# LAPLACE'S EQUATION FROM TWO PERSPECTIVES

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Abstract. We study Laplace's equation from the perspectives of partial differential equations and probability theory. We formulate the problem using both perspectives, and we obtain explicit formulas for solutions in several specific cases. We also show various qualitative properties of solutions such as uniqueness, the mean-value property, and the maximum principle.

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## 1. Introduction

A natural way to mathematically describe physical phenomena is to represent the behavior via functions which satisfy given equations. For example, a function may represent the position of a molecule, or the temperature of the medium, while the equation describes a relationship satisfied by both the function and the environment being modeled. The mathematical aspects of the function, such as its derivatives or average value, often capture this relationship. Even without knowing the specific model, we may study solutions to these equations as purely mathematical objects. Using mathematics, we can determine qualitative properties of such solutions, which in turn shed light on the phenomena modeled by the equation.

Furthermore, within mathematics there are several perspectives we may take to study equations. In this paper, we choose to examine a specific equation, Laplace's equation, from two perspectives. Laplace's equation is used in the physical sciences to model certain phenomena such as diffusion and fluid flow. We will study solutions to Laplace's equation from the point of view of partial differential equations and probability theory. The purpose of this paper is to highlight the advantages and disadvantages of each approach. Typically, partial differential equations offer more explicit information about the solutions, and require less background to be understood. Their weakness is that they do not provide intuition for the problem in mind. In contrast, the probabilistic perspective brings wonderful intuition to the problem; however, it lacks the precision of explicit formulas.

This paper is organized as follows. In Section 2, we establish the necessary probabilistic background to understand the equations we examine. We introduce the notions of expected value, conditional probability, conditional expectation, and random walks. In Section 3, we present the Laplacian as a differential operator, and we study solutions to Laplace's equation. In particular, we derive an explicit formula for solutions to Laplace's equation on a disc. As a consequence of this formula, we establish the so-called "mean value property," maximum principle, and uniqueness for solutions to Laplace's equation in more general domains. In Section 4, we look at a discrete version of Laplace's equation in probability theory. Using the probabilistic interpretation, we may determine the solution to Laplace's equation on bounded domains. We also provide an interesting interpretation of Laplace's equation through a gambler experiment.

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## 2. Probabilistic Background

In this section, we recall some fundamental concepts in probability theory and we establish some notation which will later be used in this paper. We work with experiments which produce a set of distinct, well-defined outcomes. Analyzing the outcomes of these experiments is where probability theory plays a role.

**Example 2.1.** One potential experiment could be the flip of a coin. In this experiment, one outcome is that the coin comes up heads and another possible outcome is that the coin comes up tails.

In a given experiment, we may not be interested in studying all possible outcomes. Instead, we may consider subsets of all possible outcomes, which is how we define an event.

**Definition 2.2.** An event is a subset of all of the possible outcomes of an experiment. We refer to the space of all possible outcomes of an experiment as a sample-space, $\Omega$. $\Omega$ contains a set of events, $\mathcal{F}$, where each event is a set containing zero or more outcomes. $\Omega$ also contains an assignment of probabilities to the events, which is a function $P$ from events to probabilities.

**Example 2.3.** Consider the experiment given by rolling two six-sided dice and examining their sum. One event for that experiment is that "the sum of the two dice is 4." Let $A$ denote the event, which we may represent by

$$A = \{(1,3),(2,2)\} \text{ since the dice are indistinguishable.}$$

Instead of looking at an event as a set of outcomes, it is conventional to represent the event by a function taking on a certain value. This is where we introduce the notion of a random variable.

**Definition 2.4.** A discrete random variable, $X$, is a function, $X : \Omega \to \mathbb{N}$.

A simple example of a discrete random variable would be the value of a dice roll. We can describe the event in example (2.3) with random variables.

**Example 2.5.** Let the random variable be the value of the dice roll and let $X_1$, $X_2$ represent die. In other words,

$$X_1 : \Omega \to \{1,\ 2,\ 3,\ \ldots,\ 6\}$$
$$X_2 : \Omega \to \{1,\ 2,\ 3,\ \ldots,\ 6\}.$$

Thus, $A = X_1 + X_2 = 4$ in example (2.3).

With these definitions we can turn to the probability of an event occurring.

**Definition 2.6.** Given an event $A$, the probability of the event occurring, $P(A)$, is the number of outcomes where $A$ holds over the total number of possible outcomes. That is

$$P(A) = \frac{\text{number of outcomes when } A \text{ holds}}{\text{total number of outcomes}}.$$

Sometimes we are also interested in determining if the probability of two events occurring are interrelated. The notion of conditional probability examines how an event may influence another.

**Definition 2.7.** Conditional probability is the probability that an event occurs, given that another event has occurred. Let $X$ be such that $P(X) > 0$. We define $P(Y|X)$ to be the conditional probability of $Y$ given that $X$ has occurred. We have the expression

$$P(Y|X) = \frac{P(X \cap Y)}{P(X)}.$$

Sometimes we use $P(X,Y) = P(X \cap Y)$.

Let us now examine an example of conditional probability with the roll of two dice.

**Example 2.8.** Suppose we are interested in the probability that the sum of two dice is 6 given that the first die is a 3. Let $A$ be the event that the sum of the two dice is equal to 6 and let $B$ be the event that the first die is 3 in the roll of two dice. We have that $B = \{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\}$. Then,

$P(B) = \frac{6}{36}$ since there are 36 different possibilities for the roll of the dice. By examining the possibilities for overlap between $A$ and $B$, we have that $A \cap B = \{(3,3)\}$ and so $P(A \cap B) = \frac{1}{36}$. Thus,

$$P(A|B) = \frac{\frac{1}{36}}{\frac{6}{36}} = \frac{1}{6}.$$

It is also possible that two events do not influence each other at all in probability. This is the concept of events being independent from one another.

**Definition 2.9.** Two events, $A$ and $B$, are independent if

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B).$$

Therefore, the conditional probability

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)P(B)}{P(A)} = P(B),$$

if $A$, $B$ are independent. In the discrete setting, it is often useful to recover the probability of an event by considering how it occurs with another event. In other words, given two random variables, $X, Y$, we have the following formulas:

$$P(X = x_i) = \sum_{j=1}^{N} P(X = x_i, Y = y_j) \tag{1}$$

$$P(Y = y_j) = \sum_{i=1}^{N} P(X = x_i, Y = y_j), \tag{2}$$

where $x_i$ denotes one of the possible values that $X$ can take on.

Another important and interesting notion is the "average" or expected value of a random variable, which represents the value that we predict the random variable will take on average.

**Definition 2.10.** The expected value, or mean, of random variable $X$ is

$$\mathbb{E}[X] = \sum_{x} xP(X = x).$$

We consider the following example.

**Example 2.11.** Let us look at the expected value of a roll of a six-sided die. We will call the vale of the roll $X$. Let us assume that it is a fair die. Thus, $X : \Omega \to \{1,\ 2,\ 3,\ \ldots,\ 6\}$. The expected value

$$\mathbb{E}[X] = \sum_{x=1}^{6} xP(X = x) = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6 = 3.5.$$

Say we wanted to know the expected value of the roll of a six-sided die plus two. In other words, say we wanted to know the expected value of a random variable plus a constant. Let $X : \Omega \to \{1, 2, \ldots, 6\}$ and $Y : \Omega \to \{1, 2, \ldots, 6\}$ denote random variables. Let $a,\ c \in \mathbb{R}$. In order to denote the expected value of a random variable plus a constant we write $\mathbb{E}[X + c]$. If we want to sum two random variables, we would sum up each of their possible values with each other. Thus, we have $X + Y =$

$$C : \Omega \to \{2, 3, 3, \ldots, 12\}.$$

If we want to know the expected value of random variable multiplied by some number, we would multiply each possible value of the random variable by some constant. We will now examine some of the properties of the expected value of random variables.

**Proposition 2.12.** *Let $X : \Omega \to \{x_1, x_2, \ldots, x_N\}$ and $Y : \Omega \to \{y_1, y_2, \ldots, y_N\}$ denote random variables. Let $a,\ c \in \mathbb{R}$. The expected value satisfies the following:*

*(i)* $\mathbb{E}[X + c] = \mathbb{E}[X] + c$
*(ii)* $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$

*(iii)* $\mathbb{E}[aX] = a\mathbb{E}[X]$

*Proof of (i).* We compute,

$$\begin{aligned}\mathbb{E}[X+c] &= \sum_{i=1}^{N}(x_i + c)\cdot P(X = x_i)\\ &= \sum_{i=1}^{N} x_i \cdot P(X = x_i) + c\cdot P(X = x_i) = \mathbb{E}[X] + c.\end{aligned}$$

□

*Proof of (ii).* We compute,

$$\mathbb{E}[X+Y] = \sum_{i=1}^{N}\sum_{j=1}^{B}(x_i + y_j)P(X = x_i, Y = y_j)$$

$$= \sum_{i=1,j=1}^{N} x_i P(X = x_i, Y = y_j) + \sum_{i=1,j=1}^{N} y_j P(X = x_i, Y = y_j). \tag{ii.a}$$

Note that by (1) and (2),

$$P(X = x_i) = \sum_{j=1}^{N} P(X = x_i, Y = y_j)$$

$$P(Y = y_j) = \sum_{i=1}^{N} P(X = x_i, Y = y_j).$$

This enables us to change $(ii.a)$ into

$$\sum_{i=1}^{N} x_i P(X = x_i) + \sum_{j=1}^{N} y_j P(Y = y_j) = \mathbb{E}[X] + \mathbb{E}[Y].$$

Thus, $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$. □

*Proof of (iii).* We compute:

$$\mathbb{E}[aX] = \sum_{i=1}^{N}(ax_i)P(X = x_i) = a\sum_{i=1}^{N} x_i P(X = x_i) = a\mathbb{E}[X].$$

Thus, $\mathbb{E}[aX] = a\mathbb{E}[X]$. □

We arrive at a few consequences as a result of the properties established in Proposition 2.12.

**Corollary 2.13.** *(i) For a constant, c, the expected value of the constant is the constant. That is,* $\mathbb{E}[c] = c$.
*(ii) If* $X$ *and* $Y$ *are random variables such that* $X \leq Y$ *for all outcomes, then* $\mathbb{E}[X] \leq \mathbb{E}[Y]$. *In other words,* $\mathbb{E}[Y] - \mathbb{E}[X] \geq 0$.

*Proof of (i).* We compute,

$$\mathbb{E}[c] = \sum_{i=1}^{N} cP(X = c) = c\cdot 1 = c.$$

□

*Proof of (ii).* Since $X$ and $Y$ are random variables such that $X \leq Y$ for all outcomes, then $Y - X \geq 0$. Since $Y - X \geq 0$, it follows then $\mathbb{E}[Y - X] \geq 0$. Thus, by (iii) in Proposition 2.12, we have that $\mathbb{E}[Y] - \mathbb{E}[X] \geq 0$, which allows us to conclude the proof. □

We can now also define what the expected value of a random variable is, given that another event has already occurred.

**Definition 2.14.** Let $A$ and $B$ be discrete random variables. The conditional expectation of $A$ given that $B = b$ is

$$\mathbb{E}[A|B = b] = \sum_{a\in A} aP(A = a|B = b) = \sum_{a\in A} a\frac{P(A = a, B = b)}{P(B = b)}$$

With this background in probability theory, we now introduce a well-known experiment, known as the random walk. In a $d$-dimensional walk, we would roll a $d$-sided die and let the die roll decide the direction that the walker moves. In 1-dimension, the walker flips a fair coin and moves one step to the right or one step to the left, depending on whether or not the coin is tails or heads at each step. The random variable, $X_i$, describes which direction the walker moves at each $i$ by taking on the value of "1" for heads, or "-1" for tails. If $X_i$ is "1", then the walker moves one step to the right. If $X_i$ is "$-1$" then the walker moves one step to the left. Summing up the value of each $X_i$ for all $i$, describes the position of the walker. Let $S_n$ denote the position of the random walker after $n$ steps. If the walker starts at $x$, then we can write

$$S_n = x + X_1 + \cdots + X_n.$$

If the walker is in a bounded domain, $A$, then the walker may leave this domain at some point in the experiment. We define the exit time to be

$$T_A = \min\{n \geq 0 \mid S_n \notin A\}.$$

We also let $n \wedge T$ denote the minimum of $n$ and $T$. If we write $S_{n\wedge T}$, then we are referring to the position of the walker at either time $n$ or the time the walker leaves the domain, whichever event happens first.

We can also apply the concept of expected value to predict the walker's position at time $n$. We assume that the walker starts at the origin. Since the probability of moving "$-1$" and "$+1$" are exactly equal, we have

$$\mathbb{E}[X_i] = -1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 0.$$

The walker is not expected to move at each step on average because the probability of moving "$-1$" and "$+1$" are exactly equal. By (ii) in Proposition (2.12), it follows that for all $n$

$$\mathbb{E}[S_n] = \sum_{i=1}^{n} \mathbb{E}[X_i] + \mathbb{E}[x] = x.$$

We can also use the expected value to determine the average distance of the walker from the origin. Even though the walker is expected to be at the starting position on average, that does not mean the walker never moves. In order to compute the average distance, one might compute $\mathbb{E}[|S_n|]$. However, it is actually easier to compute $\mathbb{E}[S_n^2]$ and then find the average distance from the origin by taking the square root.

**Example 2.15.** Let $X_j$ denote the direction the walker moves at each $j$ and let $X_k$ denote the direction the walker moves at each $k$. We compute,

$$\begin{aligned}
\mathbb{E}[S_n^2] &= \mathbb{E}\left[\left(\sum_{j=1}^{n} X_j\right)^2\right] \\
&= \mathbb{E}\left[\sum_{j=1}^{n}\sum_{k=1}^{n} X_jX_k\right] \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n} \mathbb{E}[X_jX_k] = \sum_{j=1}^{n} \mathbb{E}[X_j^2] + \sum_{j\neq k} \mathbb{E}[X_jX_k] \\
&= \sum_{j=1}^{n} \mathbb{E}[1] + \sum_{j\neq k} \mathbb{E}[X_jX_k] = \sum_{j=1}^{n} 1 + \sum_{j\neq k} \mathbb{E}[X_jX_k] = n + \sum_{j\neq k} \mathbb{E}[X_jX_k].
\end{aligned}$$

If $j \neq k$ then there are four possibilities for $X_jX_k$; either $X_jX_k = 1$ for two of the possibilities, or $X_jX_k = -1$ for the other two possibilities. Thus,

$$\begin{aligned}\mathbb{E}[X_jX_k] &= 1 \cdot (P(X_j = 1, X_k = 1)) + 1 \cdot (P(X_j = -1, X_k = -1)) + \\ &\quad (-1) \cdot (P((X_j = -1, X_k = 1) \cup (X_j = 1, X_k = -1))) \\ &= \frac{1}{4} + \frac{1}{4} + \frac{-1}{2} = 0.\end{aligned}$$

Therefore, $\mathbb{E}[X_jX_k] = 0$, which implies $\mathbb{E}[S_n^2] = n$. This means the typical distance away from the origin must be $\sqrt{n}$.

We can also use our notion of conditional expectation on a random walk. Let $F : \mathbb{R} \to \mathbb{R}$ describe a function. With our random walker notation, and definition (2.14), we have that

$$\mathbb{E}[F(S_{(n+1)\wedge T})] = \sum_{y=0}^{N} P(S_{n\wedge T} = y)\mathbb{E}[F(S_{(n+1)\wedge T}) \mid S_{n\wedge T} = y]. \tag{2.16}$$

## 3. The Laplacian as a Differential Operator

In this section we define the Laplacian and look at it from a differential equations viewpoint. We will derive an explicit formula for solutions to Laplace's equation on a disc. For general solutions to Laplace's equation, we will verify the "mean-value property," the maximum principle, and uniqueness.

**Definition 3.1.** For a twice differentiable function $u : \mathbb{R}^2 \to \mathbb{R}$, Laplace's Equation is:

$$\triangle u = \frac{\partial u}{\partial x^2} + \frac{\partial u}{\partial y^2} = u_{xx} + u_{yy} = 0$$

where $u_{xx}$ denotes the second partial derivative with respect to $x$. If we study this equation in a domain $D$ with prescribed boundary values h and a function $u$ that is twice differentiable and continuous in the closure of $D$, $\overline{D}$, then we generally write

$$\begin{cases} \triangle u = 0 & \text{in } D \\ u = h & \text{on } \partial D, \end{cases}$$

and we would call the problem a Dirichlet problem. We may also express Laplace's equation in polar coordinates. We recall that $x = r\cos(\theta)$ and $y = r\sin(\theta)$. This has the Jacobian

$$\mathcal{J} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}.$$

By a standard inversion, we have that

$$\mathcal{J}^{-1} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial \theta}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial \theta}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta & \frac{-\sin\theta}{r} \\ \sin\theta & \frac{\cos\theta}{r} \end{pmatrix}.$$

Therefore, in polar coordinates

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta},$$

$$\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}.$$

This yields

$$\begin{aligned} u_{xx} = \frac{\partial^2 u}{\partial x^2} &= \left[\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right]^2 \cdot u \\ &= \left[\cos^2\theta \frac{\partial^2}{\partial r^2} - 2\left(\frac{\sin\theta\cos\theta}{r}\right)\frac{\partial^2}{\partial r \partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r}\right]\cdot u, \\ u_{yy} = \frac{\partial^2 u}{\partial y^2} &= \left[\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right]^2 \cdot u \\ &= \left[\sin^2\theta \frac{\partial^2}{\partial r^2} + 2\left(\frac{\sin\theta\cos\theta}{r}\right)\frac{\partial^2}{\partial r \partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial}{\partial r}\right]\cdot u. \end{aligned}$$

By summing these equations we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

Now that we have defined the Dirichlet problem for Laplace's equation, we can solve for the solution to the Dirichlet problem in the specific case where the domain is a disc. We obtain an explicit formula for the solution to this problem which is called the Poisson formula for a disc.

**Theorem 3.2.** *(Poisson's Formula for a Disc)*
*Let $u(x, y)$ solve*

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{for} \quad D = x^2 + y^2 < a^2, \\ u = h(\theta) & \text{for} \quad D = x^2 + y^2 = a^2. \end{cases}$$

*Then the solution takes the form of*

$$u(r, \theta) = (a^2 - r^2)\int_0^{2\pi} \frac{h(\Phi)}{a^2 - 2ar\cos(\theta - \Phi)}\frac{d\Phi}{2\pi}$$

*in polar coordinates. If we write $\boldsymbol{x} = (x,\ y)$, then the solution is equivalent to*

$$u(\boldsymbol{x}) = \frac{a^2 - |\boldsymbol{x}|}{2\pi a}\int_{|\boldsymbol{z}|=a} \frac{u(\boldsymbol{z})}{|\boldsymbol{x} - \boldsymbol{z}|^2} ds'.$$

*Proof.* We first put the formula in polar coordinates. We seek separated solutions of the form $u(r, \theta) = R(r)\Theta(\theta)$. We next analyze the relations of $R(r)$, $\Theta(\theta)$ in light of this answer.

Using the equation

$$0 = u_{xx} + u_{yy} = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2},$$

we have that

$$0 = R''\Theta + \frac{R'\Theta}{r} + \frac{R\Theta''}{r^2}.$$

Dividing by $R\Theta$, multiplying by $r^2$, and subtracting the term with $\Theta''$, we have

$$\frac{r^2 R''}{R} + \frac{rR'}{R} = \frac{-\Theta''}{\Theta}.$$

Since the left side of the expression does not depend on $\theta$ and the right side of the expression does not depend on $r$, this implies that the expression must equal some constant, which we call $\lambda$. Thus,

$$\frac{r^2 R''}{R} + \frac{rR'}{R} = \frac{-\Theta''}{\Theta} = \lambda.$$

Rearranging, we have

$$\Theta'' + \lambda\Theta = 0 = r^2 R'' + rR' - \lambda R.$$

By the periodicity of $\Theta(\theta)$, we have that

$$\Theta(\theta + 2\pi) = \Theta(\theta) \text{ for } -\infty < \theta < \infty.$$

Let $\lambda = n^2$. By looking at the differentials we see that one potential solution of $\Theta'' = -\lambda\Theta$ is

$$\Theta(\theta) = A\cos(n\theta) + Bsin(n\theta) \text{ for } A,\ B \text{ constants.}$$

Note, for $\lambda = 0$, $\Theta(\theta) = A$. To determine $R(r)$, consider that if $R = r^\alpha$, with $\alpha$ chosen such that $\alpha \neq 0$, then

$$\frac{r^2 R''}{R} + \frac{rR'}{R} = \frac{r^2\alpha(\alpha-1)r^{\alpha-2}}{r^\alpha} + \frac{r\alpha r^{\alpha-1}}{r^\alpha} = \alpha(\alpha-1) + \alpha = \alpha^2 = \lambda.$$

For $\alpha^2 = \lambda = n^2$, we have $\alpha = \pm n$. Thus, $R(r) = Cr^n + Dr^{-n}$ for some constants $C$, $D$. Note that if $\lambda = 0$, then $R(r) = C + D\log(r)$ is a potential solution here. However, as $r \to 0$, $D\log(r)$ approaches $-\infty$ and $r^{-n}$ approaches $\infty$, which is impossible given that $u$ is a twice differentiable function on a bounded domain. Therefore, we reject the $r^{-n}$ solution and the $\log(r)$ solution. This analysis leads to the conclusion that $D = 0$, which implies that $u$ takes the form of

$$u = Cr^n[A\cos(n\theta) + B\sin(n\theta)] \quad \forall n \ \neq \ 0.$$

Since the analysis holds for all $n$, we have

$$u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n(A_n cos(n\theta) + B_n \sin(n\theta)).$$

Using the boundary conditions and setting $r = a$, we have

$$h(\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} a^n(A_n \cos(n\theta) + B_n \sin(n\theta)).$$

We may explicitly express $A_n$ and $B_n$ in terms of $h$ by applying the theory of Fourier series. As this exceeds the scope of this paper, we simply write these formulas without derivation (see [3] for details)

$$A_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\Phi)\cos(n\Phi)d\Phi$$
$$B_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\Phi)\sin(n\Phi)d\Phi.$$

Since $u$ can be summed explicitly, we get that

$$\begin{aligned} u(r,\theta) &= \int_0^{2\pi} h(\Phi)\frac{d\Phi}{2\pi} + \sum_{n=1}^{\infty}\frac{r^n}{\pi a^n}\int_0^{2\pi} h(\Phi)\{\cos(n\Phi)\cos(n\theta) + \sin(n\Phi)\sin(n\theta)\}d\Phi \\ &= \int_0^{2\pi} h(\Phi)\{1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos(n(\theta-\Phi))\}\frac{d\Phi}{2\pi} \end{aligned}$$

Recalling that

$$e^{ix} = \cos(x) + i\sin(x),$$
$$e^{-ix} = \cos(x) - i\sin(x),$$

we have

$$\begin{aligned} 1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos(n(\theta-\Phi)) &= 1 + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{in(\theta-\Phi)} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{-in(\theta-\Phi)} \\ &= 1 + \frac{re^{i(\theta-\Phi)}}{a - re^{i(\theta-\Phi)}} + \frac{re^{-i(\theta-\Phi)}}{a - re^{-i(\theta-\Phi)}} = \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta-\Phi) + r^2}. \end{aligned}$$

This yields

$$u(r,\theta) = (a^2 - r^2)\int_0^{2\pi} \frac{h(\Phi)}{a^2 - 2ar\cos(\theta-\Phi)}\frac{d\Phi}{2\pi} \tag{3.3}$$

which completes the derivation of Poisson's formula in polar coordinates.

An equivalent formula can be obtained in Cartesian coordinates. Let $\mathbf{x} = (x, y)$ have polar coordinates

$(r, \theta)$ and let $\mathbf{z} = (b,\, c)$ have polar coordinates $(a, \Phi)$. By connecting the origin, $\mathbf{x}$, and $\mathbf{z}$ we get a triangle with sides $r = \mathbf{x}$, $a = \mathbf{z}$, and $|\mathbf{x} - \mathbf{z}|$. Thus, by using the Law of Cosines, we have

$$|\mathbf{x} - \mathbf{z}|^2 = a^2 + r^2 - 2ar\cos(\theta - \Phi).$$

Since the arc length $s' = a\Phi$, then $ds' = a\, d\Phi$. Plugging this into equation (3.3), we get

$$u(\mathbf{x}) = \frac{a^2 - |\mathbf{x}|^2}{2\pi a} \int_{|\mathbf{z}|=a} \frac{u(\mathbf{z})}{|\mathbf{x} - \mathbf{z}|^2} ds' \tag{3.4}$$

for $\mathbf{x} \in D$, where $u(\mathbf{z}) = h(\Phi)$. □

We have obtained an explicit formula for the solution to Laplace's equation on a disc. Using this formula, we can show that a solution to Laplace's equation takes on the value of its local average throughout an arbitrary domain. This is how we define the mean-value property.

**Theorem 3.5.** *(Mean-Value Property) Let $u$ be a solution to Laplace's equation in a domain $D$. Let $B(\boldsymbol{x}, r) \subset D$ denote a ball contained in $D$ that is centered at $\boldsymbol{x}$ with radius $r$. Then the value of $u(\boldsymbol{x})$ equals the average of $u$ on the boundary, $\partial B(\boldsymbol{x}, r)$. In other words,*

$$u(\boldsymbol{x}) = \frac{1}{2\pi r} \int_{|\boldsymbol{z}-\boldsymbol{x}|=r} u(\boldsymbol{z}) ds'.$$

*Proof.* Without loss of generality, let $\mathbf{x} = 0$. Since $\triangle u = 0$ in $D$ and $\partial B(\mathbf{x}, r) \subset D$ we have that $u$ solves the Dirichlet problem

$$\begin{cases} \triangle u = 0 & \text{in } B(x, r) \\ u = u & \text{on } \partial B(\mathbf{x}, r). \end{cases}$$

Thus, we can apply Poisson's formula to the problem. Applying (3.4), we have

$$u(\mathbf{0}) = \frac{r^2}{2\pi r} \int_{|\mathbf{z}|=r} \frac{u(\mathbf{z})}{r^2} ds' = \frac{1}{2\pi r} \int_{|\mathbf{z}|=r} u(\mathbf{z}) ds',$$

which is exactly equal to the average value of $u$ on the circumference. □

We now turn to the idea of the maximum principle. Let $D$ be a connected, bounded, and open set. The weak maximum principle states that the maximum of a function solving Laplace's equation on a domain, $D$, occurs on the boundary. Therefore, to find the maximum of $u$, it is enough just to study the maximum on the boundary. It turns out there is an even stronger version, the strong maximum principle (SMP), which states that if the maximum occurs on an interior point, then the solution is actually constant. Using the mean-value property, we prove the SMP.

**Theorem 3.6.** *(Strong Maximum Principle). Let $D$ be a connected, bounded, open set. Let $u$ solve the Dirichlet problem in a domain $D$. Let $(x, y) = \mathbf{j}$ denote a point where the $\max_{\overline{D}} u = M$ is achieved. If $\mathbf{j} \in D$, this implies $u \equiv M$.*

*Proof.* Since $M$ is the maximum of $u$, it follows that

$$u(\mathbf{x}) \leq u(\mathbf{j}) = M \quad \text{for all } \mathbf{x} \in D.$$

Consider a ball around $\mathbf{j}$, $B(\mathbf{j},\, r)$, which is contained in $D$, since $D$ is open. From the mean-value property, we know that $u(\mathbf{j})$ is equal to the average on the circumference of the ball. Since the average is no greater than the maximum, we know that

$$M = u(\mathbf{j}) = \text{ average on the circumference of the ball } \leq M.$$

Therefore, $u(\mathbf{x}) = M$ for all $\mathbf{x} \in \partial B(\mathbf{j}, r)$. This is true for any ball centered on $\mathbf{j}$, and thus $u(\mathbf{x}) = M$ for all $\mathbf{x} \in B(\mathbf{j},\, r)$. Since the domain is connected, we can keep covering the domain with balls. Since the domain is bounded, we know we will only have to do this finitely many times. Thus,

$$u(\mathbf{x}) \equiv M \quad \text{throughout } D.$$

□

We will now examine the weak maximum principle as a corollary to the strong maximum principle.

**Corollary 3.7.** *(Weak Maximum Principle) Let $D$ be a connected, bounded, open set. Let $u$ solve the Dirichlet problem in a domain $D$. Let $(x, y) = \mathbf{j}$ denote a point where the $\max_{\overline{D}} u = M$ is achieved. Then $\max_{\overline{D}} u = \max_{\partial D} u$.*

*Proof.* Suppose

$$\max_{D} u > \max_{\partial D} u,$$

then there must be an interior maximum. With an interior maximum, the SMP implies that the maximum of $u$ would just be constant. However, $u$ cannot be constant for this problem. Thus, the $\max_{\overline{D}} u = \max_{\partial D} u$. □

Using the SMP, we can provide a proof of uniqueness of solutions for arbitrary domains.

**Theorem 3.8.** *(Uniqueness) Let $D$ be a connected, bounded, and open set. Let $u$ solve*

$$\begin{cases} \triangle u = 0 & \text{in } D \\ u = h & \text{on } \partial D. \end{cases} \tag{3.9}$$

*Then there exists at most one solution.*

*Proof.* Let $v$ and $w$ be solutions to (3.8). Let $z = w - v$. Thus

$$\begin{cases} \triangle z &= \triangle w - \triangle v = 0 - 0 = 0 & \text{in } D \\ z &= w - v = h - h = 0 & \text{on } \partial D. \end{cases}$$

Therefore, by the maximum principle

$$z(\mathbf{x}) \leq z(\mathbf{j}) = 0 \ \forall x \in D.$$

Moreover, $-z = v - w$ also solves Laplace's equation with the 0 boundary condition. By the maximum principle,

$$-z(\mathbf{x}) \leq -z(\mathbf{j}) = 0 \ \forall x \in D$$

which implies that $0 \leq z$. Therefore, we have the following inequality

$$0 \leq z \leq 0$$

which implies $z \equiv 0$, hence $w \equiv v$. □

## 4. Laplace's Equation In Probability

We will now look at a discrete version of Laplace's equation from a probabilistic perspective. Although Laplace's equation looks strikingly different in this setting, the two formulations are completely equivalent. We will show the properties of existence, uniqueness, and the mean-value property for solutions to Laplace's equation using different techniques from probability theory. This interpretation of Laplace's equation will offer a more intuitive understanding of the problem, even though it provides answers that are not necessarily as precise. To define the Laplacian and Laplace's equation, we will first consider a gambler's problem.

Let $N$, $x$ be positive integers. We denote $x = S_0$ as the starting number of chips for our gambler, and $N$ as the desired number of chips that the gambler wants to win. We can describe the gambler experiment using a random walk. Let $S_n$ be the number of chips currently held by the gambler at time $n$. At the next step, the gambler flips a coin. If the coin comes up heads, he loses one chip, and if the coin comes up tails, he gains one chip. The gambler will keep playing until he goes bankrupt or acquires the desired amount of money, $N$. Thus, the stopping time $T$ is defined by

$$T = \min\{n \mid S_n = 0 \text{ or } N\}.$$

Since the game is fair, and the set of answers is finite, we know that each answer must be hit once as $n \to \infty$. Thus, the gambler will hit 0 or $N$ at some step $n$. Thus, we know that $T$ must be finite. We are interested in knowing if the gambler wins his game. Define the function $F : \{0, \cdots, N\} \to [0, 1]$ as

$$F(x) = P(S_T = N \mid S_0 = x).$$

Notice that $F(x)$ is the probability that the gambler wins his game, given that he starts at $x$. Since the gambler stops when he is bankrupt, we have that

$$F(0) = P(S_T = N \mid S_0 = 0) = 0.$$

Similarly, if the gambler starts at $N$ then

$$F(N) = P(S_T = N \mid S_0 = N) = 1.$$

If $0 < x < N$, then after the first game, the gambler will have either $x + 1$ chips or $x - 1$ chips. Since both of those events are equally likely, we have

$$F(x) = \frac{1}{2}F(x+1) + \frac{1}{2}F(x-1) \quad \text{for } x = 1, \cdots, N-1. \tag{4.1}$$

This gives us a recursive formula to work with, and now we will explicitly determine $F(x)$.

**Theorem 4.2.** *Let $a, b \in \mathbb{R}$ and let $N$ be a positive integer. Then the only function $F : \{0, \cdots, N\} \to \mathbb{R}$ that satisfies (4.1) with $a = F(0)$ and $b = F(N)$ is the linear function*

$$F_0(x) = a + \frac{x(b-a)}{N}.$$

*Proof.* In the case of the gambler's problem, we have that $a = 0$, $b = 0$ so we seek to prove

$$F_0(x) = \frac{x}{N}.$$

We will present the proof for this case since the general case follows similarly. Suppose $F$ is a solution to (4.1) and let $S_n$ be the random walk that starts at $x$. We first show that for all $n$,

$$\mathbb{E}[F(S_{n\wedge T})] = F(x).$$

We will proceed by induction. For our base case with $n = 0$, we have that because $S_0 = x$,

$$F(S_0) = F(x), \text{ which implies}$$

$\mathbb{E}[F(S_0)] = F(x)$. We now assume that $\mathbb{E}[F(S_{n\wedge T})] = F(x)$ and want to show that $\mathbb{E}[F(S_{(n+1)\wedge T})] = F(x)$. To prove our inductive step we recall equation (2.16)

$$\mathbb{E}[F(S_{(n+1)\wedge T})] = \sum_{y=0}^{N} P(S_{n\wedge T} = y)\mathbb{E}[F(S_{(n+1)\wedge T})] \mid S_{n\wedge T} = y].$$

If $y = 0$ or $y = N$, and $S_{n\wedge T} = y$, then $S_{(n+1)\wedge T} = y$ since the game is over. Hence,

$$\mathbb{E}[F(S_{(n+1)\wedge T}) \mid S_{n\wedge T} = y] = F(y) \text{ for } y = 0,\ N.$$

If $0 < y < x$, and $S_{n\wedge T} = y$, then

$$\mathbb{E}[F(S_{(n+1)\wedge T} \mid S_{n\wedge T} = y] = \frac{1}{2}F(y+1) + \frac{1}{2}F(y-1) = F(y).$$

Thus, using equation (2.16) and inserting $F(y)$ for $\mathbb{E}[F(S_{(n+1)\wedge T}) \mid S_{n\wedge T} = y]$, we get

$$\mathbb{E}[F(S_{(n+1)\wedge T})] = \sum_{y=0}^{N} P(S_{n\wedge T} = y)\mathbb{E}[F(S_{(n+1)\wedge T}) \mid S_{n\wedge T} = y] = \sum_{y=0}^{N} P(S_{n\wedge T} = y)F(y).$$

By the definition of expected value, we know

$$\sum_{y=0}^{N} P\{S_{n\wedge T} = y\}F(y) = \mathbb{E}[F(S_{n\wedge T})] = F(x),$$

where the last equality holds by our inductive hypothesis, and thus

$$\mathbb{E}[F(S_{(n+1)\wedge T})] = F(x),$$

which proves the claim. Since this holds for all $n$, we may take $n \to \infty$ to get

$$F(x) = \lim_{n\to\infty} \mathbb{E}[F(S_{n\wedge T})] = \lim_{n\to\infty} \sum_{y=0}^{N} P(S_{n\wedge T} = y)F(y)$$
$$= P(S_T = 0)F(0) + P(S_T = N)F(N) = [1 - P(S_T = N)]F(0) + P(S_T = N)F(N).$$

Taking our case where $F(0) = 0$, $F(N) = 1$, and assuming $S_0 = x$, we have

$$[1 - P(S_T = N)]F(0) + P(S_T = N)F(N) = [1 - 1] \cdot 0 + P(S_T = N) \cdot 1 = P(S_T = N).$$

Thus,

$$P(S_T = N \mid S_0 = x) = F(x).$$

We will now show that

$$P(S_T = N \mid S_0 = x) = \frac{x}{N}.$$

We can rewrite (4.1), as

$$\frac{1}{2}F(x) + \frac{1}{2}F(x) = \frac{1}{2}F(x+1) + \frac{1}{2}F(x-1).$$

This yields

$$F(x+1) - F(x) = F(x) - F(x-1).$$

In particular, we know that

$$F(2) - F(1) = F(1) - F(0) = F(1)$$

since $F(0) = 0$. We also know that

$$F(3) - F(2) = F(2) - F(1) = F(1).$$

More generally, we know that $F(x+1) - F(x) = F(1)$. Therefore

$$F(x+1) - F(1) = \sum_{k=1}^{i} F(k+1) - F(k)$$
$$= \sum_{k=1}^{i} F(1)$$
$$F(x+1) = F(1) + F(1)\sum_{k=1}^{i} 1 = F(1)\sum_{k=0}^{i} 1$$
$$F(x+1) = F(1)(x+1).$$

Choosing $x = N - 1$ we get

$$F(N) = F(1) \cdot N.$$

Therefore,

$$F(1) = \frac{1}{N}.$$

In light of the fact that $F(1) = \frac{1}{N}$ and $F(x+1) = F(1)(x+1)$ we get that

$$F(x+1) = F(1)(x+1) = \frac{x+1}{N},$$

from which we conclude that

$$F(x) = \frac{x}{N}.$$

The function is unique because it is the only function which satisfies (4.1). For more general boundary conditions, similar analysis shows that

$$F(x) = F(0) + \frac{x}{N}[F(N) - F(0)] = a + \frac{x(b-a)}{N}.$$

□

Notice that the main argument was to show that for all $n$, $F(x) = \mathbb{E}[F(S_{n\wedge T})]$. Therefore, we see that the solution to the gambler experiment has the property that $F(S_n)$ has constant expectation for all $n$. Therefore, if we wanted to generalize this to higher dimensions, we would also look for this property. This summarizes how we will now define the discrete Laplacian for higher dimensions.

**Definition 4.3.** The (discrete) Laplacian is typically written as $\mathcal{L}$. Let $S_n$ represent a random walk on $\mathbb{Z}^d$. We define the Laplacian of a function to be the expected value of the difference between step $i$ and step $i-1$ given that the random walk starts at a certain location. We write that

$$\mathcal{L}F(x) \;=\; \mathbb{E}[F(S_1) - F(S_0) \mid S_0 = x] \tag{4.4}$$

If we took a continuous random variable and a continuous random walk, like those that represent the movement of a dust particle moving through a large set of other particles, then we would have what would be referred to as Brownian motion. We could also apply the Laplacian to Brownian motion. However, such a concept is beyond the scope of this paper. For details, please see [1].

In light of the gambler experiment we would like to study solutions to the equation

$$\mathcal{L}F_0(x) = 0.$$

In particular, for $A \subset \mathbb{Z}^d$, we are interested in the Dirichlet problem

$$\begin{cases} \mathcal{L}F_0(x) = 0 & x \in A \\ F_0 = F & x \in \partial A. \end{cases} \tag{4.5}$$

We now show that we may uniquely solve the Dirichlet problem in this setting.

**Theorem 4.6.** *(Dirichlet Problem for d Dimensions) If $A \subset \mathbb{Z}^d$ is finite, then for every $F : \partial A \to \mathbb{R}$, there is a unique extension of $F$ to $\overline{A}$ which satisfies (4.5). The solution is given by*

$$F_0(x) = \mathbb{E}[F(S_{T_A})|S_0 = x] = \sum_{y\in\partial A} P(S_{T_A} = y|S_0 = x)F(y).$$

*Proof.* First, we show that

$$F_0(x) = \mathbb{E}[F(S_{T_A}) \mid S_0 = x]$$

solves equation (4.5). By using Proposition (2.12) in Section 2, we have

$$\begin{aligned} \mathcal{L}F_0(x) &= \mathbb{E}[\mathbb{E}[F(S_{T_A}) \mid S_0 = S_1] - \mathbb{E}[F(S_{T_A}) \mid S_0 = S_0] \mid S_0 = x] \\ &= \mathbb{E}[\mathbb{E}[(F(S_{T_A}) \mid S_0 = S_1) \mid S_0 = x] - \mathbb{E}[(F(S_{T_A}) \mid S_0 = S_0) \mid S_0 = x)]] \\ &= \mathbb{E}[F(S_{T_A}) \mid S_1 = x] - \mathbb{E}[F(S_{T_A}) \mid S_0 = x] = 0, \end{aligned}$$

where the last equality holds because moving one point will not change the expected value for this problem. This property is exactly like how winning or losing one chip in the gambler experiment did not change the expected value of the experiment.

We now claim that any solution to (4.5) has the form of $F_0(x)$. Let $G$ satisfy (4.5), $S_0 = x \in \overline{A}$, and let $M_n = G(S_{n\wedge T_A})$. Since $G$ solves (4.5), we can rewrite (4.5) as

$$\mathbb{E}[M_{n+1} \mid S_0, \ldots, S_n] = \mathbb{E}[\mathbb{E}[M_{n+1} \mid S_n \mid S_0, \ldots, S_{n-1}]] = \mathbb{E}[M_{n+1} \mid S_n] = G(S_{n\wedge T_A}) = M_n$$

Thus, $\mathbb{E}[M_{n+1}] = \mathbb{E}[G(S_{n+1\wedge T_A})] = M_n$. By the definition of expected value, we get

$$\sum_{y\in\overline{A}} P(S_{n\wedge T_A})G(y) = \mathbb{E}[M_n] = \mathbb{E}[M_0] = G(x).$$

Since $A$ is a finite set, the random walker has to leave the domain at some point, thus $T_A < \infty$. Sending $n \to \infty$ in the above expression, we have

$$G(x) = \lim_{n\to\infty} \sum_{y\in\overline{A}} P(S_{n\wedge T_A} = y|S_0 = x)G(y) = \sum_{y\in\partial A} P(S_{T_A} = y|S_0 = x)G(y) = F_0(x)$$

where we can interchange the limit and sum since we are looking at a finite sum. Thus, not only does $F_0(x)$ satisfy (4.5), it is also unique. □

Notice that the key property we utilize is constant expectation between time steps. As in the partial differential equations version of the problem, this describes a function that is essentially constant in average, which is the mean-value property. Thus, we see that the mean-value property is built into the incorporation of the problem in probability theory, instead of having to explicitly compute anything. However, due to the lack of explicit formulas, probability theory provides a less precise solution. Note that, for convenience, this paper only dealt with the discrete cases in probability theory, however, everything done in the probability sections can be done in the continuous case with Brownian motion. For more details, please see [1].

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