

INTERPOLATION THEOREMS AND APPLICATIONS

CALISTA BERNARD

ABSTRACT. We discuss and prove two major theorems, the Riesz-Thorin Interpolation Theorem and the Marcinkiewicz Interpolation Theorem. We then address several important applications of these theorems, including the boundedness of the Hardy-Littlewood maximal operator, the boundedness of the Fourier transform, Young's inequality for convolution, and the boundedness of the Hilbert transform.

CONTENTS

1. Introduction	1
2. The Riesz-Thorin Interpolation Theorem	2
3. The Marcinkiewicz Interpolation Theorem	5
4. The Hardy-Littlewood Maximal Operator	11
5. The Fourier Transform and Convolution	12
Acknowledgments	14
References	14

1. INTRODUCTION

In this paper we present two main classical results of interpolation of operators: the Riesz-Thorin Interpolation Theorem and the Marcinkiewicz Theorem. The former allows us to show that a linear operator that is bounded on two L^p spaces is bounded on every L^p space in between the two. The latter allows us to show that a sublinear operator that satisfies two weak-type estimates is bounded on any L^p space in between the two weak L^p spaces. Thus, we may simplify the proof of the boundedness of an operator by proving the statement in two “simpler” cases (such as L^1 and L^∞ or L^1 and L^2) and then interpolating to prove the statement for every L^p space in between.

Following our discussion of the two interpolation theorems, we consider several applications: the boundedness of the Hardy-Littlewood maximal operator (section 4), the boundedness of the Fourier transform (section 5), and a proof of Young's inequality (section 5).

We assume the reader is familiar with L^p spaces, the Lebesgue integral, and basic measure theory. See Chapter one of [2] for background on L^p spaces, Chapter 2 of [1] for the Lebesgue integral, and Chapter 1 of [1] for measure theory.

2. THE RIESZ-THORIN INTERPOLATION THEOREM

We begin by proving a few useful lemmas.

Lemma 2.1. *Let $1 \leq p, q \leq \infty$ be conjugate exponents. If f is integrable over all sets of finite measure (and the measure μ is semifinite if $q = \infty$) and*

$$\sup_{\|g\|_p \leq 1, g \text{ simple}} \left| \int fg \right| = M < \infty$$

then $f \in L^q$ and $\|f\|_q = M$.

Proof. First we consider the case where $p < 1$ and $q < \infty$. Note that by Holder's inequality,

$$M \leq \|f\|_q \|g\|_p \leq \|f\|_q$$

since $\|g\|_p \leq 1$. Now we show the reverse inequality. We can find a sequence of simple functions $\{f_n\}_{n \in \mathbb{N}}$ in L^q that converges to f pointwise from below. Now define

$$g_n(x) = \frac{|f_n(x)|^{q-1} \cdot \operatorname{sgn} f}{\|f_n\|_q^{q-1}}.$$

It follows that

$$\|g_n\|_p^p = \frac{1}{\|f_n\|_q^{p(q-1)}} \int |f_n(x)|^{p(q-1)} = \frac{\|f_n\|_q^q}{\|f_n\|_q^q} = 1,$$

where the middle equality follows from the fact that p and q are conjugate exponents. Furthermore, since for large n , $\operatorname{sgn} f_n = \operatorname{sgn} f$, we see that

$$(2.2) \quad \int f_n g_n = \frac{\int |f_n|^q}{\|f_n\|_q^{q-1}} = \frac{\|f_n\|_q^q}{\|f_n\|_q^{q-1}} = \|f_n\|_q$$

for sufficiently large n . Thus, it follows from our assumption that $\int f_n g_n = \|f_n\|_q \leq M$. Then 2.2 implies that $\int |f_n|^q \leq M^q$. By Fatou's lemma, we have that

$$\int |f|^q \leq \liminf \int |f_n|^q \leq M^q.$$

It follows that $\|f\|_q \leq M$.

Now we consider the case where $p = 1$ and $q = \infty$. Fix $\epsilon > 0$ and let $E = \{x \mid |f(x)| \geq M + \epsilon\}$. Since μ is semifinite, if $\mu(E)$ were positive, then there would exist $F \subset E$ such that $0 < \mu(F) < \infty$. Let $g = \mu(F)^{-1} \chi_F \operatorname{sgn} f$. Then $\|g\|_1 = 1$ and

$$M \geq \int fg = \frac{1}{\mu(F)} \int_F |f| \geq M + \epsilon.$$

Since this is clearly impossible, we have that $\mu(E) = 0$. Then $f \in L^\infty$ and $M \geq \|f\|_\infty$. The opposite inequality is, once again, given by Holder's inequality. \square

Lemma 2.3. (Three Lines Lemma) *Suppose $\Phi : \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic on the inside of the strip $0 \leq \operatorname{Re} z \leq 1$ and continuous and bounded on the closure of the strip. Furthermore, suppose $|\Phi(z)| \leq M_0$ on the boundary $\operatorname{Re} z = 0$ and $|\Phi(z)| \leq M_1$ on the boundary $\operatorname{Re} z = 1$. Then for all $y \in \mathbb{R}$, $x \in (0, 1)$,*

$$|\Phi(x + iy)| \leq M_0^{1-x} M_1^x.$$

Proof. Let $\epsilon > 0$. Let $\Phi_\epsilon(z) = \Phi(z)M_0^{z-1}M_1^{-z}e^{\epsilon z(z-1)}$. Then if $\operatorname{Re} z = 0$, we have

$$|\Phi_\epsilon(iy)| \leq |M_0^{iy}M_1^{-iy}e^{\epsilon iy(iy-1)}| = |e^{iy \log(M_0/M_1)}e^{-\epsilon y^2}e^{-\epsilon iy}| \leq 1,$$

where the last inequality follows from the fact that $e^{iy \log M_0/M_1}$ and $e^{-\epsilon iy}$ are both on the unit circle. Similarly,

$$|\Phi_\epsilon(1+iy)| \leq |M_0^{1+iy-1}M_1^{-iy}e^{\epsilon(1+iy)(1+iy-1)}| = |e^{iy \log(M_0/M_1)}e^{-\epsilon y^2}e^{-\epsilon iy}| \leq 1.$$

Since Φ_ϵ is clearly holomorphic inside the strip, we see that it satisfies the assumptions from the statement (with 1 as M_0 and M_1). Now for any $x \in (0, 1)$,

$$\begin{aligned} |\Phi_\epsilon(x+iy)| &= |\Phi(x+iy)M_0^{x+iy-1}M_1^{-x-iy}e^{\epsilon(x+iy)(x+iy-1)}| \\ &= |\Phi(x+iy)| |e^{(x+iy-1) \log M_0}e^{(-x-iy) \log M_1}e^{\epsilon(x^2-x-y^2)}e^{\epsilon iy(2x-1)}| \\ &= |\Phi(x+iy)| |M_0^{x-1}M_1^{-x}e^{\epsilon(x^2-x)}| e^{-\epsilon y^2} \\ &\leq Ce^{-\epsilon y^2} \end{aligned}$$

for some positive constant C . This follows from the fact that Φ and e^{x^2-x} are bounded on the strip. Since $e^{-\epsilon y^2} \rightarrow 0$ as $|y| \rightarrow \infty$, it follows that $|\Phi_\epsilon(z)| \rightarrow 0$ as $|\operatorname{Im}(z)| \rightarrow \infty$. Thus, we can pick A large enough that $|\Phi_\epsilon(z)| \leq 1$ on the boundary of the rectangle $0 \leq \operatorname{Re} z \leq 1$ and $-A \leq \operatorname{Im} z \leq A$. Then the maximum modulus principle implies that $|\Phi_\epsilon(z)| \leq 1$ on the interior of the rectangle as well. Since this holds for any arbitrarily large A , we have that $|\Phi_\epsilon(z)| \leq 1$ on the entire strip.

Now let $\epsilon \rightarrow 0$. From above, we now have

$$\lim_{\epsilon \rightarrow 0} |\Phi_\epsilon(z)| = |\Phi(z)|M_0^{x-1}M_1^{-x} \leq 1$$

for $x = \operatorname{Re} z$. The result follows immediately. \square

We are now ready to prove the Riesz-Thorin Interpolation Theorem, which allows us to establish boundedness of a linear operator on certain L^p spaces.

Theorem 2.4. (Riesz-Thorin Interpolation Theorem) *Consider a linear function T , which maps the measure space (X, μ) to the measure space (Y, ν) . Suppose $p_0, q_0, p_1, q_1 \in [1, \infty]$ and*

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1} \quad \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

for $t \in (0, 1)$. If $q_0 = q_1 = \infty$, we further suppose that ν is semifinite. Suppose T maps $L^{p_0}(\mu) + L^{p_1}(\mu)$ into $L^{q_0}(\nu) + L^{q_1}(\nu)$ and we have $\|Tf\|_{q_0} \leq M_0\|f\|_{p_0}$ for $f \in L^{p_0}$ and $\|Tf\|_{q_1} \leq M_1\|f\|_{p_1}$ for $f \in L^{p_1}$, for constants $M_0, M_1 > 0$. Then T is bounded on L^p and furthermore, $\|Tf\|_q \leq M_0^{1-t}M_1^t\|f\|_p$ for all $f \in L^p$.

Our general strategy will be to construct a function Φ that satisfies the assumptions of the Three Lines Lemma, then use Φ to bound Tf in $L^q(\nu)$ for f simple. We then extend the result from f simple to all L^p .

Proof. We begin with the case $p_0 < p_1$. Let f be a simple function on X . Recall that simple functions are dense in L^p for any $p \in [1, \infty]$. Note that we can scale f because of the linearity of T , so it suffices to show the statement for $\|f\|_p = 1$. Write $f = \sum_{j=1}^n a_j \chi_{E_j} = \sum_{j=1}^n |a_j| e^{i\theta_j} \chi_{E_j}$, where $|a_j| e^{i\theta_j}$ is the polar form of a_j and the E_j are disjoint. Now let g be a simple function on Y . We further take g to

have q' -norm equal to 1 (where q' is the conjugate exponent of q). Again, we write $g = \sum_{k=1}^m b_k \chi_{F_k} = \sum_{k=1}^m |b_k| e^{i\theta_k} \chi_{F_k}$.

We now define two complex functions α and β as follows:

$$\alpha(z) = (1-z)p_0^{-1} + zp_1^{-1} \quad \beta(z) = (1-z)q_0^{-1} + zq_1^{-1}.$$

Note that if $t \in (0, 1)$, then $\alpha(t) = p^{-1}$ and $\beta(t) = q^{-1}$. Fixing $t \in (0, 1)$, we let

$$f_z = \sum_{j=1}^n |a_j|^{\alpha(z)/\alpha(t)} e^{i\theta_j} \chi_{E_j}$$

(note that $\alpha(t) > 0$) and

$$g_z = \sum_{k=1}^m |b_k|^{(1-\beta(z))/(1-\beta(t))} e^{i\theta_k} \chi_{F_k}$$

if $\beta(t) \neq 1$ and $g_z = g$ if $\beta(t) = 1$. Finally, we define

$$\Phi(z) = \int (Tf_z)g_z d\nu.$$

Then if $\beta(t) \neq 1$,

$$\Phi(z) = \sum_{j,k} |a_j|^{\alpha(z)/\alpha(t)} |b_k|^{(1-\beta(z))/(1-\beta(t))} e^{i(\theta_j+\theta_k)} \int (T\chi_{E_j})\chi_{F_k}$$

and if $\beta(t) = 1$, $\Phi(z) = \sum_{j,k} |a_j|^{\alpha(z)/\alpha(t)} |b_k| e^{i(\theta_j+\theta_k)} \int (T\chi_{E_j})\chi_{F_k}$. Then Φ is bounded and holomorphic in the strip $0 \leq \operatorname{Re} z \leq 1$. We claim that $|\Phi(z)| \leq M_0$ for $\operatorname{Re} z = 0$ and $|\Phi(z)| \leq M_1$ for $\operatorname{Re} z = 1$.

If $z = iy$ for $y \in \mathbb{R}$, then $\alpha(z) = p_0^{-1} + iy(p_1^{-1} - p_0^{-1})$ and $1 - \beta(z) = 1 - q_0^{-1} - iy(q_1^{-1} - q_0^{-1})$. Now note that since the E_j are disjoint, for any $x \in X$ at most one term of the sum equal to $f(x)$ or $f_z(x)$ may be nonzero. Let $x \in E_j$. Then we have that

$$\begin{aligned} |f_{iy}| &= ||a_j|^{\alpha(z)/\alpha(t)} e^{i\theta_j} \chi_{E_j}| \\ &= |a_j|^{\alpha(z)/\alpha(t)} \\ &= |e^{p_0^{-1}p \log |a_j|} e^{iy p(p_1^{-1} - p_0^{-1}) \log |a_j|}| \\ &= e^{p_0^{-1}p \log |a_j|} \\ &= |a_j|^{p/p_0} \end{aligned}$$

for some j . It follows that

$$(2.5) \quad |f_{iy}| = |f|^{p/p_0}.$$

A similar calculation shows that

$$(2.6) \quad |g_{iy}| = |g|^{(1-q_0^{-1})/(1-q^{-1})} = |g|^{q'/q'_0},$$

where q' and q'_0 denote the conjugate exponents of q and q_0 , respectively. Then by Holder's inequality and Equations 2.5 and 2.6, we have

$$\begin{aligned} |\Phi(iy)| &\leq \|Tf_{iy}\|_{q_0} \|g_{iy}\|_{q'_0} \\ &\leq M_0 \|f_{iy}\|_{p_0} \|g_{iy}\|_{q'_0} \\ &= M_0 \|f\|_p^{p/p_0} \|g\|_q^{q'/q_0} \\ &= M_0. \end{aligned}$$

Similarly, $|f_{1+iy}| = |f|^{p/p_1}$ and $|g_{1+iy}| = |g|^{q'/q'_1}$, so it follows that

$$\|\Phi(1+iy)\| \leq M_1 \|f_{1+iy}\|_{p_1} \|g_{1+iy}\|_{q'_1} = M_1.$$

Then Φ satisfies the assumptions of the Three Lines Lemma, so we have that $|\Phi(z)| \leq M_0^{1-t} M_1^t$ for $\operatorname{Re} z = t \in (0, 1)$. Now, by Lemma 2.1, we have that

$$(2.7) \quad \|Tf\|_q \leq M_0^{1-t} M_1^t \|f\|_p$$

for f simple.

We want to show that T satisfies this estimate for all of L^p . Let $f \in L^p$. Let $\{f_n\}$ be a sequence of measurable simple functions such that $|f_n| \leq |f|$ for all n , and $f_n \rightarrow f$ pointwise. Now let $E = \{x \mid |f(x)| > 1\}$. We define $g = f\chi_E$, $g_n = f_n\chi_E$, $h = f - g$, and $h_n = f_n - g_n$. By the Dominated Convergence Theorem, we have that $\|f_n\|_p \rightarrow \|f\|_p$. Since $f \in L^{p_0}$, and $|g| \leq |f|$, we have that $g \in L^{p_0}$. Moreover, since $f \in L^{p_1}$ and $|h| \leq 2|f|$, it follows that $h \in L^{p_1}$. Furthermore, since $|g_n| \leq |g|$ and $|h_n| \leq |h|$, we have, again by the Dominated Convergence Theorem, that $\|g_n\|_{p_0} \rightarrow \|g\|_{p_0}$ and $\|h_n\|_{p_1} \rightarrow \|h\|_{p_1}$. Then $\|Tg_n - Tg\|_{q_0} \rightarrow 0$ since $\|Tg_n - Tg\|_{q_0} \leq M_0 \|g_n - g\|_{p_0}$, and similarly, $\|Th_n - Th\|_{q_1} \rightarrow 0$. Then $Tg_n \rightarrow Tg$ in measure since for all $\epsilon > 0$,

$$\mu(x \mid |Tg_n - Tg| > \epsilon) \leq \frac{\|Tg_n - Tg\|_{q_0}^{q_0}}{\epsilon^{q_0}}$$

by Chebyshev's inequality. This implies that for some subsequence, $Tg_n \rightarrow Tg$ a.e. Similarly, we can find a subsequence such that $Th_n \rightarrow Th$ a.e. It follows that $Tf_n \rightarrow Tf$ a.e. Then by Fatou's Lemma and Equation 2.7, we have

$$\|Tf\|_q \leq \liminf \|Tf_n\|_q \leq \liminf M_0^{1-t} M_1^t \|f_n\|_p \leq M_0^{1-t} M_1^t \|f\|_p,$$

which concludes the proof for $p_0 \neq p_1$.

If $p_0 = p_1 = p$, then we claim that

$$\|Tf\|_q \leq \|Tf\|_{q_0}^{1-t} \|Tf\|_{q_1}^t \leq M_0^{1-t} M_1^t \|f\|_p.$$

This follows from the fact that if $q_0 < q < q_1$ and $q^{-1} = (1-t)q_0^{-1} + tq_1^{-1}$, then $\|f\|_q \leq \|f\|_{q_0}^{1-t} \|f\|_{q_1}^t$ (see [1] Section 6.5 for proof of this). \square

3. THE MARCINKIEWICZ INTERPOLATION THEOREM

Before proceeding to our next theorem, the Marcinkiewicz Interpolation Theorem, we introduce a few definitions and prove a few propositions.

Definition 3.1. For a measurable function f , we define the *distribution function* of f , $\lambda_f : (0, \infty) \rightarrow [0, \infty]$ by

$$\lambda_f(\alpha) = \mu(\{x \mid |f(x)| > \alpha\}).$$

We state several properties of distribution functions as facts. Proofs of these properties can be found in [1], Section 6.4.

- Fact 3.2.**
- i. If $|f| \leq |g|$, then $\lambda_f \leq \lambda_g$.
 - ii. If $\{f_n\}$ is a sequence such that $|f_n| \leq |f|$ and $f_n \rightarrow f$, then λ_{f_n} also increases to λ_f .
 - iii. If $f = g + h$, then $\lambda_f(\alpha) \leq \lambda_g(\frac{1}{2}\alpha) + \lambda_h(\frac{1}{2}\alpha)$.

Proposition 3.3. *Let $0 < p < \infty$. Then*

$$\int |f|^p d\mu = p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha) d\alpha.$$

Proof. This can be proven for simple functions using integration by parts. The general case follows from part *ii* of the above fact and the Monotone Convergence Theorem. The details can be found in [1], Section 6.4. \square

Proposition 3.4. *Let f be a measurable function and fix a constant $a > 0$. Now define $A_a = \{x \mid |f(x)| > a\}$ and let*

$$h_a = f\chi_{A_a^c} + a(\operatorname{sgn}f)\chi_{A_a} \quad g_a = f - h_a = (\operatorname{sgn}f)(|f| - a)\chi_{A_a}.$$

Then $\lambda_{g_a}(\alpha) = \lambda_f(\alpha + a)$ and

$$\lambda_{h_a}(\alpha) = \begin{cases} \lambda_f(\alpha) & \alpha < a \\ 0 & \alpha \geq a \end{cases}.$$

Proof. Note that if $|f(x)| \leq a$, then $h_a(x) = f(x)$ and if $|f(x)| > a$, then $h_a(x) = a(\operatorname{sgn}f)$. Thus, $|h_a| \leq a$, so $\lambda_{h_a}(\alpha) = 0$ when $\alpha \geq a$. Moreover, $|h_a(x)| = |f(x)|$ when $x \in A_a^c$ and $|h_a(x)| = a \geq |f(x)|$ when $x \in A_a$. Thus, if $\alpha < a$, then $\lambda_{h_a}(\alpha) = \lambda_f(\alpha)$.

We note that $g_a(x) = 0$ for $x \in A_a^c$, so $\{x \mid |g_a(x)| > \alpha\} \subset A_a$. Thus we want to find x such that $(\operatorname{sgn}f)(|f| - a) > \alpha$. If $x \in A_a$, then $|f(x)| - a > 0$, so $(\operatorname{sgn}f(x))(|f(x)| - a) > \alpha$ iff $\operatorname{sgn}f = 1$. Thus,

$$\lambda_{g_a}(\alpha) = \mu(\{x \mid |f(x)| - a > \alpha\}) = \mu(\{x \mid |f(x)| > \alpha + a\}) = \lambda_f(\alpha + a).$$

\square

Definition 3.5. For a measurable function f and $0 < p < \infty$, let

$$[f]_p = (\sup_{\alpha > 0} \alpha^p \lambda_f(\alpha))^{1/p}.$$

We define *weak* L^p , denoted $L^{p,w}$, to be the set of functions f such that $[f]_p < \infty$.

We note that $[\cdot]_p$ fails the triangle inequality, so it is not a norm. However, we do have that $[cf]_p = |c|[f]_p$. This follows from the fact that

$$\mu(\{x \mid |cf(x)| > \alpha\}) = \mu(\{x \mid |f(x)| > \frac{\alpha}{|c|}\}),$$

so $\lambda_{cf}(\alpha) = \lambda_f(\frac{\alpha}{|c|})$. Thus,

$$[cf]_p = \left(\sup_{\alpha > 0} \alpha^p \lambda_f\left(\frac{\alpha}{|c|}\right) \right)^{1/p} = \left(\sup_{\beta > 0} \beta^p |c|^p \lambda_f(\beta) \right)^{1/p} = |c| \left(\sup_{\beta > 0} \beta^p \lambda_f(\beta) \right)^{1/p} = |c|[f]_p$$

Clearly, $L^p \subset L^{p,w}$ and we have $[f]_p \leq \|f\|_p$ by Chebychev's inequality.

Definition 3.6. Let T be a sub-linear map from a vector space \mathcal{V} of measurable functions on (X, μ) to the space of measurable functions on (Y, ν) . We say T is *strong type* (p, q) if $L^p(\mu) \subset \mathcal{V}$, T takes $L^p(\mu)$ to $L^q(\nu)$, and $\|Tf\|_q \leq C\|f\|_p$ for all $f \in L^p$ and some $C > 0$.

Definition 3.7. Let T be a sub-linear map from a vector space \mathcal{V} of measurable functions on (X, μ) to the space of measurable functions on (Y, ν) . We say T is *weak type* (p, q) for $p \in [1, \infty], q \in [1, \infty)$ if T maps $L^p(\mu)$ into $L^{q,w}$ and for some $C > 0$, $\|Tf\|_q \leq C\|f\|_p$ for all $f \in L^p$. We say T is weak type (p, ∞) iff T is strong type (p, ∞) .

We are now ready to prove the Marcinkiewicz Interpolation Theorem, which will allow us to work with sublinear operators. Furthermore, we will only require that the operator satisfy weak rather than strong-type estimates. However, we will need more stringent restrictions on our p 's and q 's, and we will have a less specific estimate for the operator norm.

Theorem 3.8. (Marcinkiewicz Interpolation Theorem) *Consider a sublinear function T , which maps the measure space (X, μ) to the measure space (Y, ν) . Suppose $p_0, q_0, p_1, q_1 \in [1, \infty]$ such that $p_0 \leq q_0, p_1 \leq q_1$, and $q_0 \neq q_1$ and*

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1} \quad \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

for $t \in (0, 1)$. If T maps $L^{p_0}(X, \mu) + L^{p_1}(X, \mu)$ into the space of ν -measurable functions on Y and $\|Tf\|_{q_i} \leq C_i\|f\|_{p_i}$ for $C_i > 0$ and $i = 0, 1$, then there exists a constant C that is dependent on $p, q, p_0, q_0, p_1, q_1, C_0, C_1$ such that $\|Tf\|_q \leq C\|f\|_p$. More simply stated, if T is weak types (p_0, q_0) and (p_1, q_1) , then T is strong type (p, q) .

For all but the simplest cases, our strategy will be to split f into g_a and h_a from Proposition 3.4. We then rewrite their norms in terms of the distribution function of f , giving each integral a bound of a . We then strategically choose a in terms of α so that we end up with an appropriate constant C bounding the q -norm of Tf .

Proof. There are several cases to consider. First we prove for $p_0 = p_1 = p$ and $q_1 = \infty$. First we note that for $\alpha > 0$,

$$\alpha^{q_0} \lambda_{Tf}(\alpha) \leq [Tf]_{q_0}^{q_0} \leq C_0^{q_0} \|f\|_p^{q_0},$$

so $\lambda_{Tf}(\alpha) \leq (C_0\|f\|_p/\alpha)^{q_0}$. Then $\|Tf\|_\infty \leq C_1\|f\|_p$, so if we let $a = C_1\|f\|_p$, we find that $\lambda_{Tf}(\alpha) = 0$ if $\alpha > a$. Thus, by Proposition 3.3 we have

$$\begin{aligned} \|Tf\|_q^q &= q \int_0^\infty \alpha^{q-1} \lambda_{Tf}(\alpha) d\alpha \\ &= q \int_0^a \alpha^{q-1} \lambda_{Tf}(\alpha) d\alpha \\ &\leq q C_0^{q_0} \|f\|_p^{q_0} \int_0^a \alpha^{q-1-q_0} d\alpha \\ &= C \|f\|_p^{q_0} \end{aligned}$$

for a positive constant C since $q-1-q_1 > -1$. Note that this argument also holds when $q_0 = \infty$ simply by switching the q_0 's and q_1 's.

For the case where both $q_i < \infty$ and $p_0 = p_1$, we have a very similar argument. Here we assume that $q_0 < q_1$ (otherwise, we may simply switch the 1's and 0's in the following argument). We have that $\lambda_{Tf} \leq (C_i \|f\|_p / \alpha)^{q_i}$ for $i = 0, 1$. Then

$$\begin{aligned} [Tf]_q^q &= q \int_0^\infty \alpha^{q-1} \lambda_{Tf}(\alpha) d\alpha \\ &\leq q \int_0^1 \alpha^{q-q_0-1} C_0^{q_0} \|f\|_p^{q_0} d\alpha + q \int_1^\infty \alpha^{q-q_1-1} C_1^{q_1} \|f\|_p^{q_1} d\alpha \\ &= \frac{q C_0^{q_0} \|f\|_p^{q_0}}{q - q_0} + \frac{q C_1^{q_1} \|f\|_p^{q_1}}{q_1 - q} \\ &= C \|f\|_p^q \end{aligned}$$

for a positive constant C .

Thus, we may assume that $p_0 < p_1$. Now we will consider the case where $q_0, q_1 < \infty$. Recall the functions g_a and h_a from Proposition 3.4. By Propositions 3.3 and 3.4 and change of variables, we can write

$$\begin{aligned} \int |g_a|^{p_0} &= p_0 \int_0^\infty \alpha^{p_0-1} \lambda_{g_a}(\alpha) d\alpha \\ &= p_0 \int_0^\infty \alpha^{p_0-1} \lambda_f(\alpha + a) d\alpha \\ (3.9) \quad &= p_0 \int_a^\infty (\alpha - a)^{p_0-1} \lambda_f(\alpha) d\alpha \\ &\leq p_0 \int_a^\infty \alpha^{p_0-1} \lambda_f(\alpha) d\alpha \end{aligned}$$

and

$$\begin{aligned} \int |h_a|^{p_1} &= p_1 \int_0^\infty \alpha^{p_1-1} \lambda_{h_a}(\alpha) d\alpha \\ (3.10) \quad &= p_1 \int_0^a \alpha^{p_1-1} \lambda_f(\alpha) d\alpha. \end{aligned}$$

Furthermore, we know by Fact 3.2 that

$$(3.11) \quad \lambda_{Tf}(2\alpha) \leq \lambda_{Tg_a}(\alpha) + \lambda_{Th_a}(\alpha).$$

Thus, using Proposition 3.3, Equation 3.11, and the given weak type estimates, we have

$$\begin{aligned} \int |Tf|^q &= q \int_0^\infty \beta^{q-1} \lambda_{Tf}(\beta) d\beta \\ &= 2^q q \int_0^\infty \beta^{q-1} \lambda_{Tf}(2\beta) d\beta \\ (3.12) \quad &\leq 2^q q \int_0^\infty \beta^{q-1} (\lambda_{Tg_a}(\beta) + \lambda_{Th_a}(\beta)) d\beta \\ &\leq 2^q q \int_0^\infty \beta^{q-1} \left(\frac{[Tg_a]_{q_0}^{q_0}}{\beta^{q_0}} + \frac{[Th_a]_{q_1}^{q_1}}{\beta^{q_1}} \right) d\beta \\ &\leq 2^q q \int_0^\infty \beta^{q-1} \left[\left(\frac{C_0 \|g_a\|_{p_0}}{\beta} \right)^{q_0} + \left(\frac{C_1 \|h_a\|_{p_1}}{\beta} \right)^{q_1} \right] d\beta. \end{aligned}$$

Now, by Equations 3.9 and 3.10, we have that

$$\begin{aligned}
(3.13) \quad & 2^q q \int_0^\infty \beta^{q-1} \left[\left(\frac{C_0 \|g_a\|_{p_0}}{\beta} \right)^{q_0} + \left(\frac{C_1 \|h_a\|_{p_1}}{\beta} \right)^{q_1} \right] d\beta \\
& \leq 2^q q C_0^{q_0} p_0^{q_0/p_0} \int_0^\infty \beta^{q-q_0-1} \left(\int_a^\infty \alpha^{p_0-1} \lambda_f(\alpha) d\alpha \right)^{q_0/p_0} d\beta \\
& \quad + 2^q q C_1^{q_1} p_1^{q_1/p_1} \int_0^\infty \beta^{q-q_1-1} \left(\int_0^a \alpha^{p_1-1} \lambda_f(\alpha) d\alpha \right)^{q_1/p_1} d\beta.
\end{aligned}$$

Since this holds for any $a > 0$, we can let $a = \beta^\sigma$, where $\sigma = \frac{p_0(q_0-q)}{q_0(p_0-p)}$. By Minkowski's integral inequality (since $q_i/p_i \geq 1$), we have

$$\begin{aligned}
& \int_0^\infty \beta^{q-q_i-1} \left(\int_0^\infty \chi_i \alpha^{p_i-1} \lambda_f(\alpha) d\alpha \right)^{q_i/p_i} d\beta \\
& \leq \left[\int_0^\infty \left[\int_0^\infty \chi_i \beta^{q-q_i-1} (\alpha^{p_i-1} \lambda_f(\alpha))^{q_i/p_i} d\beta \right]^{p_i/q_i} d\alpha \right]^{q_i/p_i}
\end{aligned}$$

for $i = 0, 1$, where χ_0 is the characteristic function of $\{(\alpha, \beta) \mid \alpha > \beta^\sigma\}$ and χ_1 is the characteristic function of $\{(\alpha, \beta) \mid \alpha < \beta^\sigma\}$.

Now, either $q_1 > q > q_0$ or $q_1 < q < q_0$. In the first case, $q - q_0 > 0$, so $\sigma > 0$ (since we have assumed that $p_0 < p < p_1$.) If $\alpha > \beta^\sigma$, then $\alpha^{1/\sigma} > \beta$, so we have

$$(3.14) \quad 2^q q C_0^{q_0} p_0^{q_0/p_0} \left[\int_0^\infty \left[\int_0^{\alpha^{1/\sigma}} \beta^{q-q_0-1} (\alpha^{p_0-1} \lambda_f(\alpha))^{q_0/p_0} d\beta \right]^{p_0/q_0} d\alpha \right]^{q_0/p_0}$$

$$(3.15) \quad = 2^q q C_0^{q_0} p_0^{q_0/p_0} (q - q_0)^{-1} \left[\int_0^\infty \alpha^{p_0-1+(q-q_0)p_0/q_0\sigma} \lambda_f(\alpha) d\alpha \right]^{q_0/p_0}$$

$$(3.16) \quad = 2^q q C_0^{q_0} p_0^{q_0/p_0} |q - q_0|^{-1} \left[\int_0^\infty \alpha^{p-1} \lambda_f(\alpha) d\alpha \right]^{q_0/p_0}$$

$$(3.17) \quad = 2^q q p_0^{q_0/p_0} C_0^{q_0} |q - q_0|^{-1} \left(\frac{\|f\|_p^p}{p} \right)^{q_0/p_0}$$

where the second-to-last equality follows from the fact that $\sigma = \frac{p_0(q_0-q)}{q_0(p_0-p)}$.

Similarly, if $q_1 < q < q_0$, then $q - q_0 < 0$, so $\sigma < 0$ and thus if $\alpha > \beta^\sigma$, we have $\beta > \alpha^{1/\sigma}$. Then we have the same proof as above, except the bounds on the second integral in Equation 3.14 will change from 0 and $\alpha^{1/\sigma}$ to $\alpha^{1/\sigma}$ and ∞ , respectively and we will have a minus sign in front of the α in Equation 3.15. However, since $|q - q_0| = (q_0 - q) = -(q - q_0)$, this does not change the result in Equation 3.17.

Now note that

$$\begin{aligned}\sigma &= \frac{p_0(q_0 - q)}{q_0(p_0 - p)} = \frac{1 - \left(\frac{q_0}{q}\right)^{-1}}{1 - \left(\frac{p_0}{p}\right)^{-1}} = \frac{p^{-1}(q^{-1} - q_0^{-1})}{q^{-1}(p^{-1} - p_0^{-1})} \\ &= \frac{p^{-1}(q^{-1} - q_1^{-1})}{q^{-1}(p^{-1} - p_1^{-1})} = \frac{1 - \left(\frac{q_1}{q}\right)^{-1}}{1 - \left(\frac{p_1}{p}\right)^{-1}} = \frac{p_1(q_1 - q)}{q_1(p_1 - p)},\end{aligned}$$

so $q - q_1 < 0$ corresponds to $\sigma > 0$ and $\alpha^{1/\sigma} < \beta$ and $q - q_1 > 0$ corresponds to $\sigma < 0$ and $\alpha^{1/\sigma} > \beta$. Then we can do a similar calculation to find

$$\begin{aligned}(3.18) \quad & 2^q q C_1^{q_1} p_1^{q_1/p_1} \left[\int_0^\infty \left[\int_0^\infty \chi_1(\beta^{q-q_1-1} \alpha^{p_1-1} \lambda_f(\alpha))^{q_1/p_1} d\beta \right]^{p_1/q_1} d\alpha \right]^{q_1/p_1} \\ &= 2^q q C_1^{q_1} (p_1/p)^{q_1/p_1} |q - q_1|^{-1} (\|f\|_p^p)^{q_1/p_1}\end{aligned}$$

Thus, from (3.17) and (3.18), we see that

$$\|Tf\|_q \leq 2q^{1/q} \left[\sum_{i=0}^1 (p_i/p)^{q_i/p_i} C_i^{q_i} |q - q_i|^{-1} \|f\|_p^{p q_i/p_i} \right]^{1/q}.$$

This implies that

$$\sup_{\|f\|_p=1} \|Tf\|_q \leq C = 2q^{1/q} \left[\sum_{i=0}^1 (p_i/p)^{q_i/p_i} C_i^{q_i} |q - q_i|^{-1} \right]^{1/q},$$

so for all $f \in L^p$, $\|Tf\|_q \leq C\|f\|_p$ (we can scale f because of the sublinearity of T). This completes the proof.

We can use a similar argument in the remaining cases. If $p_1 = q_1 = \infty$, then we let $a = \beta/C_1$. Then $\|Th_a\|_\infty \leq C_1\|h_a\|_\infty \leq \beta$, so $\lambda_{Th_a} = 0$. Making this adjustment to the third line of Equation 3.12, we arrive at Equation 3.13 with only the first term (since the second is 0). Then in Equation 3.14, we replace $\alpha^{1/\sigma}$ with αC_1 and we have

$$\|Tf\|_q \leq 2 \left[q(p_0/p)^{q_0/p_0} C_0^{q_0} C_1^{q-q_1} |q - q_1|^{-1} \right]^{1/q} \|f\|_p.$$

In the case where $q_0 < q_1 = \infty$, we let $a = \left(\frac{\beta}{d}\right)^\tau$, where $d = C_1(p_1\|f\|_p^p/p)^{1/p_1}$ and $\tau = p_1/(p_1 - p)$. Then we have

$$\begin{aligned}\|Th_a\|_\infty^{p_1} &\leq C_1^{p_1} \|h_a\|_{p_1}^{p_1} \\ &= C_1^{p_1} p_1 \int_0^a \beta^{p_1-1} \lambda_f(\beta) d\beta \\ &\leq C_1^{p_1} p_1 a^{p_1-p} \int_0^a \beta^{p-1} \lambda_f(\beta) d\beta \\ &\leq C_1^{p_1} \frac{p_1}{p} a^{p_1-p} \|f\|_p^p \\ &= C_1^{p_1} \frac{p_1}{p} \left(\frac{\beta}{d}\right)^{p_1} \|f\|_p^p \\ &= \beta^{p_1}\end{aligned}$$

Then, as in the previous case, we have that $\lambda_{Th_a} = 0$ and we can follow the same steps above to find an appropriate constant C .

For our final case, $q_1 < q_0 = \infty$, we follow essentially the same proof as in the previous case, choosing $a = (\beta/d)^\tau$ with d chosen so that $\lambda_{Th_a} = 0$. \square

4. THE HARDY-LITTLEWOOD MAXIMAL OPERATOR

We now proceed to a useful application of the Marcinkiewicz Theorem.

Definition 4.1. We say a complex-valued, Lebesgue-measurable function f is *locally integrable* on $X \subset \mathbb{R}^n$ if for any measurable, bounded set $E \subset X$, $\int_E |f| < \infty$. The space of locally integrable functions is denoted L^1_{loc} .

Note that L^1_{loc} is similar to L^1 , but locally integrable functions need not vanish at infinity.

Definition 4.2. Let $f \in L^1_{\text{loc}}$. The *Hardy-Littlewood maximal function* of f , denoted Hf , is given by

$$Hf(x) = \sup_{r>0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y)| dy,$$

where $B_r(x)$ is the open ball of radius r centred at x .

Note that the maximal function takes the supremum of the average value of $|f|$ over balls centred at x . We would like to show that H is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. However, we first prove an interesting lemma.

Lemma 4.3. *Suppose \mathcal{U} is a collection of open balls in \mathbb{R}^n and $V = \bigcup_{U \in \mathcal{U}} U$. If $m(V) > a$ for some nonnegative constant a , then there exist disjoint $U_1, \dots, U_n \in \mathcal{U}$ such that $\sum_{j=1}^k m(U_j) > \frac{a}{3^n}$.*

Proof. Recall that by the regularity of Lebesgue measure we may choose $K \subset V$ such that K is compact and $a < m(K) \leq m(V)$. Then \mathcal{U} forms an open covering of K , so there exists a finite subcollection $V_1, \dots, V_m \in \mathcal{U}$ that also covers K . Let W_1 be the V_i with largest radius. Let U_2 be the V_i with largest radius such that $U_2 \cap U_1 = \emptyset$. Let U_3 be the V_i with largest radius such that U_3 is disjoint from U_2 and U_1 . We continue selecting U_j to be the ball with the largest radius among the remaining V_i that is disjoint from the previous U_j until we have no more V_i . Note that each V_i is either equal to some U_j or intersects some U_j with larger radius nontrivially (we can ensure that this U_j has larger radius by selecting the smallest j such that $V_i \cap U_j \neq \emptyset$). Now denote by W_j the ball concentric with U_j that has radius three times that of U_j . By our previous statement, we know that each V_i is contained in some such W_j . It follows that $K \subset \bigcup_{j=1}^k W_j$. We then have

$$a < m(K) \leq \sum_{j=1}^k m(W_j) = 3^n \sum_{j=1}^k m(U_j).$$

Thus, $\sum_{j=1}^k m(U_j) > \frac{a}{3^n}$. \square

Theorem 4.4. *The Hardy-Littlewood maximal function is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. That is, for some $C > 0$,*

$$\|Hf\|_p \leq C\|f\|_p.$$

Proof. First we see that $|H(cf)| = |c||Hf|$ and by the triangle inequality for absolute value, $|H(f+g)| \leq |Hf| + |Hg|$, so H must be sub-linear. Furthermore, H is weak-type (∞, ∞) since

$$\sup_{r>0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y)| dy \leq \sup_{r>0} \frac{m(B_r(x))}{m(B_r(x))} \|f\|_\infty = \|f\|_\infty.$$

We claim that H is weak-type $(1, 1)$. That is,

$$\sup_{\alpha>0} \alpha \lambda_{Tf}(\alpha) \leq C_1 \int |f|$$

for all $f \in L^1$ and some positive constant C_1 .

Proof of claim. Let $A_\alpha = \{x \mid Hf(x) > \alpha\}$. If A_α is empty, then $\lambda_{Tf}(\alpha) = 0$, so the inequality holds. If A_α is nonempty, then we can choose $r_x > 0$ for each $x \in A_\alpha$ such that $\frac{1}{m(B_{r_x}(x))} \int_{B_{r_x}(x)} |f(y)| dy > \alpha$. Then $\{B_{r_x}(x)\}$ covers A_α .

Now fix $\beta > 0$ such that $m(A_\alpha) > \beta$. By Lemma 4.3, there we can choose $B_{r_{x_1}}(x_1), \dots, B_{r_{x_k}}(x_k)$, which we will denote by B_1, \dots, B_n , such that

$$\sum_{i=1}^k m(B_i) > 3^{-n} \beta.$$

This gives us

$$\beta < 3^n \sum_{i=1}^k m(B_i) \leq \frac{3^n}{\alpha} \int_{B_i} |f(y)| dy \leq \frac{3^n}{\alpha} \int_{\mathbb{R}^n} |f(y)| dy.$$

Then, letting $\beta \rightarrow m(A_\alpha)$, we find that $m(A_\alpha) \leq \frac{3^n}{\alpha} \|f\|_1$, so

$$\sup_{\alpha>0} \alpha \lambda_{Hf}(\alpha) \leq 3^n \|f\|_1.$$

Thus, H is weak-type $(1, 1)$.

Thus, by the Marcinkiewicz Interpolation Theorem, we have that for all $p \in (1, \infty)$, $\|Hf\|_p \leq C \|f\|_p$. \square

5. THE FOURIER TRANSFORM AND CONVOLUTION

We now turn to an application of the Riesz-Thorin Theorem. We assume the reader is already familiar with the basics of Fourier analysis, but state some definitions for clarity.

Definition 5.1. Let $f \in L^1(\mathbb{T}^n)$. We define the *Fourier transform* $\hat{f} : \mathbb{Z}^n \rightarrow \mathbb{C}$ of f to be

$$\hat{f}(k) = \int_{\mathbb{T}^n} f(x) e^{-2\pi i k x} dx.$$

Furthermore, we call the map $f \mapsto \hat{f}$ the *Fourier transform operator*.

Note that the Fourier transform operator is linear since for $f_1, f_2 \in L^1(\mathbb{T}^n)$ and $c \in \mathbb{C}^n$, we have

$$\begin{aligned} (\widehat{cf_1 + f_2})(k) &= \int_{\mathbb{T}^n} (cf_1 + f_2)(x) e^{-2\pi i k x} dx \\ &= c \int_{\mathbb{T}^n} f_1(x) e^{-2\pi i k x} dx + \int_{\mathbb{T}^n} f_2(x) e^{-2\pi i k x} dx \\ &= c\widehat{f_1}(k) + \widehat{f_2}(k). \end{aligned}$$

We would like to show that this operator is bounded on $L^p(\mathbb{T}^n)$ for $1 \leq p \leq 2$ and q the conjugate exponent of p .

Theorem 5.2. *The Fourier transform is bounded on $L^p(\mathbb{T}^n)$ for $1 \leq p \leq 2$. That is, if $f \in L^p(\mathbb{T}^n)$, then $\widehat{f} \in \ell^q(\mathbb{Z}^n)$ and $\|\widehat{f}\|_q \leq \|f\|_p$ for any $p \in [1, 2]$.*

Proof. If $f \in L^1$, then

$$\begin{aligned} \|\widehat{f}\|_\infty &= \sup_{\mathbb{Z}^n} |\widehat{f}(k)| \leq \sup_{\mathbb{Z}^n} \int_{\mathbb{T}^n} |f(x) e^{-2\pi i k x}| dx \\ &= \int_{\mathbb{T}^n} |f(x)| dx = \|f\|_1. \end{aligned}$$

Moreover, by Parseval's identity, we have $\|\widehat{f}\|_2 = \|f\|_2$ for $f \in L^2$. Then by the Riesz-Thorin Interpolation Theorem, we have that $\|\widehat{f}\|_q \leq \|f\|_p$ for any $p \in [1, 2]$. \square

We now wish to find a similar result for $L^p(\mathbb{R}^n)$. Since we define the Fourier transform for $f \in L^1(\mathbb{R}^n)$ to be $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi x} dx$, we see that the Fourier transform operator is also linear on $L^1(\mathbb{R}^n)$.

Theorem 5.3. *The Fourier transform is bounded on $L^p(\mathbb{R}^n)$ for $1 \leq p \leq 2$. That is, if $f \in L^p(\mathbb{R}^n)$, then $\widehat{f} \in L^q(\mathbb{R}^n)$ and $\|\widehat{f}\|_q \leq \|f\|_p$ for $1 \leq p \leq 2$.*

Proof. As is the case on the torus, $\|\widehat{f}\|_\infty \leq \|f\|_1$, and by Plancherel's Theorem, $\|\widehat{f}\|_2 = \|f\|_2$. The result then follows from the Riesz-Thorin Interpolation Theorem. \square

Another application of the Riesz-Thorin Theorem, also with applications in Fourier analysis and harmonic analysis, is in proving Young's inequality. Although a proof that does not use this theorem can be given, it is very tedious. The Riesz-Thorin Theorem allows for a much simpler proof.

Definition 5.4. For $f, g \in L^1(\mathbb{R}^n)$, we define the *convolution* of f and g to be

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y) g(x - y) dy.$$

Now we are ready for Young's inequality.

Theorem 5.5. (Young's Inequality) *Let $f \in L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$. If $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$ for $p, q, r \in [1, \infty]$, then $f * g \in L^r(\mathbb{R}^n)$ and*

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Proof. We denote by T_g the convolution operator of g . Thus, $T_g f = f * g$ for $f \in L^1$. Clearly, T_g is linear due to the linearity of the integral. By Minkowski's Integral Inequality, we have

$$\begin{aligned} \|T_g f\|_q &= \left(\int \left| \int f(y)g(x-y) dy \right|^q dx \right)^{1/q} \\ &\leq \int \left(\int |f(y)g(x-y)|^q dx \right)^{1/q} dy \\ &= \|f\|_1 \|g\|_q. \end{aligned}$$

Thus, T_g is strong type $(1, q)$. Furthermore, by Holder's inequality, we have

$$\begin{aligned} \|T_g f\|_\infty &= \sup_{\mathbb{R}^n} \left| \int f(y)g(x-y) dy \right| dx \\ &\leq \|f\|_{q'} \|g\|_q, \end{aligned}$$

where q' is the conjugate exponent of q . Then T_g is strong type (q', ∞) . Now let $\frac{1}{r} = \frac{1-t}{q} + \frac{t}{\infty}$ and $\frac{1}{p} = \frac{1-t}{1} + \frac{t}{q'}$ for $t \in (0, 1)$. Note that

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1-t}{q} = \frac{1}{r} + 1.$$

Then by the Riesz-Thorin Interpolation Theorem, T_f is strong type (p, r) , that is

$$\|f * g\|_r \leq \|f\|_p \|g\|_q$$

and we are done. □

Acknowledgments. I would like to thank my mentor, Rachel Vishnepolsky, for all of her guidance, answers to my numerous questions, and multiple reviews of this paper. I also owe thanks Casey Rodriguez for suggesting the topic of this paper. Finally, I would like to thank Peter May for his support and for organizing this REU.

REFERENCES

- [1] Gerald Folland. Real Analysis: Modern Techniques and Their Applications. John Wiley & Sons Inc. 1999.
- [2] Elias M. Stein and Rami Shakarchi. Functional Analysis: Introduction to Further Topics in Analysis. Princeton University Press. 2011.