_{p,q} = K_p((x_1, \dots, x_{r-1}); M) \otimes K_q(x_r)$, since exterior powers converts direct sums into tensor products.

We now induct on r. The case $r = 1$ is easy to see, since $K_0(x_1; M) = M$ and $K_1(x_1; M) = Ae_1 \otimes M$, and the differential is $d^1 : e_1 \mapsto x_1$. So suppose $r > 1$ and let $I' = (x_1, \dots, x_{r-1})$. Then by the induction hypothesis,

$$E^1_{p,q} = H^{\mathrm{hor}}_p(K^{*,q}) = \begin{cases} M/I'M & \text{for } p = 0 \text{ and } q = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

Next, by assumption, $x_r : M/I'M \to M/I'M$ is injective, so

$$E^2_{p,q} = H^{\mathrm{vert}}_q(E^1_{p,*}) = \begin{cases} M/IM & \text{for } p = q = 0 \\ 0 & \text{otherwise.} \end{cases}$$

As the spectral sequence abuts to $H^{p+q}(\mathrm{Tot}(K_{**})) = H^{p+q}(K_*(\underline{x}; M))$, we see that $H^0(K_*(\underline{x}; M)) = M/IM$ and $H^i(K_*(\underline{x}; M)) = 0$ for $i > 0$, whence (a).

The proof of (b) is straightforward. It is easy to observe the following: (i) $\varphi'_{\underline{x}}$ takes coboundaries to IM, so induces a well-defined map $\varphi_{\underline{x}}$ in cohomology, (ii) $\varphi'_{\underline{x}}$ is surjective, and (iii) if $\varphi'_{\underline{x}}(a) = 0$, then $a(e_1 \wedge \cdots \wedge e_r) = \sum_j x_j m_j$ for some $m_j \in M$. In this case, $a = db$, where $b : \wedge^{r-1}(A^r) \to M$ takes $e_1 \wedge \cdots \wedge \widehat{e_j} \wedge \cdots \wedge e_r$ to $(-1)^j m_j$. Hence $\varphi_{\underline{x}}$ is injective.

Finally, for (c), simply follow the maps. □

Recall our notational convention in (2.23)

Theorem 2.34. *Let P be an S-scheme and X a closed subscheme with sheaf of ideals $\mathscr{I}$, with relative dimensions n and r respectively.*
(a) If X is regularly immersed in P and $\mathscr{F}$ is a quasicoherent $\mathscr{O}_X$-module, then there exists an isomorphism

$$\varphi : \mathscr{E}\mathrm{xt}_P^{n-r}(\mathscr{O}_X, \mathscr{F}) \xrightarrow{\sim} \mathscr{H}\mathrm{om}_X(\wedge^r(\mathscr{I}/\mathscr{I}^2), \mathscr{F}/\mathscr{I}\mathscr{F}).$$

(b) If X and P are smooth over S, then the canonical sheaf $\omega_{X/S}$ is isomorphic to $\mathscr{E}\mathrm{xt}_P^{n-r}(\mathscr{O}_X, \omega_{P/S})$. In particular, if P is the k-scheme $\mathbb{P}_k^n$, then the canonical sheaf ω_X is isomorphic to $\mathscr{E}\mathrm{xt}_P^{n-r}(\mathscr{O}_X, \omega_P) = \omega_X^\circ$, the dualizing sheaf.

Proof.
(a) Let $U = \operatorname{Spec} A$ be an affine open subset of P on which $\mathscr{I}$ is regular. Let $M = \Gamma(U, \mathscr{F})$ and $I = \Gamma(U, \mathscr{I})$. Then I is generated by an A-regular sequence $(x_1, \dots, x_{n-r})$ and I/I^2 is free of rank $n-r$ over A/I. Let x_i' be the residue class of x_i in I/I^2; then the element $x_1' \wedge \cdots \wedge x_{n-r}'$ generates $\wedge^r(I/I^2)$. Define

$$\begin{gathered}\varphi : \mathrm{Ext}_A^{n-r}(A/I, M) \to \mathrm{Hom}_{A/I}(\wedge^r(I/I^2), M/IM) \\ \varphi(a)(x_1' \wedge \cdots \wedge x_{n-r}') = \varphi_{\overline{x}}(a).\end{gathered}$$

This is well-defined independent of the choice of A-regular sequence: if we have another A-regular sequence $(y_1, \dots, y_{n-r})$ that generates I, then there exists $c_{ij} \in A$ such that $y_i = \sum_j c_{ij} x_j$, and so $y_1 \wedge \cdots \wedge y_{n-r} = \det(c_{ij}) x_1 \wedge \cdots \wedge x_{n-r}$. Then

$$\begin{aligned}\varphi(a)(y_1 \wedge \cdots \wedge y_{n-r}) &= \det(c_{ij})\varphi(a)(x_1 \wedge \cdots \wedge x_{n-r}) \\ &= \det(c_{ij})\varphi_{\underline{x}}(a) = \varphi_{\underline{y}}(a).\end{aligned}$$

The maps φ on a regularizing affine open cover glue together to form an isomorphism between $\mathrm{Ext}_P^{n-r}(\mathscr{O}_X, \mathscr{F})$ and $\mathrm{Hom}_X(\wedge^r(\mathscr{I}/\mathscr{I}^2), \mathscr{F}/\mathscr{I}\mathscr{F})$.

(b) The assumptions imply that X is regularly immersed in P. On the one hand, by part (a) we have

$$\mathscr{E}\mathrm{xt}_P^{n-r}(\mathscr{O}_X, \omega_{P/S}) \cong (\wedge^{n-r}\mathscr{I}/\mathscr{I}^2)^\vee \otimes_P \omega_{P/S}$$

On the other hand, since X and P are smooth, we have the following short exact sequence of locally free sheaves

$$0 \to \mathscr{I}/\mathscr{I}^2 \to \Omega_{P/S} \otimes_P \mathscr{O}_X \to \Omega_{X/S} \to 0,$$

whence we deduce that $\wedge^n \Omega_{P/S} \cong \wedge^{n-r}(\mathscr{I}/\mathscr{I}^2) \otimes \wedge^r \Omega_{X/S}$, i.e.,

$$\omega_{X/S} \cong (\wedge^{n-r}\mathscr{I}/\mathscr{I}^2)^\vee \otimes_P \omega_{P/S}.$$

Hence $\mathscr{E}\mathrm{xt}_P^{n-r}(\mathscr{O}_X, \omega_{P/S}) \cong \omega_{X/S}$.

□

To conclude this section, we prove the following theorem about some properties of the pairing $H^i(X, \mathscr{F}) \times \mathrm{Ext}_X^{r-i}(\mathscr{F}, \omega_X) \to k$. This will make use of the Grothendieck spectral sequence to transfer results about duality on $\mathbb{P}_k^n$ to an arbitrary projective scheme X.

Theorem 2.35. *Let k be a field, $P = \mathbb{P}_k^n$, and $\omega_P = \mathscr{O}_P(-n-1)$ the dualizing sheaf for P. Let X be a closed subscheme of dimension r, $\omega_X = \mathscr{E}\mathrm{xt}_P^{n-r}(\mathscr{O}_X, \omega_P)$ the dualizing sheaf for X, and $\mathscr{F}$ a coherent $\mathscr{O}_X$-module. Then for all $s \leq r$, the following are equivalent.*

(i) The map $\mathrm{Ext}_X^{r-i}(\mathscr{F},\omega_X) \to H^i(X,\mathscr{F})^*$ *is an isomorphism for all* $r-s \le i \le r$.
(ii) $H^i(X,\mathscr{O}_X(-m)) = 0$ *for large* m *and* $r-s \le i < r$.
(iii) $\mathscr{E}\mathrm{xt}_P^{n-i}(\mathscr{O}_X,\omega_P) = 0$ *for* $r-s \le i < r$.

Proof sketch. Assume (i). Then by the pairing, $H^i(X,\mathscr{O}_X(-m)) = 0$ if and only if $\mathrm{Ext}_X^{r-i}(\mathscr{O}_X(-m),\omega_X) = 0$. We have $\mathrm{Ext}_X^{r-i}(\mathscr{O}_X(-m),\omega_X) \cong H^{r-i}(X,\omega_X(m))$ by (2.22), and this is zero for large m by Serre vanishing. Thus (i) implies (ii).

Next, consider the spectral sequence in (2.21):

$$E_2^{p,q} = H^p(P,\mathscr{E}\mathrm{xt}_P^q(\mathscr{O}_X(-m),\omega_P)) \Rightarrow \mathrm{Ext}_P^{p+q}(\mathscr{O}_X(-m),\omega_P).$$

We have $\mathscr{E}\mathrm{xt}_P^q(\mathscr{O}_X(-m),\omega_P) = \mathscr{E}\mathrm{xt}_P^q(\mathscr{O}_X,\omega_P)(m)$. So by Serre vanishing again, the E_2 page of the spectral sequence for large m is zero except for $p=0$. So the spectral sequence degenerates. Let $q = n-i$; we get

$$H^0(P,\mathscr{E}\mathrm{xt}_P^{n-i}(\mathscr{O}_X,\omega_P)(m)) \cong \mathrm{Ext}_P^{n-i}(\mathscr{O}_X(-m),\omega_P).$$

By Serre's finite generation theorem, $\mathscr{E}\mathrm{xt}_P^{n-i}(\mathscr{O}_X,\omega_P)(m)$ is generated by the global sections $H^0(P,\mathscr{E}\mathrm{xt}_P^{n-i}(\mathscr{O}_X,\omega_P)(m))$ for large enough m, and by Serre duality we have $y_i(\mathscr{O}_X(-m)) : \mathrm{Ext}_P^{n-i}(\mathscr{O}_X(-m),\omega_P) \xrightarrow{\sim} H^i(P,\mathscr{O}_X(-m))^*$ is an isomorphism. So

$$\begin{aligned} H^i(P,\mathscr{O}_X(-m)) = 0 &\Leftrightarrow \mathrm{Ext}_P^{n-i}(\mathscr{O}_X(-m),\omega_P) = 0 \\ &\Leftrightarrow \mathscr{E}\mathrm{xt}_P^{n-i}(\mathscr{O}_X,\omega_P)(m) = 0 \\ &\Leftrightarrow \mathscr{E}\mathrm{xt}_P^{n-i}(\mathscr{O}_X,\omega_P) = 0. \end{aligned}$$

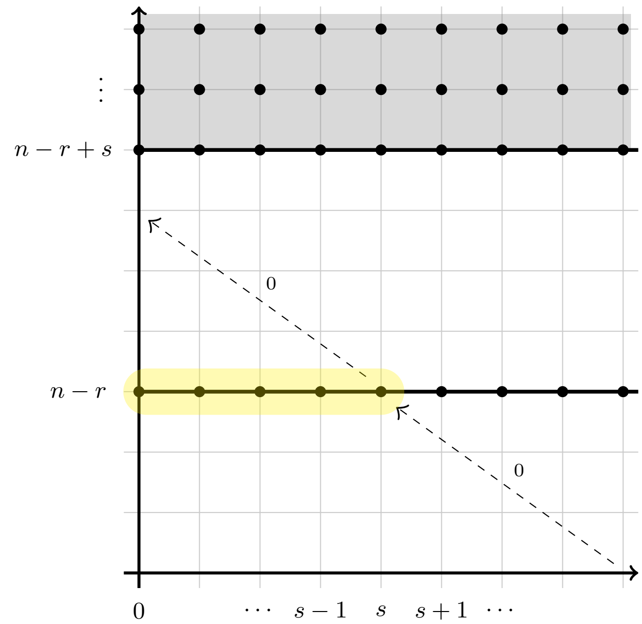
Thus (ii) $\Leftrightarrow$ (iii).

Finally assume (iii). Consider the spectral sequence in (2.24)

$$E_2^{p,q} = \mathrm{Ext}_X^p(\mathscr{F},\mathscr{E}\mathrm{xt}_P^q(\mathscr{O}_X,\omega_P)) \Rightarrow \mathrm{Ext}_P^{p+q}(\mathscr{F},\omega_P).$$

Recall the natural transformation ε^0 we defined in the proof of theorem 2.30 using this spectral sequence. In general we can define the edge homomorphisms $\varepsilon^{r-i}(\mathscr{F}) : \mathrm{Ext}_X^{r-i}(\mathscr{F},\omega_X) \to \mathrm{Ext}_P^{n-i}(\mathscr{F},\omega_P)$. Together these form a δ-morphism $\varepsilon^* : \mathrm{Ext}_X^*(-,\omega_X) \to \mathrm{Ext}_P^*(-,\omega_P)$ of degree $n-r$.

Now, by lemma 2.28, we have $\mathscr{E}\mathrm{xt}_P^q(\mathscr{O}_X,\omega_P) = 0$ for all $q < n-r$ and by (iii), $\mathscr{E}\mathrm{xt}_P^{n-i}(\mathscr{O}_X,\omega_P) = 0$ for all $r-s \le i < r$, i.e., for all $n-r < q = n-i \le n-r+s$. So the only nonzero rows in the spectral sequence are rows $n-r$ and $n-r+s+1$ and above. By these lacunary considerations, the edge homomorphism $\varepsilon^{r-i}(\mathscr{F})$ is an isomorphism.



Now suppose the following diagram commutes.

$$\begin{tikzcd}
\operatorname{Ext}_X^{r-i}(\mathscr{F}, \omega_X) \arrow[r] \arrow[d, "\varepsilon^{r-i}(\mathscr{F})"', "\sim"] & H^i(X, \mathscr{F})^* \arrow[d, dash, "\mathrm{id}"] \\
\operatorname{Ext}_P^{n-i}(\mathscr{F}, \omega_P) \arrow[r] & H^i(P, \mathscr{F})^*
\end{tikzcd}$$

The bottom row is an isomorphism by Serre duality, whence $\operatorname{Ext}_X^{r-i}(\mathscr{F}, \omega_X)$ is isomorphic to $H^i(X, \mathscr{F})^*$. This proves (i), modulo the proof that the diagram indeed commutes. This last part of the proof is somewhat long and technical and so is omitted; see lemma IV.5.3 in [1] for a proof. $\square$

Corollary 2.36. *Suppose the same conditions from the theorem above hold. Then*
(a) $\operatorname{Hom}_X(\mathscr{F}, \omega_X) \xrightarrow{\sim} H^r(X, \mathscr{F})^*$ *is always an isomorphism.*
(b) The map $\operatorname{Ext}_X^{r-i}(\mathscr{F}, \omega_X) \to H^i(X, \mathscr{F})^*$ *is an isomorphism for all i iff X is Cohen-Macaulay.*

3. The Riemann-Roch theorem for curves

Now our goal is to persuade the reader that all the time spent on developing Serre duality is worth it. After all the abstraction in the previous section, we now present two concrete applications. The first is the Riemann-Roch theorem, which is an easy consequence of Serre duality with a wide range of applications.

In this section, by a curve X we mean an integral regular scheme of dimension 1 that is proper over k. First we define a well-known invariant of the curve.

Proposition/Definition 3.1. *If X is a curve, its* genus *is* $g = \dim_k H^1(X, \mathscr{O}_X)$.

Remark 3.2. The reason why this is not a *bona fide* definition is because generally one defines the arithmetic genus of a variety using the Hilbert polynomial and the geometric genus using the dimension of the global sections of ω_X. In the case for curves, it is not too difficult using Serre duality to show that both quantities are equal to the dimension of $H^1(X, \mathscr{O}_X)$.

Given a (Weil) divisor D on a curve X, we denote by $\mathscr{L}(D)$ the associated invertible sheaf. In particular, we let K be the canonical divisor corresponding to the canonical sheaf ω_X. Also, define $l(D)$ be $\dim_k H^0(X, \mathscr{L}(D))$. The complete linear system $|D|$ can be parametrized by the projective space $(H^0(X, \mathscr{L}(D)) \setminus \{0\})/k^*$, and so the dimension of $|D|$ is $l(D) - 1$. Using this interpretation of $l(D)$ and the fact that every divisor with $|D| \neq \emptyset$ is linearly equivalent to some effective divisor by definition, we can prove the following lemma.

Lemma 3.3. *Let D be a divisor on a curve X such that $l(D) \neq 0$. Then $\deg D \geq 0$. If we have equality, then $D \sim 0$.*

Theorem 3.4 (Riemann-Roch)**.** *Let D be a divisor on a curve X of genus g. Then*

$$l(D) - l(K - D) = \deg D + 1 - g.$$

Proof. The divisor $K - D$ corresponds to the invertible sheaf $\omega_X \otimes \mathscr{L}(D)^\vee$. Since X is projective, Serre duality implies $H^0(X, \omega_X \otimes \mathscr{L}(D)^\vee) \cong H^1(X, \mathscr{L}(D))^*$. So in terms of the Euler characteristic

$$\chi(\mathscr{F}) := \dim H^0(X, \mathscr{F}) - \dim H^1(X, \mathscr{F}),$$

we wish to show

$$\chi(\mathscr{L}(D)) = \deg D + 1 - g.$$

Consider the case $D = 0$. We have $\mathscr{L}(D) \cong \mathscr{O}_X$, and thus the formula holds since

$$\begin{gathered} \chi(\mathscr{L}(D)) = 1 - \dim H^1(X, \mathscr{O}_X) \\ \deg D + 1 - g = 0 + 1 - \dim H^1(X, \mathscr{O}_X). \end{gathered}$$

Now we will show that the Riemann-Roch formula holds for D iff it holds for $D + P$, where P is any point of X. Consider the point P as a closed subscheme of X with structure sheaf $k(P)$, the skyscraper sheaf k sitting at P. Its ideal sheaf is $\mathscr{L}(-P)$. So we have the short exact sequence

$$0 \to \mathscr{L}(-P) \to \mathscr{O}_X \to k(P) \to 0.$$

Tensoring with $\mathscr{L}(D + P)$ gives

$$0 \to \mathscr{L}(D) \to \mathscr{L}(D + P) \to k(P) \to 0.$$

Since the Euler characteristic χ is additive on short exact sequences and

$$\chi(k(P)) = \dim H^0(X, k(P)) - \dim H^1(X, k(P)) = 1 - 0 = 1,$$

we get $\chi(\mathscr{L}(D+P)) = \chi(\mathscr{L}(D)) + 1$. On the other hand, $\deg(D+P) = \deg D + 1$, so the Riemann-Roch formula holds for D iff it holds for $D + P$.

Note the role played by Serre duality in this proof. Using duality, we transformed the group $H^0(X, \omega_X \otimes \mathscr{L}(D)^\vee)$ into $H^1(X, \mathscr{L}(D))^*$, which is much, much more tractable. □

The Riemann-Roch theorem together with the preceding lemma quite easily solves the *Riemann-Roch problem* for curves, which asks to determine the behaviour of $\dim|nD|$ as n grows large, given a divisor D. We leave it to the reader to work out the case-by-case analysis, which can also be found in [12].

The Riemann-Roch theorem also allows us to define a group structure on an elliptic curve X together with a point $P_0 \in X$ using the Picard group. Recall that a curve X is elliptic if it has genus one. In this case, the Riemann-Roch theorem applied to the canonical divisor K shows that $\deg K = 0$.

Let $\mathrm{Pic}^0(X)$ denote the subgroup of $\mathrm{Pic}\, X$ corresponding to divisors of degree 0. We claim that the map $P \mapsto \mathscr{L}(P - P_0)$ is a bijection. To show this, we need to prove there exists a unique $P \in X$ such that $D \sim P - P_0$ for any divisor D with $\deg D = 0$. Applying Riemann-Roch to $D + P_0$, we get

$$l(D + P_0) - l(K - D - P_0) = \deg(D + P_0) + 1 - g = 1.$$

Moreover, since $\deg K = 0$, we have $\deg(K-D-P_0) = -1$, so $l(K-D-P_0) = 0$. Then $l(D + P_0) = 1$ and $\dim|D + P_0| = 0$. Hence there exists a unique effective divisor P linearly equivalent to $D + P_0$. Since $\deg P = 1$, it is a point, and it is the unique point such that $P - P_0 \sim D$. So we have a bijective correspondence between points of X and $\mathrm{Pic}^0(X)$, and as the latter forms a group, we have an induced group structure on the elliptic curve (X, P_0).

4. Bott's theorem

In this section, we use the techniques and ideas developed above to give a proof of Bott's theorem (of Borel-Weil-Bott fame) in representation theory. The outline for the proof is a two-page article by Demazure [8], which we combine with a clever argument from Professor Nori.

4.1. **Statement and proof.**

Let k be a field of characteristic zero, and G a split reductive simply-connected algebraic group over k, T a split maximal torus of G, and B a Borel subgroup of G containing T. Let X be the group of characters of B and $W = N_G(T)/T$ be the Weyl group, and let $R \subseteq X$ denote the set of all roots of G with respect to T. Denote by $s_\alpha \in W$ the simple reflection corresponding to the simple root α. Also, let $\rho \in X$ be the half-sum of all the positive roots. Then $\langle \alpha^\vee, \rho \rangle = 1$ for every simple root α. We center the action of the Weyl group at ρ, i.e., define the dot action $w \cdot \lambda = w(\lambda + \rho) - \rho$.

Given an integral weight λ of G, let $\mathscr{L}_{-\lambda}$ be the invertible sheaf over the flag variety G/B associated with the equivariant geometric line bundle $L_{-\lambda}$ given by λ. This correspondence between roots and invertible sheaves is rather involved, so we work through a simple concrete example to illustrate the construction.

Example 4.1 (The construction of $\mathscr{L}_{-\lambda}$ for $G = SL(3, \mathbb{C})$)**.** Let $G = SL(3, \mathbb{C})$ and $T \leq G$ be the maximal torus consisting of diagonal matrices. Let $B \leq G$ be the Borel subgroup of upper triangular matrices. Before we move on to the actual construction, let's pause a while to figure out the character group of B. We can write $B = TU$, where U is the unipotent part of B. Since irreducible representations of U are trivial, every character of B restricts to a unique character on T, and since $T \cong \mathbf{G}_m \times \mathbf{G}_m$ ($\mathbf{G}_m$ is the multiplicative group $\mathbb{C}^\times$ thought of as an algebraic group), we see that its character group $X(T)$ is $\mathbb{Z} \times \mathbb{Z}$, and every character of B is of the form

$$\lambda_{m_1,m_2}\begin{pmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{pmatrix} = a^{m_1}b^{m_2}.$$

for some integers m_1 and m_2.

Next, the root system $SL(3,\mathbb{C})$ consists of two simple roots. Let $\alpha \in X(T)$ be the simple root given by

$$\alpha\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = ab^{-1}.$$

Then the root subgroup and the parabolic subgroup corresponding to α are

$$U_\alpha = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ z & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : z \in \mathbb{C} \right\} \quad \text{and} \quad P_\alpha = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \in SL(3,\mathbb{C}) \right\}.$$

Now let $\lambda \in X$ be a root, i.e., a character of B, say $\lambda = \lambda_{m_1,m_2}$. We define the (geometric) line bundle $L_{-\lambda}$ associated to λ by

$$\begin{tikzcd} L_{-\lambda} := (G \times \mathbb{C})/(gb,v) \sim (g,\lambda(b)v) \arrow[d, "p"] \\ G/B \end{tikzcd}$$

Denote by $\mathscr{L}_{-\lambda}$ the sheaf of sections of $L_{-\lambda}$. This is a locally free sheaf of rank one, and thus is an invertible sheaf on G/B.

Now consider the fibration $\pi : G/B \to G/P_\alpha$. For $y \in G/P_\alpha$, denote by $(G/B)_y \subseteq G/B$ the fibre above y and $(\mathscr{L}_{-\lambda})_y$ the restriction of $\mathscr{L}_{-\lambda}$ to $(G/B)_y$. By the Bruhat decomposition $P_\alpha = B \sqcup Bs_\alpha B$, we have $(G/B)_y \cong \mathbb{P}^1$. So $(\mathscr{L}_{-\lambda})_y$ is an invertible sheaf on $\mathbb{P}^1$. As $\operatorname{Pic}\mathbb{P}^1 \cong \mathbb{Z}$, we know that $(\mathscr{L}_{-\lambda})_y \cong \mathscr{O}_{\mathbb{P}^1}(m)$, where m is the degree of $(\mathscr{L}_{-\lambda})_y$.

To figure out what m is, we explicitly realize the isomorphism $(G/B)_y \cong \mathbb{P}^1$ as follows. Let P_α act on $\mathbb{P}^1$, thought of as the set of lines in $\mathbb{A}^2$, by

$$\begin{pmatrix} a & b & * \\ c & d & * \\ 0 & 0 & * \end{pmatrix} \cdot [x_0 : x_1] = [ax_0 + bx_1 : cx_0 + dx_1].$$

Then $\operatorname{Stab}_{P_\alpha}([1:0]) = B$. Since the action is transitive, this gives a map $P_\alpha/B \xrightarrow{\sim} \mathbb{P}^1$ by $gB \mapsto g \cdot [1:0]$. Now define the map $\pi^{-1}(yP_\alpha) = y \cdot (P_\alpha/B) \xrightarrow{\sim} \mathbb{P}^1$ by $y(gB) \mapsto g \cdot [1:0]$. This reduces our analysis to the case where y is the identity element in G/P_α.

Next, we introduce a trivializing cover for $(\mathscr{L}_{-\lambda})_y$ on $\mathbb{P}^1$ as follows:

$$U_0 = \{[1:x_1] : x_1 \in \mathbb{A}^1\} \quad \text{and} \quad U_1 = \{[x_0:1] : x_0 \in \mathbb{A}^1\}.$$

The line bundle $(\mathscr{L}_{-\lambda})_y$ trivializes over U_0 and U_1 since they are affine. We now construct the local trivializations. First consider the open set U_0. Each element in $p^{-1}(U_0)$ can be uniquely represented as

$$[\overline{g}, v] \in (P_\alpha/B) \times \mathbb{C}, \text{ where } g = \begin{pmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, v \in \mathbb{C}.$$

Now define the homeomorphism

$$\varphi_0 : \mathbb{A}^1 \times \mathbb{C} \to p^{-1}(U_0), \quad ([1 : x_1], v) \mapsto \left[\begin{pmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, v \right].$$

It has the inverse $\varphi_0^{-1} : p^{-1}(U_0) \to \mathbb{A}^1 \times \mathbb{C}$, where

$$\begin{aligned} \varphi_0^{-1}(\left[\begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & g \end{pmatrix}, v \right]) &= \varphi_0^{-1}(\left[\begin{pmatrix} 1 & 0 & 0 \\ \frac{c}{a} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & e \\ 0 & \frac{1}{ag} & \frac{af-ce}{a} \\ 0 & 0 & g \end{pmatrix}, v \right]) \\ &= \varphi_0^{-1}(\left[\begin{pmatrix} 1 & 0 & 0 \\ \frac{c}{a} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \lambda \begin{pmatrix} a & b & e \\ 0 & \frac{1}{ag} & \frac{af-ce}{a} \\ 0 & 0 & g \end{pmatrix} v \right]) \\ &= \varphi_0^{-1}(\left[\begin{pmatrix} 1 & 0 & 0 \\ \frac{c}{a} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, a^{m_1 - m_2} g^{-m_2} v \right]) \\ &= ([1 : \frac{c}{a}], a^{m_1 - m_2} g^{-m_2} v). \end{aligned}$$

Similarly on U_1, every element in $p^{-1}(U_1)$ can be uniquely represented by

$$[\overline{g}, v] \in (P_\alpha / B) \times \mathbb{C}, \text{ where } g = \begin{pmatrix} x_0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, v \in \mathbb{C}$$

and we have the local trivialization

$$\begin{aligned} \varphi_1 : &\mathbb{A}^1 \times \mathbb{C} \to p^{-1}(U_1) \\ &([x_0 : 1], v) \mapsto \left[\begin{pmatrix} x_0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, v \right] \\ \varphi_1^{-1} : &p^{-1}(U_1) \to \mathbb{A}^1 \times \mathbb{C} \\ &\left[\begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & g \end{pmatrix}, v \right] \mapsto ([\frac{a}{c} : 1], c^{m_1 - m_2} g^{-m_2} v). \end{aligned}$$

Now let's figure out the gluing condition on the sheaf $(\mathscr{L}_{-\lambda})_y$. Consider a section $s : (G/B)_y \cong \mathbb{P}^1 \to L_{-\lambda}$. Then from the condition $p \circ s = \mathrm{id}$, we require that s takes the following forms on the affine cover.

$$\begin{aligned} s|_{U_0} : &[x_0 : x_1] \mapsto [\begin{pmatrix} 1 & 0 & 0 \\ \frac{x_1}{x_0} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, h_0([x_0 : x_1]) v] \\ s|_{U_1} : &[x_0 : x_1] \mapsto [\begin{pmatrix} \frac{x_0}{x_1} & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, h_1([x_0 : x_1]) v]. \end{aligned}$$

where $h_0 : U_0 \to \mathbb{C}$ and $h_1 : U_1 \to \mathbb{C}$ are some rational functions.

If $[x_0 : x_1] \in U_0 \cap U_1$, then we need the section to agree on the intersection. We have

$$\begin{aligned} s|_{U_0}[x_0 : x_1] &= \left[\begin{pmatrix} 1 & 0 & 0 \\ \frac{x_1}{x_0} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, h_0([x_0 : x_1])v \right] \\ &= \left[\begin{pmatrix} \frac{x_0}{x_1} & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{x_1}{x_0} & 1 & 0 \\ 0 & \frac{x_0}{x_1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, h_0([x_0 : x_1])v \right] \\ &= \left[\begin{pmatrix} \frac{x_0}{x_1} & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \underbrace{(\frac{x_1}{x_0})^{m_1 - m_2} h_0([x_0 : x_1])}_{h_1([x_0:x_1])} v \right] \end{aligned}$$

So the gluing condition is $h_1([x_0 : x_1]) = (\frac{x_1}{x_0})^{m_1-m_2} h_0([x_0 : x_1])$. Note that $[x_0 : x_1] \mapsto (\frac{x_1}{x_0})^{m_1-m_2} \in GL(1, \mathbb{C})$ is the transition function for $\varphi_1^{-1} \circ \varphi_0$.

Now let's take a step back. We showed that the sections of $\mathscr{L}_{-\lambda}$ is

$$\mathscr{L}_{-\lambda}(U) = \{(h_0 : U \cap U_0 \to \mathbb{C}, h_1 : U \cap U_1 \to \mathbb{C}) \mid h_1 = (\frac{x_1}{x_0})^{m_1-m_2} h_0\}.$$

We now recognize that $\mathscr{L}_{-\lambda}$ is simply the line bundle $\mathscr{O}_{\mathbb{P}^1}(-(m_1-m_2))$. One way to see this is to calculate the degree of a meromorphic section by summing the orders of the zeroes and poles. Let $z_1 = \frac{x_1}{x_0}$ be a coordinate on U_0 and $z_0 = \frac{x_0}{x_1}$ be a coordinate on U_1. Consider the section $h_0(z_1) = z_1$ over U_0. It has a zero of order 1 at $[0 : 1]$. Also, h_0 extends to the rational section $h_1(z_0) = (\frac{1}{z_0})^{m_1-m_2} \frac{1}{z_0} = \frac{1}{z_0^{m_1-m_2+1}}$ over U_1, which has a pole of order $m_1 - m_2 + 1$ at $[1 : 0]$. So overall, the sum of the orders is $-(m_1 - m_2)$.

We leave it to the reader to work out the case for the other simple root of $SL(3, \mathbb{C})$.

Remark 4.2. The correspondence $\lambda \mapsto \mathscr{L}_{-\lambda}$ is a *homomorphism* $X(T) \to \operatorname{Pic} G/B$.

Proposition 4.3. *The degree of the invertible sheaf* $(\mathscr{L}_{-\lambda})_y$ *is* $-\langle \alpha^\vee, \lambda \rangle$, *i.e.,* $(\mathscr{L}_\lambda)_y \cong \mathscr{O}(\langle \alpha^\vee, \lambda \rangle)$.

Proof (in the $SL(3, \mathbb{C})$ case). In the example given above, α was the root given by

$$\alpha \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = ab^{-1}.$$

Its coroot is the cocharacter

$$\alpha^\vee(t) = \begin{pmatrix} t & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which should be highly reminiscent of the matrix

$$\begin{pmatrix} \frac{x_1}{x_0} & 1 & 0 \\ 0 & \frac{x_0}{x_1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

used above to figure out the transition function. Applying the character λ gives $(\lambda \circ \alpha^\vee)(t) = t^{m_1 - m_2}$, so $\langle \alpha^\vee, \lambda \rangle = m_1 - m_2$, which agrees with what we obtained above.

It should not be too surprising that the calculation for general reductive groups G would be similar. Indeed, the calculation above already easily generalizes to all $SL(n, \mathbb{C})$. $\square$

Let G be a reductive algebraic group, etc. Let C be the dominant Weyl chamber with boundary, i.e., $C = \{\mu \in X : \langle \alpha^\vee, \mu \rangle \geq 0\}$ consisting of the dominant roots of G with respect to T. There are two possibilities for the root $\lambda + \rho \in X$. Either $\lambda + \rho$ is singular and lies on the boundary of some Weyl chamber, i.e., $\langle \alpha^\vee, \lambda + \rho \rangle = 0$ for some root α, or $\lambda + \rho$ is regular and is in the interior of some Weyl chamber, so there is a unique w in the Weyl group W such that $w \cdot \lambda \in C$.

To see this, suppose $\lambda + \rho$ is regular and $\langle \alpha^\vee, \lambda + \rho \rangle \neq 0$ for all roots α. Then choose $w \in W$ to maximize $\langle \rho^\vee, w(\lambda + \rho) \rangle$. We have

$$\begin{aligned} \langle \rho^\vee, w(\lambda + \rho) \rangle &\geq \langle \rho^\vee, s_\alpha w(\lambda + \rho) \rangle = \langle s_\alpha(\rho)^\vee, w(\lambda + \rho) \rangle \\ &= \langle \rho^\vee - \alpha^\vee, w(\lambda + \rho) \rangle = \langle \rho^\vee, w(\lambda + \rho) \rangle - \langle \alpha^\vee, w(\lambda + \rho) \rangle, \end{aligned}$$

so $\langle \alpha^\vee, w(\lambda + \rho) \rangle \geq 0$ for all α. Since $\lambda + \rho$ is not singular, in particular it does not lie on the hyperplane perpendicular to $w^{-1}(\alpha)$, so $\langle \alpha^\vee, w(\lambda + \rho) \rangle \neq 0$. Hence $\langle \alpha^\vee, w(\lambda + \rho) \rangle \geq 1$ and thus $\langle \alpha^\vee, w \cdot \lambda \rangle = \langle \alpha^\vee, w(\lambda + \rho) - \rho \rangle \geq 0$ for all α, so $w \cdot \lambda \in C$.

Denote the length of an element w in W by $\ell(w)$, i.e., the minimum number of simple reflections needed in an expression for w. Now we can state Bott's theorem.

Theorem 4.4 (Bott). *Let G be a split reductive simply-connected algebraic group over an algebraically closed field, and fix a split maximal torus T along with a Borel subgroup B containing T. Let W be the Weyl group acting via the shifted action on the integral weights. Let λ be an integral weight of G, and let $\mathscr{L}_\lambda$ be the line bundle constructed above. Then*

(a) If $\lambda \in C$, then $H^i(G/B, \mathscr{L}_\lambda) = 0$ for $i \neq 0$.
(b) If $\lambda + \rho$ lies on some hyperplane defined by the root α, then $H^i(G/B, \mathscr{L}_\lambda) = 0$ for all i.
(c) Otherwise, if there exists $w \in W$ (which is unique) such that $w^{-1} \cdot \lambda = \mu \in C$, then

$$H^i(G/B, \mathscr{L}_\lambda) = \begin{cases} H^0(G/B, \mathscr{L}_\mu) & \text{if } i = \ell(w) \\ 0 & \text{if } i \neq \ell(w). \end{cases}$$

Bott's theorem can be deduced from the following theorem.

Theorem 4.5. *Let α be a simple root and λ a root of G be such that $\langle \alpha^\vee, \lambda + \rho \rangle \geq 0$. Then there exists an isomorphism*

$$H^i(G/B, \mathscr{L}_\lambda) \xrightarrow{\sim} H^{i+1}(G/B, \mathscr{L}_{s_\alpha \cdot \lambda})$$

of G-modules for $i \in \mathbb{Z}$.

The proof of theorem 4.5 in [8] uses the Borel-Weil theorem to construct representations of the parabolic subgroup P_α with special properties. The proof we give below eschews the Borel-Weil theorem and proceeds using algebraic geometry (cf. [6]). But before we do so, let us see how we can deduce Bott's theorem from (4.5). We start with a corollary.

Corollary 4.6. *Let $\lambda \in X$ such that $\lambda+\rho$ is dominant, i.e., $\langle \alpha^\vee, \lambda+\rho\rangle \geq 0$ for all simple roots α. If $w \in W$ has length n, then $H^i(G/B, \mathscr{L}_\lambda)$ and $H^{n+i}(G/B, \mathscr{L}_{w\cdot\lambda})$ as isomorphic as G-modules.*

Proof. Induct on the length of w and use theorem 4.5. The only tricky part is to show that if $\lambda+\rho \in C$ and $w = s_\alpha w'$ where $\ell(w') = \ell(w)-1$, then $\langle \alpha, (w'\cdot\lambda)+\rho\rangle \geq 0$, but this follows since $w'^{-1}(\alpha)$ is a positive root (see [5]). □

Now we prove Bott's theorem.

Proof of Bott's theorem.

(a) Let w_0 be the longest element in the Weyl group, so $\ell(w_0) = \dim G/B =: n$. Then $\lambda+\rho$ is dominant, and so we may apply the corollary to obtain
$$H^i(G/B, \mathscr{L}_\lambda) \cong H^{i+n}(G/B, \mathscr{L}_{w_0\cdot\lambda}).$$
But if $i > 0$, then $i+n > \dim G/B$, so by Grothendieck vanishing, this cohomology group is zero.

(b) Since $\lambda+\rho$ lies on the hyperplane defined by α, we have $\langle \alpha^\vee, \lambda+\rho\rangle = 0$ and $s_\alpha \cdot \lambda = \lambda$. So we may apply the theorem. Hence we see that
$$H^i(G/B, \mathscr{L}_\lambda) \cong H^{i+1}(G/B, \mathscr{L}_{s_\alpha\cdot\lambda}) = H^{i+1}(G/B, \mathscr{L}_\lambda).$$
We can repeat this involution to get $H^i(G/B, \mathscr{L}_\lambda) \cong H^{i+k}(G/B, \mathscr{L}_\lambda)$ for all $k \in \mathbb{Z}^+$. But once $i+k > \dim G/B$, the cohomology group is again zero by Grothendieck vanishing.

(c) We have $\lambda = w\cdot\mu$, where μ is dominant. Applying the corollary, we get
$$H^i(G/B, \mathscr{L}_\lambda) = H^i(G/B, \mathscr{L}_{w\cdot\mu}) = H^{i-\ell(w)}(G/B, \mathscr{L}_\mu).$$
By part (a), if $i \neq \ell(w)$, we have $H^i(G/B, \mathscr{L}_\lambda) = 0$. Otherwise, we have $H^{\ell(w)}(G/B, \mathscr{L}_\lambda) = H^0(G/B, \mathscr{L}_\mu)$.

□

Now that we have proved Bott's theorem, the natural question (if one does not already know the answer) to ask is what is special about $H^0(G/B, \mathscr{L}_\mu)$ when μ is a dominant weight.

Theorem 4.7 (Borel-Weil)**.** *$H^0(G/B, \mathscr{L}_\mu)$ is dual to the irreducible highest-weight representation of G with highest weight μ.*

Bott's theorem implies that the Euler characteristic $\chi(G/B, \mathscr{L}_\mu)$ is equal to $\dim H^0(G/B, \mathscr{L}_\mu)$. It is now possible to apply some version of Hirzebruch-Riemann-Roch to $\chi(G/B, \mathscr{L}_\mu)$ and deduce the Weyl dimension formula. Furthermore, we can combine theorem 4.4 with the Atiyah-Bott fixed point theorem to obtain the Weyl character formula. See [7], [2], and [4].

4.2. **Some facts from algebraic geometry.** The goal for this section is to develop some general algebraic geometric facts that will be needed to prove theorem 4.5. First, we prove several "projection"-type formulae.

Fact 4.8 ([12], Ex. III.8.3)**.** *Let $f : X \to Y$ be a morphism of ringed spaces, let $\mathscr{F}$ be an $\mathscr{O}_X$-module, and let $\mathscr{E}$ be a locally free $\mathscr{O}_Y$-module of finite rank. Then for $i \geq 0$, we have the projection formula $R^i f_*(\mathscr{F} \otimes f^*\mathscr{E}) \cong R^i f_*(\mathscr{F}) \otimes \mathscr{E}$.*

Proof sketch. As usual with proving facts about derived functors, first consider the case $i = 0$. We have

$$\begin{aligned} f_*(\mathscr{F}) \otimes \mathscr{E} &\cong f_*(\mathscr{F}) \otimes \mathscr{E}^{\vee\vee} \cong \mathscr{H}\mathrm{om}(\mathscr{E}^\vee, f_*\mathscr{F}) \\ &\cong f_* \mathscr{H}\mathrm{om}(f^*\mathscr{E}^\vee, \mathscr{F}) \cong f_*(\mathscr{F} \otimes (f^*\mathscr{E}^\vee)^\vee) \\ &\cong f_*(\mathscr{F} \otimes f^*\mathscr{E}). \end{aligned}$$

Now let $0 \to \mathscr{F} \to \mathscr{I}^\bullet$ be an injective resolution of $\mathscr{F}$.

$$\begin{tikzcd} 0 \arrow[r] & f_*(\mathscr{F} \otimes f^*\mathscr{E}) \arrow[r] \arrow[d, "\sim"] & f_*(\mathscr{I}^0 \otimes f^*\mathscr{E}) \arrow[r] \arrow[d, "\sim"] & f_*(\mathscr{I}^1 \otimes f^*\mathscr{E}) \arrow[r] \arrow[d, "\sim"] & \cdots \\ 0 \arrow[r] & f_*(\mathscr{F}) \otimes \mathscr{E} \arrow[r] & f_*(\mathscr{I}^0) \otimes \mathscr{E} \arrow[r] & f_*(\mathscr{I}^1) \otimes \mathscr{E} \arrow[r] & \cdots \end{tikzcd}$$

The isomorphisms on the cochain level induce an isomorphism on cohomology, i.e., $R^i f_*(\mathscr{F} \otimes f^*\mathscr{E}) \cong R^i f_*(\mathscr{F}) \otimes \mathscr{E}$ as wanted. □

Fact 4.9 ([12], Ex. III.8.4a–c)**.** *Let Y be a noetherian scheme, and let $\mathscr{E}$ be a locally free $\mathscr{O}_Y$-module of rank $n+1$, $n \geq 1$. Let $X = \mathbb{P}(\mathscr{E})$, with the invertible sheaf $\mathscr{O}_X(1)$ and the projection morphism $\pi : X \to Y$. Then*

(i) We have $\pi_*(\mathscr{O}(l)) \cong S^l(\mathscr{E})$ for $l \geq 0$, $\pi_*(\mathscr{O}(l)) = 0$ for $l < 0$, $R^j\pi_*(\mathscr{O}(l)) = 0$ for $0 < j < n$ and all $l \in \mathbb{Z}$, and $R^n\pi_*(\mathscr{O}(l)) = 0$ for $l > -n-1$ (cf. proposition II.7.11 in [12]*).*

(ii) For any $l \in \mathbb{Z}$, $R^n\pi_*(\mathscr{O}(l)) \cong \pi_*(\mathscr{O}(-l-n-1))^\vee \otimes (\wedge^{n+1}\mathscr{E})^\vee$.

Proof sketch.

(i) Let $\{U_i\}$ be a trivializing cover for $\mathscr{E}$ on Y. Assume without loss of generality that U_i is affine for each i, so $U_i = \operatorname{Spec} A_i$ for some noetherian ring A_i. We have $\mathscr{E}(U_i) \cong A_i^{n+1}$ and $\pi^{-1}(U_i) = \mathbb{P}^n_{A_i}$. By proposition III.8.1 in [12] and theorem 2.4, we have

$$\begin{aligned} R^j\pi_*(\mathscr{O}(l))(U_i) &= H^j(\pi^{-1}(U_i), \mathscr{O}(l)|_{\pi^{-1}(U_i)}) = H^j(\mathbb{P}^n_{A_i}, \mathscr{O}(l)) \\ &= \begin{cases} S^l(A_i) & \text{for } j = 0 \text{ and } l \geq 0 \\ 0 & \text{for } j = 0 \text{ and } l < 0 \\ 0 & \text{for } j \neq 0, n \text{ and } l \in \mathbb{Z} \\ 0 & \text{for } j = n \text{ and } l > -n-1. \end{cases} \end{aligned}$$

(ii) By proposition II.7.11, we have the surjection $\pi^*\mathscr{E} \to \mathscr{O}_X(1) \to 0$. Twisting by $\mathscr{O}(-1)$, we get the short exact sequence

$$0 \to \mathscr{F} \to (\pi^*\mathscr{E})(-1) \to \mathscr{O}_X \to 0.$$

Let U be any open affine subscheme of Y trivializing $\mathscr{E}$, i.e., $\pi^{-1}(U) \cong \mathbb{P}^n_A$ for some noetherian ring A. Restricting to $\pi^{-1}(U)$, we get

$$0 \to \mathscr{F}|_{\mathbb{P}^n_A} \to \mathscr{O}_X(-1)^{n+1}|_{\mathbb{P}^n_A} \to \mathscr{O}_X|_{\mathbb{P}^n_A} \to 0.$$

Comparing this to the short exact sequence in theorem II.8.13, we see that $\mathscr{F}|_{\mathbb{P}^n_A} \cong \Omega_{\mathbb{P}^n_A/Y}$. Gluing over an open cover, we get $\mathscr{F} \cong \Omega_{X/Y}$. So we have the short exact sequence

$$0 \to \Omega_{X/Y} \to (\pi^*\mathscr{E})(-1) \to \mathscr{O}_X \to 0.$$

Applying exercise II.5.16d, we get

$$\wedge^{n+1}(\pi^*\mathscr{E}(-1)) = (\pi^* \wedge^{n+1} \mathscr{E})(-n-1) \cong \wedge^n \Omega_{X/Y} \otimes \wedge^1 \mathscr{O}_X \cong \wedge^n \Omega_{X/Y}.$$

The sheaf $\omega_{X/Y} = \wedge^n \Omega_{X/Y}$ is called the relative canonical sheaf.

Now cover X with open subsets of the form $U = \mathbb{P}^n_A$ over which $\mathscr{E}$ trivializes, i.e., $\mathscr{E}(U) \cong A^{n+1}$ and $\pi^{-1}(U) \cong \mathbb{P}^n_A$. Then $\omega_{X/Y}|_{\pi^{-1}(U)} \cong \mathscr{O}_X(-n-1)|_{\pi^{-1}(U)}$ and

$$\begin{aligned} R^n\pi_*(\omega_{X/Y})|_{\operatorname{Spec} A} &\cong R^n\pi_*(\omega_{X/Y}|_{\pi^{-1}(U)}) \\ &\cong R^n\pi_*(\mathscr{O}_X(-n-1)|_{\pi^{-1}(U)}) \cong \mathscr{O}_Y|_{\operatorname{Spec} A}. \end{aligned}$$

Gluing these isomorphisms, we get $R^n\pi_*(\omega_{X/Y}) \cong \mathscr{O}_Y$.

Finally,

$$\begin{aligned} R^n\pi_*(\mathscr{O}(l)) &= R^n\pi_*(\omega_{X/Y} \otimes \omega_{X/Y}^\vee \otimes \mathscr{O}(l)) \\ &= R^n\pi_*(\omega_{X/Y} \otimes \mathscr{H}\mathrm{om}(\omega_{X/Y}, \mathscr{O}(l))) \\ &= R^n\pi_*(\omega_{X/Y} \otimes \mathscr{H}\mathrm{om}(\pi^* \wedge^{n+1} \mathscr{E} \otimes \mathscr{O}(-n-1), \mathscr{O}(l))) \\ &= R^n\pi_*(\omega_{X/Y} \otimes \pi^* \mathscr{H}\mathrm{om}(\wedge^{n+1}\mathscr{E}, \pi_*\mathscr{O}(l+n+1))) \\ &= (R^n\pi_*\omega_{X/Y}) \otimes \mathscr{H}\mathrm{om}(\wedge^{n+1}(\mathscr{E}), \pi_*\mathscr{O}(l+n+1)) \\ &= (\wedge^{n+1}\mathscr{E})^\vee \otimes \pi_*\mathscr{O}(l+n+1), \end{aligned}$$

where $\pi_*\mathscr{O}(l+n+1) = (\pi_*\mathscr{O}(-l-n-1))^\vee$.

□

Now we need a correspondence between $\mathbb{P}^n$-fibrations and the projective space bundle construction in algebraic geometry.

Definitions 4.10. Let X be a noetherian scheme.

(1) Let $\mathscr{S}$ be a quasicoherent sheaf of $\mathscr{O}_X$-modules, which has a structure of a sheaf of graded $\mathscr{O}_X$-algebras. Then $\mathscr{S} \cong \bigoplus_{d \geq 0} \mathscr{S}_d$, where $\mathscr{S}_d$ is the homogeneous part of degree d. Assume that $\mathscr{S}_0 = \mathscr{O}_X$, that $\mathscr{S}_1$ is a coherent $\mathscr{O}_X$-module, and that $\mathscr{S}$ is locally generated by $\mathscr{S}_1$ as an $\mathscr{S}_0$-algebra.

   We construct the **Proj** of $\mathscr{S}$ over X as follows (this is called the relative Proj construction). For each open affine subset $U = \operatorname{Spec} A$ of X, let $\mathscr{S}(U) = \Gamma(U, \mathscr{S}|_U)$, and consider $\operatorname{Proj} \mathscr{S}(U)$ with its natural morphism $\pi_U : \operatorname{Proj} \mathscr{S}(U) \to U$. It can be shown that if V is another open affine subset of X, then $\pi_U^{-1}(U \cap V)$ is naturally isomorphic to $\pi_V^{-1}(U \cap V)$, which allows us to glue the schemes $\operatorname{Proj} \mathscr{S}(U)$ together. This gives a scheme $\mathbf{Proj}\, \mathscr{S}$ together with a morphism $\pi : \mathbf{Proj}\, \mathscr{S} \to X$ such that $\pi^{-1}(U) \cong \operatorname{Proj} \mathscr{S}(U)$ for each open affine U. Furthermore, the invertible sheaves $\mathscr{O}(1)$ on each $\operatorname{Proj} \mathscr{S}(U)$ glue together to give an invertible sheaf $\mathscr{O}(1)$ on $\mathbf{Proj}\, \mathscr{S}$.

(2) Let $\mathscr{E}$ be a locally free coherent sheaf on X. We define the associated *projective space bundle* $\mathbb{P}(\mathscr{E})$ as follows. Let $\mathscr{S} = S(\mathscr{E})$ be the symmetric algebra of $\mathscr{E}$ and $\mathscr{S} = \bigoplus_{d \geq 0} S^d(\mathscr{E})$. Define $\mathbb{P}(\mathscr{E}) = \mathbf{Proj}\, \mathscr{S}$.

Fact 4.11 ([12], Ex. II.7.10c)**.** *Let X be a regular noetherian scheme. Then every $\mathbb{P}^n$-bundle P over X is isomorphic to $\mathbb{P}(\mathscr{E})$ for some locally free sheaf $\mathscr{E}$ on X.*

Proof sketch. Let $\pi : P \to X$ be the $\mathbb{P}^n$-bundle. Let $U \subseteq X$ be an open set such that $\pi^{-1}(U) = U \times \mathbb{P}^n$. Let $\mathscr{L}_0 = \mathscr{O}(1)$ on $U \times \mathbb{P}^n$. Then using the construction in exercise II.5.15, $\mathscr{L}_0$ extends to an invertible sheaf $\mathscr{L}$ on P. Moreover, if we set $\mathscr{E} = \pi_*\mathscr{L}$, then $\mathscr{E}$ is locally free.

We claim that P is isomorphic to $\mathbb{P}(\mathscr{E})$. Consider the adjunction between π^* and π_*; the counit $\varepsilon(\mathscr{L}) : \pi^*\pi_*\mathscr{L} = \pi^*(\mathscr{E}) \to \mathscr{L}$ is a surjective map, given by "projection". Locally $\mathscr{E}$ is free, so this amounts to giving $n+1$ sections of $\mathscr{L}$ that generate. Theorem II.7.12 furnishes an morphism $\varphi : P \to \mathbb{P}(\mathscr{E})$, which is an isomorphism. $\square$

Next, we decompose the group of line bundles on $\mathbb{P}(\mathscr{E})$.

Fact 4.12 ([12], Ex. II.7.9a). *Let X be a regular noetherian scheme and $\mathscr{E}$ a locally free coherent sheaf of rank ≥ 2 on X. Then* $\operatorname{Pic}\mathbb{P}(\mathscr{E}) \cong \operatorname{Pic} X \times \mathbb{Z}$.

Proof sketch. Let $\pi : \mathbb{P}(\mathscr{E}) \to X$ be the projection map. Consider the homomorphism $\varphi : \operatorname{Pic} X \times \mathbb{Z} \to \operatorname{Pic}\mathbb{P}(\mathscr{E})$ given by $(\mathscr{M}, m) \mapsto \pi^*\mathscr{M}(m)$. We claim that φ is an isomorphism.

For injectivity, suppose $(\mathscr{M}, m) \in \ker\varphi$. Then $\pi^*\mathscr{M}(m) = \mathscr{O}_{\mathbb{P}(\mathscr{E})}$. So,

$$\mathscr{O}_X = \pi_*(\mathscr{O}_{\mathbb{P}(\mathscr{E})}) = \pi_*(\pi^*\mathscr{M}(m)) = \mathscr{M} \otimes \pi_*\mathscr{O}_{\mathbb{P}(\mathscr{E})}(m),$$

where we have used the projection formula (4.8) for the last equality. Using (4.9), if $m < 0$, then $\pi_*\mathscr{O}_{\mathbb{P}(\mathscr{E})}(m) = 0$. If $m > 0$, then $\pi_*\mathscr{O}_{\mathbb{P}(\mathscr{E})}(m) = S^m(\mathscr{E})$, the m-th graded part of the symmetric algebra on $\mathscr{E}$. Since $\operatorname{rk}(S^m(\mathscr{E})) > 1$ if $m > 0$, we must have $m = 0$. So $\mathscr{M} \otimes \mathscr{O}_X = \mathscr{O}_X$. Therefore $\mathscr{M} \cong \mathscr{O}_X$, whence the kernel of φ is trivial.

For surjectivity, let $\mathscr{L} \in \operatorname{Pic}\mathbb{P}(\mathscr{E})$ be any line bundle. Choose an open, integral, separated set $U \subseteq X$ over which $\mathscr{E}$ trivializes. Let $V = \pi^{-1}(U)$. Shrinking U further we may assume $V = U \times \mathbb{P}^r$. Then $\operatorname{Pic} V = \operatorname{Pic}(U \times \mathbb{P}^r) \cong \operatorname{Pic}\mathbb{P}^r \cong \mathbb{Z}$ (see exercise II.6.1), where r is the rank of $\mathscr{E}$ minus one. Hence $\mathscr{L}|_V = \mathscr{O}_{\mathbb{P}(\mathscr{E})}(m)|_V$ for some $m \in \mathbb{Z}$. The value of m is locally constant, so if X is connected, then we obtain the same m for every open set in an open cover of X.

Replacing $\mathscr{L}$ by $\mathscr{L}(-m)$, we may assume $m = 0$. For a trivializing cover $\{V_i\}$ for $\mathscr{L}$ of $\mathbb{P}(\mathscr{E})$, we have the transition maps $\psi_{ij} : \pi^*(\mathscr{O}_X|_{U_i}) \to \pi^*(\mathscr{O}_X|_{U_j})$ satisfying the cocycle condition. Since $\pi_*(\pi^*(\mathscr{O}_X|_{U_i})) = \mathscr{O}_X|_{U_i}$, we get induced transition maps $\pi_*\psi_{ij} : \mathscr{O}_X|_{U_i} \to \mathscr{O}_X|_{U_j}$ that satisfy the cocycle condition (exercise II.1.22), and so glue together to give an invertible sheaf $\mathscr{M}$ on X that pulls back to $\mathscr{L}$. Thus φ is surjective as desired. $\square$

Remark 4.13. The projective space bundle we are interested in is the $\mathbb{P}^1$-fibration $\pi : G/B \to G/P_\alpha$. In this case, $\operatorname{Pic} G/B \cong \operatorname{Pic} G/P_\alpha \times \mathbb{Z}$ and in fact, given the invertible sheaf $\mathscr{L}_\lambda \in \operatorname{Pic} G/B$, we have $\mathscr{L}_\lambda \cong \pi^*\mathscr{M}(m)$, where $\mathscr{M} \in \operatorname{Pic} G/P_\alpha$ and $m = \langle \alpha^\vee, \lambda \rangle$. This is apparent from the construction given in (4.1). See [6] for more details.

Finally, as we are dealing with fibre bundles, we need the theorem on semicontinuity. The following lemma and theorem come from [19], where they are neatly proven using the Grothendieck complex.

Lemma 4.14. *Let Y be a reduced scheme and $\mathscr{F}$ a coherent sheaf on Y such that $\dim_{k(y)} \mathscr{F} \otimes_{\mathscr{O}_Y} k(y) = r$ for all $y \in Y$. Then $\mathscr{F}$ is locally free of rank r on Y.*

Theorem 4.15 (Semicontinuity). *Let $f : X \to Y$ be a proper morphism of noetherian schemes, and $\mathscr{F}$ a coherent sheaf on X that is flat over Y. Assume Y is reduced and connected. Then for all q, the following are equivalent:*

(i) $y \mapsto \dim_{k(y)} H^q(X_y, \mathscr{F}_y)$ *is a constant function.*
(ii) $R^q f_*(\mathscr{F})$ *is a locally free sheaf on* Y *and for all* $y \in Y$*, the natural map* $R^q f_*(\mathscr{F}) \otimes_{\mathscr{O}_Y} k(y) \to H^q(X_y, \mathscr{F}_y)$ *is an isomorphism.*

4.3. **Proof of theorem 4.5.** We are almost ready to prove theorem 4.5. Consider the $\mathbb{P}^1$-fibration $\pi : G/B \to G/P_\alpha$. First, we identify the relative canonical sheaf in terms of the roots of G.

Lemma 4.16. *The relative canonical sheaf* $\omega := \omega_{(G/B)/(G/P_\alpha)}$ *is* $\mathscr{L}_{-\alpha}$.

Proof sketch. Consider the short exact sequence of G/B-modules (see proposition II.8.11 in [12])

$$0 \to \pi^* \Omega_{(G/P_\alpha)/k} \to \Omega_{(G/B)/k} \to \Omega_{(G/B)/(G/P_\alpha)} \to 0.$$

Given $y \in G/B$, the maximal torus T acts on the fibres of these modules above y. The weights of T in the first of these fibres are the positive roots excluding α, and the weights in the second are all the positive roots. Therefore, the fibre of the third is a simple T-module with root α, whence the assertion, since $\omega_{(G/B)/(G/P_\alpha)} = \wedge^1 \Omega_{(G/B)/(G/P_\alpha)} = \Omega_{(G/B)/(G/P_\alpha)}$. For more details, see [6]. $\square$

We now prove theorem 4.5. Let us recall the setting. We have an algebraically closed field of characteristic zero, and a simply-connected reductive group G. Let ρ be the half-sum of the positive roots, so $\langle \alpha^\vee, \rho \rangle = 1$ for any simple root α. We wish to show that if α is a simple root and λ is a root such that $\langle \alpha^\vee, \lambda + \rho \rangle \geq 0$, then $H^i(G/B, \mathscr{L}_\lambda)$ and $H^{i+1}(G/B, \mathscr{L}_{s_\alpha \cdot \lambda})$ are isomorphic as G-modules for all $i \in \mathbb{Z}$.

Proof of theorem 4.5. We divide the proof into three main steps.

Step 1: $H^p(G/P_\alpha, \pi_* \mathscr{L}_\lambda) \cong H^p(G/B, \mathscr{L}_\lambda)$**.** Let $m := \deg_\pi \mathscr{L}_\lambda$ be the degree of $\mathscr{L}_\lambda$ restricted to a fibre. Since $\langle \alpha^\vee, \lambda + \rho \rangle \geq 0$, we have $m = \langle \alpha^\vee, \lambda \rangle \geq -1$. Then for all $q \geq 0$,

$$h^q(y, \mathscr{L}_\lambda) = \dim_{k(y)} H^q((G/B)_y, (\mathscr{L}_\lambda)_y) = \dim_{k(y)} H^q(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(m))$$

is constant in y. So the semicontinuity theorem (4.15) gives us

$$R^q \pi_*(\mathscr{L}_\lambda) \otimes k(y) \cong H^q((G/B)_y, (\mathscr{L}_\lambda)_y) \cong H^q(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(m)),$$

which is zero for $q > 0$. So by lemma 4.14, $R^q \pi_*(\mathscr{L}_\lambda) = 0$ for all $q > 0$.

Consider the Leray spectral sequence

$$H^p(G/P_\alpha, R^q \pi_*(\mathscr{L}_\lambda)) \Rightarrow H^{p+q}(G/B, \mathscr{L}_\lambda).$$

This degenerates on the second page, and thus $H^p(G/P_\alpha, \pi_* \mathscr{L}_\lambda)$ is isomorphic to $H^p(G/B, \mathscr{L}_\lambda)$.

Step 2: $H^p(G/P_\alpha, R^1 \pi_* \mathscr{L}_{s_\alpha \cdot \lambda}) \cong H^{p+1}(G/B, \mathscr{L}_{s_\alpha \cdot \lambda})$**.** The proof of this is similar to step 1. Set $m' = \deg_\pi \mathscr{L}_{s_\alpha \cdot \lambda}$; then using $\langle \alpha^\vee, \alpha \rangle = 2$, we have

$$m' = \langle \alpha^\vee, s_\alpha \cdot \lambda \rangle = \langle \alpha^\vee, \lambda - \langle \alpha^\vee, \lambda + \rho \rangle \alpha \rangle = -\langle \alpha^\vee, \lambda + \rho \rangle - 1 \leq -1.$$

Next,

$$h^q(y, \mathscr{L}_{s_\alpha \cdot \lambda}) = \dim H^q((G/B)_y, (\mathscr{L}_{s_\alpha \cdot \lambda})_y) = \dim H^q(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(m'))$$

is constant in y, so semicontinuity (4.15) implies

$$R^q \pi_*(\mathscr{L}_{s_\alpha \cdot \lambda}) \otimes k(y) \cong H^q(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(m')),$$

which is zero for all $q \neq 1$. By lemma 4.14, $R^q \pi_*(\mathscr{L}_{s_\alpha \cdot \lambda}) = 0$ for all $q \neq 1$.

The Leray spectral sequence

$$H^p(G/P_\alpha, R^q\pi_*\mathscr{L}_{s_\alpha\cdot\lambda}) \Rightarrow H^{p+q}(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(m'))$$

collapses on the second page, and we get an isomorphism of $H^p(G/P_\alpha, R^1\pi_*\mathscr{L}_{s_\alpha\cdot\lambda})$ with $H^{p+1}(G/B, \mathscr{L}_{s_\alpha\cdot\lambda})$.

Step 3: $\pi_*\mathscr{L}_\lambda \cong R^1\pi_*\mathscr{L}_{s_\alpha\cdot\lambda}$**.** This is most involved step. By (4.11), we have $G/B \cong \mathbb{P}(\mathscr{E})$ for some $\mathscr{E}$ of rank 2. By (4.12), we can write $\mathscr{L}_\lambda = \pi^*\mathscr{M} \otimes \mathscr{O}(m)$ for some $\mathscr{M} \in \operatorname{Pic} G/P_\alpha$, where $m = \langle\alpha^\vee, \lambda\rangle$. We first claim that

$$\mathscr{L}_{s_\alpha\cdot\lambda} = \pi^*(\mathscr{M} \otimes (\wedge^2\mathscr{E})^{\otimes m+1}) \otimes \mathscr{O}(-m-2).$$

First, we have $\mathscr{L}_{s_\alpha\cdot\lambda} = \mathscr{L}_{\lambda-\langle\alpha^\vee,\lambda+\rho\rangle\alpha} = \mathscr{L}_\lambda \otimes \omega^{\otimes m+1}$. Recall that $\omega \cong \mathscr{L}_{-\alpha}$ from the previous lemma. So using (4.9),

$$\begin{aligned}
\mathscr{L}_{s_\alpha\cdot\lambda} &= \mathscr{L}_\lambda \otimes \omega^{\otimes m+1} \\
&= (\pi^*\mathscr{M} \otimes \mathscr{O}(m)) \otimes (\pi^* \wedge^2 \mathscr{E} \otimes \mathscr{O}(-2))^{\otimes m+1} \\
&= \pi^*(\mathscr{M} \otimes (\wedge^2\mathscr{E})^{\otimes m+1}) \otimes \mathscr{O}(-m-2).
\end{aligned}$$

Now, $\pi_*\mathscr{L}_\lambda = \pi_*(\pi^*\mathscr{M} \otimes \mathscr{O}(m)) \cong \mathscr{M} \otimes \pi_*\mathscr{O}(m)$ by the projection formula. On the other hand, using the projection formula and (4.9),

$$\begin{aligned}
R^1\pi_*\mathscr{L}_{s_\alpha\cdot\lambda} &\cong R^1\pi_*(\pi^*(\mathscr{M} \otimes (\wedge^2\mathscr{E})^{\otimes m+1}) \otimes \mathscr{O}(-m-2)) \\
&\cong \mathscr{M} \otimes (\wedge^2\mathscr{E})^{\otimes m+1} \otimes R^1\pi_*\mathscr{O}(-m-2) \\
&\cong \mathscr{M} \otimes (\wedge^2\mathscr{E})^{\otimes m+1} \otimes ((\pi_*\mathscr{O}(m))^\vee \otimes (\wedge^2\mathscr{E})^\vee) \\
&\cong \mathscr{M} \otimes ((\pi_*\mathscr{O}(m))^\vee \otimes (\wedge^2\mathscr{E})^{\otimes m})
\end{aligned}$$

Using (4.9) and the fact that the perfect pairing

$$\begin{aligned}
S^m(\mathscr{E}) \otimes S^m(\mathscr{E}) &\to (\wedge^2\mathscr{E})^{\otimes m} \\
(x^a y^b, x^c y^d) &\mapsto \begin{cases} (x \wedge y)^{\otimes m} & \text{if } a+c = b+d = m \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

induces an isomorphism

$$\pi_*\mathscr{O}(m)^\vee \otimes (\wedge^2\mathscr{E})^{\otimes m} \cong S^m(\mathscr{E})^\vee \otimes (\wedge^2\mathscr{E})^{\otimes m} \xrightarrow{\sim} S^m(\mathscr{E}) \cong \pi_*\mathscr{O}(m),$$

we see that $R^1\pi_*\mathscr{L}_{s_\alpha\cdot\lambda} \cong \mathscr{M} \otimes \pi_*\mathscr{O}(m) \cong \pi_*\mathscr{L}_\lambda$.

Combining the three steps of the proof, we have

$$\begin{aligned}
H^i(G/B, \mathscr{L}_\lambda) &\cong H^i(G/P_\alpha, \pi_*\mathscr{L}_\lambda) \\
&\cong H^i(G/P_\alpha, R^1\pi_*\mathscr{L}_{s_\alpha\cdot\lambda}) \\
&\cong H^{i+1}(G/B, \mathscr{L}_{s_\alpha\cdot\lambda}).
\end{aligned}$$

This concludes the proof of theorem 4.5. □

Acknowledgments. Firstly, I would like to thank my mentor Tianqi Fan for her hard work guiding me through the many aspects of algebraic geometry and putting up with my naïve questions about the subject. In particular, I am grateful for her expertise that she shared with me during our lengthy biweekly discussions and the unwavering encouragement she gave me to pursue this subject. I also very much enjoyed the cultural articles that she provided me to broaden my perspectives about mathematics.

Next, I am also indebted to Professor Nori for his note on the proof of Bott's theorem presented here. (All mistakes here are of course mine.)

Last but not least, I want to thank Professor May for organizing the REU. I witnessed firsthand (twice, being a two-time participant) the tremendous effort that Professor May invested in running the REU – from organizing it to teaching in it – and the many sacrifices that he made to ensure that the REU ran smoothly. I also want to thank Professor May for his invaluable comments on this paper. Thank you for everything: I had fun.

References

[1] Altman, Allen and Steven Kleiman. *Introduction to Grothendieck duality theory.* New York: Springer-Verlag, 1970.
[2] Atiyah, Michael and Raoul Bott. "A Lefschetz fixed point formula for elliptic complexes: II. Applications." *Ann. of math. (2)* 88:3 (1968), 451–491.
[3] Atiyah, Michael and Ian Macdonald. *Introduction to commutative algebra.* Reading: Addison-Wesley, 1969.
[4] Bott, Raoul. "On induced representations." *Collected Papers* 48 (1994): 402–414.
[5] Bourbaki, Nicolas. *Groupes et algèbres de Lie – Chapitres 4 à 6.* New York: Springer, 2007.
[6] Demazure, Michel. "Une démonstration algébrique d'un théorème de Bott." *Invent. math.* 5:4 (1968), 349–356.
[7] Demazure, Michel. "Sur la formule des caractères de H. Weyl." *Invent. math.* 9:3 (1970), 249–252.
[8] Demazure, Michel. "A very simple proof of Bott's theorem." *Invent. math.* 33:3 (1976), 271–272.
[9] Eisenbud, David. *Commutative algebra with a view toward algebraic geometry.* New York: Springer-Verlag, 1995.
[10] Gelfand, Sergei I. and Yuri I. Manin. *Methods of homological algebra.* New York: Springer, 2003.
[11] Greenlees, J.P.C. and J.P. May. "Derived functors of I-adic completion oand local homology." *J. algebra* 149 (1992), 438–453.
[12] Hartshorne, Robin. *Algebraic geometry.* New York: Springer-Verlag, 2006.
[13] Humphreys, James E. *Introduction to Lie algebras and representation theory.* New York: Springer-Verlag, 1972.
[14] Lipman, Joseph. "Notes on Derived Functors and Grothendieck Duality." In *Foundations of Grothendieck duality for diagrams of schemes.* New York: Springer-Verlag, 2009.
[15] Lurie, Jacob. "A proof of the Borel-Weil-Bott theorem." `http://www.math.harvard.edu/~lurie/papers/bwb.pdf`.
[16] Malle, Gunther and Donna Testerman. *Linear algebraic groups and finite groups of Lie type.* New York: Cambridge UP, 2011.
[17] Matsumura, Hideyuki. *Commutative ring theory.* New York: Cambridge UP, 1989.
[18] May, J.P. "Notes on Tor and Ext." `http://math.uchicago.edu/~may/MISC/TorExt.pdf`.
[19] Mumford, David. *Abelian varieties.* London: Oxford UP, 1970.
[20] Mumford, David. *The red book on varieties and schemes.* New York: Springer, 1999.
[21] Rotman, Joseph J. *An introduction to homological algebra.* New York: Springer, 2009.
[22] Springer, T.A. *Linear algebraic groups.* New York: Birkhäuser Boston, 2009.
[23] Weibel, Charles A. *An introduction to homological algebra.* New York: Cambridge UP, 1994.