ISOMETRY GROUPS OF COMPACT RIEMANN SURFACES

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Abstract. We explore the structure of compact Riemann surfaces by studying their isometry groups. First we give two constructions due to Accola [1] showing that for all $g \geq 2$, there are Riemann surfaces of genus g that admit isometry groups of at least some minimal size. Then we prove a theorem of Hurwitz giving an upper bound on the size of any isometry group acting on any Riemann surface of genus $g \geq 2$. Finally, we briefly discuss Hurwitz surfaces – Riemann surfaces with maximal symmetry – and comment on a method for computing isometry groups of Riemann surfaces.

Contents

1. Introduction

A Riemann surface is a topological surface equipped with a conformal structure, which determines a notion of orientation and angle on the surface. This additional structure allows us to do complex analysis on the surface, so that Riemann surfaces are central objects in geometry, analysis, and mathematical physics, for example string theory. Felix Klein, who did foundational work on Riemann surfaces, proposed in 1872 that group theory should be used to understand geometries by studying their symmetries (i.e. isometry groups). We will study the symmetries of compact Riemann surfaces to understand one aspect of the rigidity provided by their conformal structure. For us, a "symmetry" will be a conformal isometry, which is a topological homeomorphism that preserves arc length and angle, including orientation.

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The simplest compact Riemann surface is the Riemann sphere, the sphere embedded in $\mathbb{R}^3$ with the inherited metric. The Riemann sphere admits infinite isometry groups; for example, the rotations about some fixed axis form a group isomorphic to the unit circle as a multiplicative subgroup of $\mathbb{C}$. The torus Y_1 can be realized as a quotient of $\mathbb{C}$ by a lattice Λ of isometries generated by two independent translations. If for example we take Λ generated by the translations by 1 and by i, then $\mathbb{C}/\Lambda$ is the torus obtained by identifying the opposite edges of the square $[0,1] \times [0,i]$. Conformal isometries of $\mathbb{C}$ are then also conformal isometries of the quotient space, so that $\mathbb{C}/\Lambda$ admits the infinite group $(\mathbb{R} \times \mathbb{R})/(\mathbb{Z} \times \mathbb{Z})$ of all translations up to equivalence by a translation in Λ.

Thus when $g < 2$, the compact Riemann surface Y_g of genus g may admit huge isometry groups. However, when $g \geq 2$, the conformal structure on Y_g constrains the possible isometry groups, so we can describe their possible sizes more precisely. Indeed, we will prove tight upper and lower bounds on $N(g)$, the maximum size of any isometry group admitted by any compact Riemann surface of genus g.

In Section 2 we collect the notation and background facts we will need. In Section 3 we develop some machinery for dealing with isometries and automorphisms of covering spaces. We use this machinery to prove two lower bounds on $N(g)$ via constructions due to Accola [1]:

Theorem 3.6. *For every $g \geq 2$, $N(g) \geq 8(g+1)$. That is, there exists a Riemann surface Y_g of genus g which has an isometry group of size $8(g+1)$.*

Theorem 3.7. *For every $g \geq 2$, if g is divisible by 3, then $N(g) \geq 8(g+3)$. That is, there exists a Riemann surface Y_g of genus g which has an isometry group of size $8(g+3)$.*

Then we discuss the possibility of extending Accola's construction, which uses the dihedral and octahedral symmetries of the sphere, to the dodecahedral symmetries of the sphere.

In Section 4 we prove a striking upper bound, due to Hurwitz, on the possible sizes of the isometry group $\mathrm{Isom}^+(Y_g)$ of Y_g:

Theorem 4.2. *Let Y_g be any Riemann surface of genus $g \geq 2$, equipped with any conformal metric. Then we have $|Isom^+(Y_g)| \leq 84(g-1)$. In other words, $N(g) \leq 84(g-1)$ for all $g \geq 2$.*

Then we discuss the tightness of this bound and the construction of Klein's quartic and its coverings. We conclude with a brief note on the methods used by Kuribayashi [7] to compute isometry groups of Riemann surfaces.

2. Preliminaries

This section outlines the background knowledge assumed in this paper, records the notation used, and recalls the most crucial facts for reference.

Let τ denote the ratio of the circumference of a circle to its radius.

We assume familiarity with the fundamental group $\pi_1(X, x_0)$ of a space X at basepoint x_0 and the notion of a covering $\Pi : \widetilde{X} \to X$ of the space X by a covering space $\widetilde{X}$. We denote by $\Pi_* : \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ the homomorphism of fundamental groups induced by the covering map Π. **Deck transformations** of a covering $\Pi : \widetilde{X} \to X$, sometimes call covering transformations, are maps $f : \widetilde{X} \to \widetilde{X}$

such that $\Pi \circ f = \Pi$. The most important fact for us is the Galois correspondence between coverings of X and subgroups of the fundamental group of X.

Theorem 2.1 (Galois correspondence)**.** *Let X be a path-connected, locally path-connected, and semilocally simply-connected space. (In particular, X may be a surface.) Associating a covering $\Pi : \widetilde{X} \to X$ to the subgroup $\Pi_*(\pi_1(\widetilde{X}))$ gives a bijective correspondence between isomorphism classes of path-connected coverings of X and conjugacy classes of subgroups of $\pi_1(\widetilde{X})$. If the spaces are based and the maps are basepoint-preserving, we obtain a bijective correspondence between equivalence classes of based path-connected coverings of (X, x_0) up to basepoint-preserving isomorphisms, and subgroups of $\pi(X, x_0)$. If $\Pi_*(\pi_1(\widetilde{X}))$ is a normal subgroup of $\pi_1(X)$, then the deck transformation group of $\Pi : \widetilde{X} \to X$ is isomorphic to $\pi_1(X)/\Pi_*(\pi_1(\widetilde{X}))$.*

For a leisurely introduction to covering spaces, see Hatcher [4], Chapter 1, Section 3. For a more concise treatment from a categorical point of view, see May [11], Chapter 3.

We assume familiarity with the geometry of the hyperbolic half-plane $\mathbb{H}$ and the hyperbolic disc, which are conformally equivalent. Recall that the group $PSL(2, \mathbb{R})$ acts on $\mathbb{H}$ by orientation-preserving isometries by taking $z \in \mathbb{H}$ to $\frac{az+b}{cz+d}$.

Definition 2.2 (Fuchsian group)**.** A **Fuchsian group** is a discrete subgroup of $PSL(2, \mathbb{R})$ acting on $\mathbb{H}$.

We will also refer to Fuchsian groups as discrete subgroups of the orientation-preserving isometry group $\mathrm{Isom}^+(\mathbb{H})$ of the hyperbolic half-plane. See Katok [6] for an exposition of hyperbolic geometry and Fuchsian groups.

A topological surface is a 2-manifold – a space locally homeomorphic to $\mathbb{R}^2$. We will denote the topological surface of genus g by S_g. A *chart* on S_g is a homeomorphism $p : U \to \mathbb{C}$, where U is an open set in S_g. An *atlas* on S_g is a family $p_\alpha : U_\alpha \to \mathbb{C}$ of charts, so that the U_α cover S_g. An atlas is *conformal* if all the "change of coordinates" maps

$$\begin{tikzcd} U_\alpha \cap U_\beta \arrow[r, hook] & U_\beta \arrow[d, "p_\beta"] \\ \mathbb{C} \arrow[u, "p_\alpha^{-1}|_{U_\alpha \cap U_\beta}"] & \mathbb{C} \end{tikzcd}$$

are holomorphic. A *conformal structure* on S_g is a maximal conformal atlas, in the sense that the atlas has all holomorphically compatible charts, so that adding any new charts would make the atlas stop being conformal.

Definition 2.3 (Riemann surface)**.** A Riemann surface of genus g is a topological surface S_g equipped with a conformal structure.

For us, "Riemann surface" will mean "compact Riemann surface". We will denote a Riemann surface of genus g by Y_g. A *conformal metric* on a Riemann surface Y_g is defined in local coordinates by

$$\lambda(z)^2 dz d\bar{z},$$

where $\lambda(z) > 0$ is assumed to be C^∞ and consistent under change of coordinates. Then arc lengths and areas are the same in different local coordinates. This lets us define an *isometry* between Riemann surfaces as a continuous map that preserves arc-length. In this paper, isometries will be *conformal*, i.e. orientation-preserving.

Proposition 2.4. *Every compact Riemann surface admits a conformal metric.*

The idea of the proof of Proposition 2.4 is to take, for each $z \in Y_g$, a chart $p_z : U_z \to \mathbb{C}$ with $z \in U_z$, and a disc $D_z \subset p_z(U_z)$ with $p_z(z) \in D_z$. Since Y_g is compact, it can be covered by finitely many $p_z^{-1}(D_z)$. Smooth metrics, each positive and conformal on a single D_z and zero elsewhere, can be pulled back and summed to a positive metric on all of Y_g, giving a conformal metric. See Jost [5], Lemma 2.3.3.

Theorem 2.5 (Uniformization)**.** *For every compact Riemann surface Y_g of genus $g \geq 2$, there exists a conformal diffeomorphism $\phi : Y_g \to \mathbb{H}/F$, where F is some Fuchsian group.*

This says that every Y_g with $g \geq 2$ is conformally equivalent to a quotient of the hyperbolic plane. Up to conformal equivalence, the only conformal metrics on Y_g are the hyperbolic metrics of constant curvature, inherited from $\mathbb{H}$ quotiented by some Fuchsian group isomorphic to $\pi_1(S_g)$. See Jost [5] for standard material on Riemann surfaces.

We briefly recall some facts about orbifolds; for an exposition, see Farb and Margalit [3], Chapter 7.

Definition 2.6 (Orbifolds)**.** An orbifold is a quotient space Y_g/F for some finite group F of orientation-preserving isometries of a Riemann surface Y_g.

If F does not act freely on Y_g, the resulting covering of the quotient orbifold by Y_g is called a *ramified covering*. Every point in an orbifold has a neighborhood homeomorphic to the quotient of an open ball in $\mathbb{H}$ by a finite group of rotational isometries. The order of that finite group for a given point is the *branching number* of that point; the *ramification* at that point is one less than the branching number. A point with positive ramification is a *ramification point*.

Since we take Y_g to be compact, Y_g/F has finitely many ramification points. An orbifold is uniquely determined up to conformal equivalence by its signature $(g; r_1, r_2, \dots, r_m)$, where g is the genus of the orbifold and the r_i are the ramification at each ramification point. We denote (ambiguously) an orbifold with signature $(g; r_1, r_2, \dots, r_m)$ by $S_{g,m}$, and we will also call topological surfaces "orbifolds". Punctured orbifolds – orbifolds with their ramification points removed – will be denoted by $S'_{g,m}$.

Now we can state a few crucial theorems.

Theorem 2.7 (Gauss-Bonnet formula)**.** *Let T be a geodesic triangle in the hyperbolic plane with angles α, β, γ. Then the hyperbolic area of T is*

$$\tau/2 - \alpha - \beta - \gamma.$$

Consequently, if Y_g is a compact Riemann surface of genus g, then Area$(Y_g) = \tau(2g-2)$.

Hyperbolic area interacts naturally with quotients, so that the area of a quotient orbifold is the area of the covering space divided by the number of sheets in the covering.

Theorem 2.8 (Riemann-Hurwitz formula)**.** *Let $\phi : Y_{g_1} \to Y_{g_2}$ be a conformal map of compact Riemann surfaces. Then there is a natural number m such that for every point $p \in Y_{g_2}$ that is not a ramification point of ϕ, p has exactly m preimages under ϕ. Furthermore, letting s denote the sum of the ramification at all ramification points of Y_{g_2}, we have that*

$$2g_1 - 2 = m(2g_2 - 2) + s.$$

Theorem 2.9 (Orbifold Gauss-Bonnet formula)**.** *Let $S_{g,m}$ be a hyperbolic orbifold with signature $(g; r_1, r_2, \ldots, r_m)$. Then the hyperbolic area of $S_{g,m}$ is*

$$\tau \left(2g - 2 + \sum_{i=1}^{m} \left(1 - \frac{1}{r_m} \right) \right).$$

Proofs of these theorems appear in Jost [5] and Farb and Margalit [3].

3. Lower bounds on the maximum size of $|\mathrm{Isom}^+(Y_g)|$

3.1. **Automorphisms to isometries.**

Proposition 3.1. *For $g \geq 2$, the isometry group $Isom^+(Y_g)$ of any Riemann surface Y_g of genus g is finite.*

See Farb and Margalit [3], Chapter 7, for a proof.

For $g \geq 2$, not only is $\mathrm{Isom}^+(Y_g)$ a finite group, but also its size $|\mathrm{Isom}^+(Y_g)|$ can be *bounded* as a function of g. This allows us to define the maximum size $N(g)$ of any isometry group acting on any Riemann surface of genus g; that is,

$$N(g) = \max_{Y_g} |\mathrm{Isom}^+(Y_g)|,$$

where Y_g ranges over the set of all Riemann surfaces of genus g. We will prove two lower bounds on $N(g)$.

As an illustration, we can immediately obtain a lower bound on $N(g)$ for all g:

Example 3.2. Realize the Riemann surface Y_g of genus g as a regular hyperbolic $4g+2$-gon, where each internal angle is $\tau/(4g+2)$, with opposite sides identified. The rotations of the $4g+2$-gon through integer multiples of $\tau/(4g+2)$ give $4g+2$ isometries of Y_g. See Figure 1. (Farb and Margalit, [3], Chapter 7.)

The family of isometry groups of Y_g constructed in Example 3.2 gives that $N(g) \geq 4g+2$ for all $g \geq 2$. To obtain a tighter lower bound on $N(g)$, we will use topological methods to construct slightly bigger automorphism groups of the underlying topological space of Y_g – that is, the surface S_g. By an **automorphism** of S_g we mean a homeomorphism from S_g to itself. To get isometries of Y_g from these automorphisms of S_g, we need the Metric Symmetrization Lemma.

Lemma 3.3. *Let A be a finite group of orientation-preserving automorphisms acting on the surface S_g of genus $g \geq 2$. Then there exists a Riemann surface Y_g of genus g with a hyperbolic metric so that A acts on Y_g by isometries.*

Proof. We know (Proposition 2.4) that we can put a conformal structure and a conformal metric λ' on S_g to obtain a Riemann surface Y'_g. Now, for each automorphism $\alpha \in A$, define the pullback metric $\alpha^*(\lambda')$, where

$$\alpha^*(\lambda')(z) = \lambda'(\alpha(z)),$$

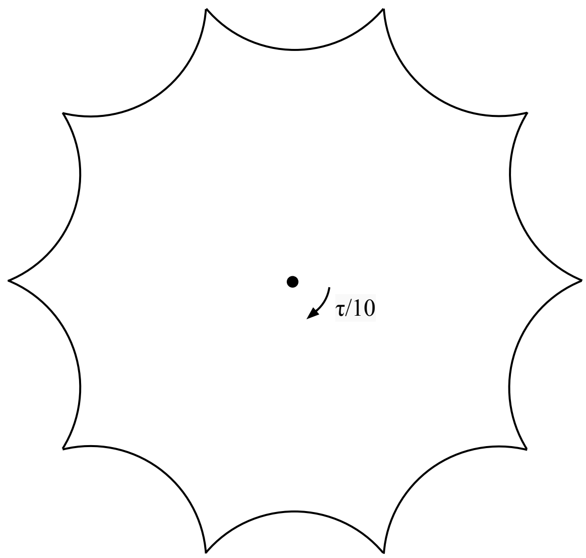

Figure 1. The 10-gon realization of Y_2 and a generator of its rotation isometry group.

for $z \in Y_g'$. Then set

$$\lambda = \sum_{\alpha \in A} \alpha^*(\lambda'),$$

so that λ is the "average" of the pullback metrics of λ' under the action of A. This definition makes sense because $|A| < \infty$.

Now A acts by isometries on the Riemann surface Y_g'', the surface S_g equipped with the average metric λ: since left multiplication by $\alpha \in A$ permutes the elements of A, for all $\beta \in A$ and $z \in Y_g''$ we have

$$\begin{aligned} \lambda(\beta(z)) &= \sum_{\alpha \in A} \alpha^*(\lambda')(\beta(z)) \\ &= \sum_{\alpha \in A} \lambda'((\alpha \circ \beta)(z)) \\ &= \sum_{\alpha \in A} \lambda'(\alpha(z)) \\ &= \lambda(z). \end{aligned}$$

To obtain a hyperbolic Riemann surface admitting the action of A, we invoke the Uniformization Theorem (Theorem 2.5). This gives us a conformal diffeomorphism $\phi : Y_g'' \to Y_g$, where Y_g is a Riemann surface of the form $\mathbb{H}/F$, for some Fuchsian group F. The surface Y_g has genus g and inherits the hyperbolic metric from $\mathbb{H}$. We can then have A act by isometry on Y_g through the uniformizing map ϕ; that is, for each $\alpha \in A$, $\phi \circ \alpha \circ \phi^{-1}$ is an isometry of Y_g, by definition of conformal diffeomorphism. Thus $\phi \circ A \circ \phi^{-1} \cong A$ acts by isometry on the hyperbolic Riemann surface g, as desired. ◫

So, with a finite automorphism group of $S_g, g \geq 2$, in hand, we can produce a hyperbolic isometry group of Y_g of the same size. If the automorphism group is orientation-preserving, then the isometry group will be conformal, as opposed to possibly being anti-conformal. Since conformal isometries of a hyperbolic Riemann surface are in particular orientation-preserving, we've shown that we can study conformal hyperbolic isometries of Y_g interchangeably with orientation-preserving automorphisms of S_g. For $g \geq 2$, conformal hyperbolic isometry groups of $Y_g = \mathbb{H}/F$ are exactly quotient groups G/F, where G is a Fuchsian group with the Fuchsian group F as a normal subgroup.

Thus, we can improve on the lower bound $N(g) \geq 4g+2$ from Example 3.2 by finding a finite automorphism group of S_g which is larger than $4g+2$. To produce such an automorphism group of S_g, we will consider S_g as a ramified covering space of an orbifold. Then we will lift an automorphism group of the quotient orbifold up to an automorphism group of the covering space; to accomplish this, we need the following proposition.

Proposition 3.4. *Let S be a surface with a (not ramified) covering $\Pi : \widetilde{S} \to S$, and let $\alpha : S \to S$ be an automorphism of S. Fix a basepoint p_0 in S, and define $q_0 = \alpha^{-1}(p_0)$. Then fix $\widetilde{p}_0 \in \Pi^{-1}(p_0)$ and $\widetilde{q}_0 \in \Pi^{-1}(q_0)$. Suppose that α is such that*

$$\alpha_*(\Pi_*(\pi_1(\widetilde{S}, \widetilde{q}_0))) = \Pi_*(\pi_1(\widetilde{S}, \widetilde{p}_0)).$$

Then there exists a unique automorphism $\widetilde{\alpha} : \widetilde{S} \to \widetilde{S}$ such that the following diagram commutes:

$$\begin{tikzcd}
(\widetilde{S}, \widetilde{q}_0) \arrow[r, "\widetilde{\alpha}"] \arrow[d, "\Pi"'] & (\widetilde{S}, \widetilde{p}_0) \arrow[d, "\Pi"] \\
(S, q_0) \arrow[r, "\alpha"'] & (S, p_0)
\end{tikzcd}$$

When this is the case, we say that $\widetilde{\alpha}$ is the **lift automorphism** *of α relative to the basepoints q_0 and p_0.*

Proof. Except for uniqueness, this is just Hatcher [4], Proposition 1.37.

Since the lift automorphism $\widetilde{\alpha}$ must send $\widetilde{q}_0$ to $\widetilde{p}_0$, uniqueness follows from the unique lifting property.

◫

The choice of $\widetilde{q}_0 \in \Pi^{-1}(\widetilde{p}_0)$ corresponds to the choice of a deck transformation. That is, the lift automorphism $\widetilde{\alpha} : \widetilde{S} \to \widetilde{S}$ of $\alpha : S \to S$ is unique up to a deck transformation, if we ignore basepoints. This is because $\iota \circ \widetilde{\alpha}$ and $\widetilde{\alpha} \circ \iota$ are also lifts of α, for any deck transformation $\iota : \widetilde{S} \to \widetilde{S}$. Choosing the basepoints $\widetilde{p}_0$ and $\widetilde{q}_0$ determines a unique covering isomorphism taking $\widetilde{q}_0$ to $\widetilde{p}_0$; if we change our minds and pick $\widetilde{q}_0'$ instead, we get the new covering isomorphism from the old one by precomposing with the unique deck transformation that takes $\widetilde{q}_0'$ to $\widetilde{q}_0$.

This suggests the following corollary of Proposition 3.4, which we will use to actually produce lifted automorphism groups of covering spaces.

Corollary 3.5. *Let A be an automorphism group of S, a surface with a normal covering $\Pi : \widetilde{S} \to S$. Let C_Π be the deck transformation group of $\Pi : \widetilde{S} \to S$. Suppose each $\alpha \in A$ satisfies the criterion in Proposition 3.4, so that we can lift A to a fixed set $\widetilde{A}$ which contains a single (arbitrarily chosen) lift of each automorphism of $\widetilde{S}$; denote by $\widetilde{\alpha}$ the unique lift of α in $\widetilde{A}$. Then the set of automorphisms $\widetilde{A}C_\Pi$ forms an automorphism group of $\widetilde{S}$ of size $|\widetilde{A}||C_\Pi| = |A||\pi_1(S)/\Pi_*(\pi_1(\widetilde{S}))|$.*

Proof. The set $\widetilde{A}$ consists of $|A|$ elements, each of which is a lift of a distinct automorphism in A. To show that $\widetilde{A}C_\Pi$ is closed under composition, take any two elements $\widetilde{\alpha} \circ \iota$ and $\widetilde{\beta} \circ \zeta$ in $\widetilde{A}C_\Pi$, with $\alpha, \beta \in A$ and $\iota, \zeta \in C_\Pi$, and where $\widetilde{\alpha}$ and $\widetilde{\beta}$ are the two unique elements in $\widetilde{A}$ that lift α and β, respectively. Therefore $\widetilde{\alpha} \circ \iota$ lifts α and $\widetilde{\beta} \circ \zeta$ lifts β. This implies that $\widetilde{\beta} \circ \zeta \circ \widetilde{\alpha} \circ \iota$ is *one* lift of $\beta \circ \alpha$, if not the exact lift $\widetilde{\beta \circ \alpha}$ of $\beta \circ \alpha$ that was arbitrarily chosen to be in the set $\widetilde{A}$. Then, by the discussion above, for some possibly nontrivial $\eta \in C_\Pi$ we have that

$$\widetilde{\beta} \circ \zeta \circ \widetilde{\alpha} \circ \iota \circ \eta = \widetilde{\beta \circ \alpha}.$$

The following diagram illustrates the situation:

$$\begin{tikzcd}
\widetilde{S} \arrow[r, "\eta"] \arrow[d, "\Pi"'] \arrow[rrr, bend left, "\widetilde{\beta\circ\alpha}"] & \widetilde{S} \arrow[r, "\widetilde{\alpha}\circ\iota"] \arrow[d, "\Pi"'] & \widetilde{S} \arrow[r, "\widetilde{\beta}\circ\zeta"] \arrow[d, "\Pi"'] & \widetilde{S} \arrow[d, "\Pi"] \\
S \arrow[r, equal] & S \arrow[r, "\alpha"'] & S \arrow[r, "\beta"'] & S
\end{tikzcd}$$

Thus we have

$$\widetilde{\beta} \circ \zeta \circ \widetilde{\alpha} \circ \iota = \widetilde{\beta \circ \alpha} \circ \eta^{-1} \in \widetilde{A}C_\Pi,$$

so that $\widetilde{A}C_\Pi$ is closed under composition. The identity of $\widetilde{A}C_\Pi$ is $\widetilde{\mathbb{1}}_S \circ \widetilde{\mathbb{1}}_S^{-1} = \mathbb{1}_{\widetilde{S}}$, where the lift $\widetilde{\mathbb{1}}_S$ of $\mathbb{1}_S$ is a possibly nontrivial deck transformation. Associativity and the existence of inverses is straightforward. Therefore, $\widetilde{A}C_\Pi$ is a group of automorphisms of $\widetilde{S}$.

Finally, suppose $\widetilde{\beta} \circ \zeta = \widetilde{\alpha} \circ \iota$ for some $\widetilde{\beta}, \widetilde{\alpha} \in \widetilde{A}$ and $\zeta, \iota \in C_\Pi$. Then $\widetilde{\beta} = \widetilde{\alpha} \circ \iota \circ \zeta^{-1}$, which says that $\widetilde{\alpha}$ and $\widetilde{\beta}$ lift the same automorphism of S. If $\widetilde{\alpha} \neq \widetilde{\beta}$, this contradicts the construction of $\widetilde{A}$. If $\widetilde{\alpha} = \widetilde{\beta}$, then $\iota = \zeta$. Thus each distinct element of the set $\widetilde{A}C_\Pi$ gives a distinct element of the group $\widetilde{A}C_\Pi$. Therefore, by the construction of $\widetilde{A}$ and by the Galois correspondence, we have that

$$|\widetilde{A}C_\Pi| = |\widetilde{A}||C_\Pi| = |A||\pi_1(S)/\Pi_*(\pi_1(\widetilde{S}))|.$$

$\square$

3.2. **Lower bounds on $N(g)$.** It is easy to find huge automorphism groups of the surface S_g. For example, we can pick any small closed disc in S_g, and consider all automorphisms of that disc which fix the boundary. These automorphisms of the disc, extended to S_g by acting as the identity elsewhere, give an uncountably large group of automorphisms of S_g. However, by Proposition 3.1, we know that there is no Riemann surface structure on S_g that admits this huge group as a group of isometries; the Metric Symmetrization Lemma does not apply because averaging a metric over an infinite automorphism group will not even give a continuous function. Instead, we want large *finite* groups of automorphisms of S_g.

With this in mind, we are ready to improve on the $N(g) \geq 4g+2$ bound by constructing bigger groups of isometries for Y_g.

Theorem 3.6. *For every $g \geq 2$, $N(g) \geq 8(g+1)$. That is, there exists a Riemann surface Y_g of genus g which has an isometry group of size $8(g+1)$.*

Proof. The strategy of the proof is to construct a punctured sphere S_0' covered by the punctured surface S_K'. The punctured sphere will have a large automorphism group, which we will lift to the covering space S_K' using Proposition 3.4. The lifted automorphisms, together with the deck transformations of the covering $S_K' \to S_0'$, will produce a large automorphism group of S_K', by Corollary 3.5. Filling in the punctures will give us a ramified covering of S_0 by S_K. Once we use the Riemann-Hurwitz formula to calculate the genus g of S_K, Lemma 3.3 will produce a large isometry group acting on Y_g.

Step 1. We start by taking $S_{0,2g+2}'$, the sphere punctured at the $2g+2$ vertices of the regular $2g+2$-gon. This surface admits the dihedral group D_{2g+2} of order $4g+4$ as automorphisms, induced by the dihedral group acting on the inscribed $2g+2$-gon – that is, the group generated by the cyclic group of rotations of order $2g+2$, along with the rotation through $\tau/2$ about a diameter of the $2g+2$-gon. Figure 2 depicts the case for $g=2$. Note that D_{2g+2} permutes the punctures of $S_{0,2g+2}'$. (To talk about the "regular $2g+2$-gon", we view $S_{0,2g+2}'$ as embedded in $\mathbb{R}^3$.)

The fundamental group $\pi_1(S_{0,2g+2}', p_0)$ at basepoint p_0 has the presentation

$$\langle x_1, x_2, \ldots, x_{2g+2} \mid x_1 x_2 \ldots x_{2g+2} = 1 \rangle,$$

where x_i is the loop going (say) counterclockwise around the i-th puncture. Consider the normal subgroup $K \lhd \pi_1(S_{0,2g+2}', p_0)$ generated by products of two generators (or inverses of generators) of the fundamental group; that is, $K = \langle x_i^{\pm 1} x_j^{\pm 1} \rangle$. By the Galois correspondence between connected coverings and the subgroups of the fundamental group, we may take the connected covering space S_K' of $S_{0,2g+2}'$ corresponding to the normal subgroup K. Since $S_{0,2g+2}'$ is a surface, so is S_K'.

Denote the covering map by $\Pi : S_K' \to S_{0,2g+2}'$. Since K is a subgroup of $\pi_1(S_{0,2g+2}', p_0)$ of index 2, S_K' is two-sheeted; each point $p \in S_{0,2g+2}'$ has a corresponding set $\Pi^{-1}(p)$ of two preimages. S_K' is a normal covering, so the deck transformation group of S_K' is

$$\pi_1(S_{0,2g+2}', p_0)/K \cong \mathbb{Z}/2\mathbb{Z}.$$

Fixing an element $\widetilde{p}_0 \in \Pi^{-1}(p_0)$, we have that $\Pi_*(\pi_1(S_K', \widetilde{p}_0)) = K$.

Step 2. As in Proposition 3.4, for a given element α in D_{2g+2}, fix $q_0 = \alpha^{-1}(p_0)$ and $\widetilde{q}_0 \in \Pi^{-1}(q_0)$. We want to apply Proposition 3.4 to lift D_{2g+2} to automorphisms acting on $S_{0,2g+2}'$, so we have to check that each element α in D_{2g+2} takes $\Pi_*(\pi_1(S_K', \widetilde{q}_0))$ to exactly $\Pi_*(\pi_1(S_K', \widetilde{p}_0))$.

Take any loop $\gamma \in \Pi_*(\pi_1(S_K', \widetilde{q}_0))$. Then γ is the product of an even number of generators (inverses of generators) of $\pi_1(S_{0,2g+2}', q_0)$. As above, a generator is a single loop around one puncture. By inspection, α sends a generator (inverse of a generator) to a generator (inverse of a generator). See Figure 3. Thus, α sends γ to a product of an even number of generators of $\pi_1(S_{0,2g+2}', p_0)$; that is, α sends γ to an element of $K = \Pi_*(\pi_1(S_K', \widetilde{p}_0))$.

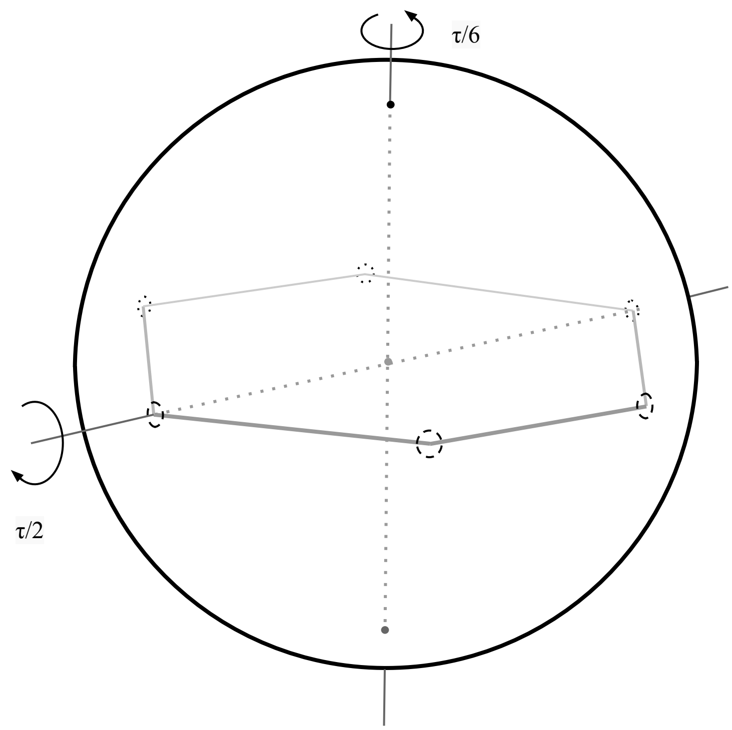

FIGURE 2. The punctured sphere $S'_{0,6}$ and generators of the automorphism group D_6.

So α takes $\Pi_*(\pi_1(S'_K, \widetilde{q}_0))$ into $\Pi_*(\pi_1(S'_K, \widetilde{p}_0))$. Running the same reasoning in the other direction, we see that α takes $\Pi_*(\pi_1(S'_K, \widetilde{q}_0))$ to exactly $\Pi_*(\pi_1(S'_K, \widetilde{p}_0))$.

Step 3. By Corollary 3.5, S'_K has an automorphism group G of size

$$|D_{2g+2}||\mathbb{Z}/2\mathbb{Z}| = (4g+4)(2) = 8(g+1).$$

Since the action of D_{2g+2} on $S'_{0,2g+2}$ (embedded in $\mathbb{R}^3$) is isometric and orientation-preserving, we can put a conformal structure on $S'_{0,2g+2}$ and have D_{2g+2} act on the resulting (non-compact) Riemann surface conformally. The conformal structure on $S'_{0,2g+2}$ can be pulled back to S'_K, and then the automorphism group G acts conformally on the resulting Riemann surface (see Jost [5], Chapter 2).

Step 4. The punctured sphere $S'_{0,2g+2}$ is the quotient space $S'_K/(\mathbb{Z}/2\mathbb{Z})$. We can repair the punctures in $S'_{0,2g+2}$ by adding $2g+2$ points back to $S'_{0,2g+2}$ in the natural way, one for each puncture, and then adding $2g+2$ corresponding points to the covering S'_K. The neighborhoods of a new point p in $S_{0,2g+2}$ will be precisely the neighborhoods of the point before it was originally removed, and the neighborhoods in S_K of the single point $\Pi^{-1}(p)$ are precisely the preimages under Π of the neighborhoods of p in $S_{0,2g+2}$. Then D_{2g+2} acting on $S_{0,2g+2}$ will permute the repaired punctures, and so the lifts of D_{2g+2} will permute the preimages in S_K of the repaired punctures.

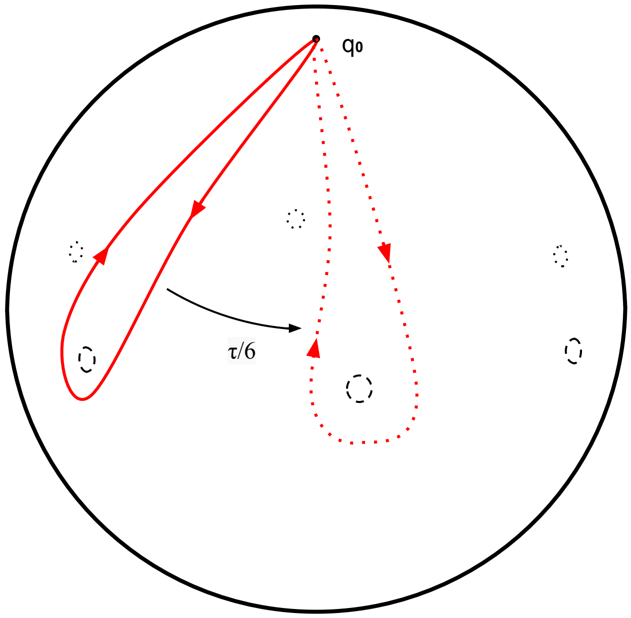

FIGURE 3. The image of x_i under the rotation through $\tau/6$, in the case $g = 2$. In this figure the basepoint is fixed by the automorphism, so p_0 and q_0 coincide.

Step 5. We put a conformal structure on S_K as in step 3. The map Π is now a ramified covering, and $S_{0,2g+2}$ is the quotient orbifold $S_K/(\mathbb{Z}/2\mathbb{Z})$, with signature

$$(0; \underbrace{2, 2, \ldots, 2}_{2g+2 \text{ times}}).$$

This can be visualized as the result of skewering S_g lengthwise, and taking the quotient by the rotation through $\tau/2$, fixing $2g+2$ points. This automorphism is also known as a hyperelliptic involution of a hyperelliptic curve of genus g. The covering Π has a branching number of 2 at each of the $2g+2$ points of the regular $2g+2$-gon, since there are two distinct lifts of a path from a basepoint $p_0 \in S'_{0,2g+2}$ to one of the ramification points – one lifted path for each lift $\Pi^{-1}(p_0)$.

Thus the ramification at each ramified point is 1, and the total order of ramification is $2g+2$. Let g_K denote the genus of S_K. Then by the Riemann-Hurwitz formula, we have that

$$2g_K - 2 = 2(-2) + (1)(2g+2),$$

which implies $g_K = g$.

Step 6. By Lemma 3.3 (Metric Symmetrization), the hyperbolic Riemann surface Y_g admits an isometry group of order $8(g+1)$, for all $g \geq 2$.

$\square$

The construction used in the proof of Theorem 3.6 can also be carried out by choosing a homomorphism $\phi : \pi_1(S'_{0,2g+2}) \to G$, where G is some group, and then taking the normal covering S_K corresponding to $K = \ker \phi$. By covering space theory, we then have that the deck transformation group of S_K is $\pi_1(S'_{0,2g+2})/K = G$. In the above construction we started with an automorphism group downstairs that grew with g, and lifted it upstairs, to the covering surface S_K. Then we got a slightly bigger automorphism group on S_K by multiplying in the two deck transformations – the kernel K was always index 2 in $\pi_1(S'_0)$, so that $|G| = 2$.

A similar construction proves the next theorem. This time, however, the automorphism group of the orbifold will be fixed for all g. Instead, the deck transformation group will grow with g; so we want a homomorphism $\phi : \pi_1(S'_{0,8}) \to G$ such that $|G|$ grows with g.

Theorem 3.7. *For every $g \geq 2$, if g is divisible by 3, then $N(g) \geq 8(g+3)$. That is, there exists a Riemann surface Y_g of genus g which has an isometry group of size $8(g+3)$.*

Proof. We follow the same strategy as in the proof of Theorem 3.6.

Step 1. We take $S'_{0,8}$, the sphere punctured at the 8 vertices of the cube, again viewed as embedded in $\mathbb{R}^3$. This surface admits the octahedral group O_{24} of order 24 as automorphisms, induced by the octahedral group acting on the inscribed cube. The group O_{24} consists of the the rotations of order 2 fixing a pair of edges, the rotations of order 3 fixing a pair of vertices, and the rotations of order 4 fixing a pair of faces (see Figure 4).

The octahedral group permutes the punctures of $S'_{0,8}$. Consider the set $\{1, 3, 5, 7\}$ of vertices of the cube, no two of which are adjacent (see Figure 5). The sets $M = \{1, 3, 5, 7\}$ and $W = \{2, 4, 6, 8\}$ are called **sets of imprimitivity**, meaning that O_{24} acting on the set of vertices does not mix the two sets together. That is, every element α of O_{24} either sends M to itself and W to itself (e.g. the rotations fixing a pair of vertices), or sends M to W and W to M (e.g. the rotations fixing a pair of edges).

The fundamental group $\pi_1(S'_{0,8}, p_0)$ has the presentation

$$\langle x_1, x_2, \ldots, x_8 \mid x_1 x_2 \ldots x_8 = 1 \rangle,$$

where a generator is a single loop counterclockwise around one puncture. Let X_M be the set of generators that loop around a point in M, and let X_W be the set of generators that loop around a point in W. The fact that M and W are sets of imprimitivity under the action of O_{24} implies that O_{24} also acts nicely on the sets X_M and X_W. That is, every $\alpha \in O_{24}$ either sends X_M to X_W and X_W to X_M, or sends each set to itself.

Now, define a homomorphism $\phi : \pi_1(S'_{0,8}, p_0) \to G$ by setting

$$\phi(x_i) = \begin{cases} 1 \mod n & : x_i \in X_M \\ -1 \mod n & : x_i \in X_W \end{cases}$$

and extending to all of $\pi_1(S'_{0,8}, p_0)$. Here G is the cyclic group of order n; that is, $G = \langle \phi(x_i) \mid n \cdot \phi(x_i) = 1 \rangle$, for any one generator x_i. The homomorphism ϕ is well-defined because

$$\phi(x_1 x_2 \ldots x_8) = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 = 0 = \phi(1).$$

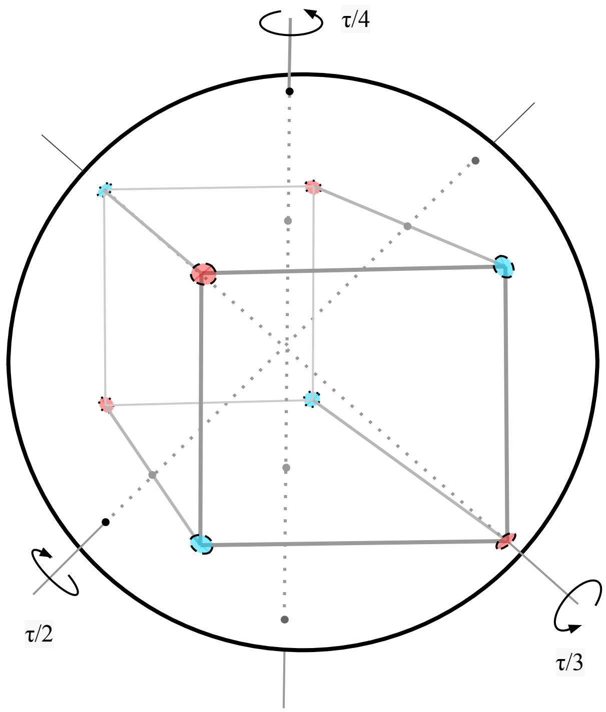

FIGURE 4. The octahedral group O_{24} acting on the sphere with 8 punctures.

Name $K = \ker\phi$, so that K is a normal subgroup of $\pi_1(S'_{0,8}, p_0)$. By the Galois correspondence, we may take the connected covering space S'_K of $S'_{0,8}$ corresponding to the normal subgroup K. The deck transformation group of S'_K is (isomorphic to) $G \cong \pi_1(S'_{0,8}, p_0)/K$.

Step 2. We want to apply Proposition 3.4 to lift O_{24} to automorphisms of S'_K, so we must check that every element α in O_{24} sends K to itself. This is easier and more enjoyable to see directly by visualization than to prove, but we include a detailed argument for completeness.

Any element of $\pi_1(S'_{0,8}, p_0)$ can be written as a product $y_1 y_2 \dots y_k$, where each y_j is of the form $(x_{i(j)})^{r_j}, r_j \in \mathbb{Z}$. Define

$$M(y_1 y_2 \dots y_k) = \sum_{j : x_{i(j)} \in X_M} r_j$$

$$W(y_1 y_2 \dots y_k) = \sum_{j : x_{i(j)} \in X_W} r_j$$

Then

$$\phi(y_1 y_2 \dots y_k) = M(y_1 y_2 \dots y_k) - W(y_1 y_2 \dots y_k) \mod n.$$

Thus, by definition, the normal subgroup K consists of elements $y_1 y_2 \dots y_k$ such that

$$M(y_1 y_2 \dots y_k) - W(y_1 y_2 \dots y_k) \equiv 0 \mod n.$$

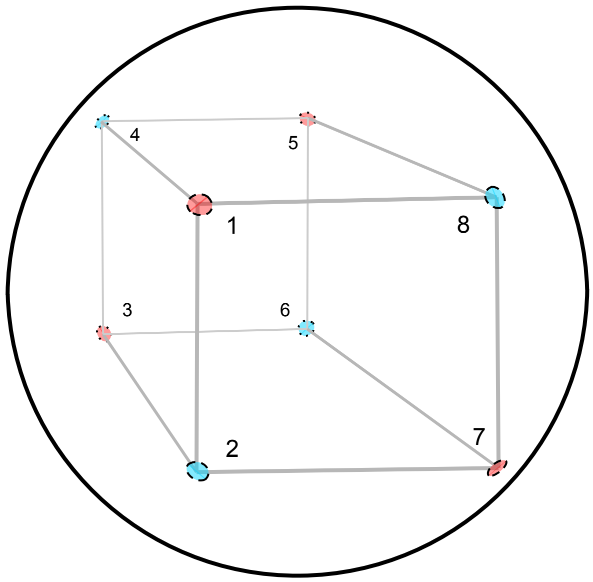

FIGURE 5. The sets of imprimitivity under the action of O_{24}; M is in red, W is in blue.

Suppose $\alpha \in O_{24}$ takes X_M to itself and X_W to itself. Then for any loop $y_1 y_2 \dots y_k$, we have

$$M(\alpha(y_1 y_2 \dots y_k)) - W(\alpha(y_1 y_2 \dots y_k)) = M(y_1 y_2 \dots y_k) - W(y_1 y_2 \dots y_k),$$

so that if $y_1 y_2 \dots y_k$ satisfies the condition above for membership in K, then so does $\alpha(y_1 y_2 \dots y_k)$. The same reasoning applies if α swaps X_M with X_W; in this case, we have

$$M(\alpha(y_1 y_2 \dots y_k)) - W(\alpha(y_1 y_2 \dots y_k)) = W(y_1 y_2 \dots y_k) - M(y_1 y_2 \dots y_k),$$

so that, again, $y_1 y_2 \dots y_k \in K$ implies $\alpha(y_1 y_2 \dots y_k) \in K$. By the construction of X_M and X_W, these two cases account for all the automorphisms in O_{24}. Therefore, every $\alpha \in O_{24}$ takes K to itself.

Step 3. By Corollary 3.5, S'_K has an automorphism group of size

$$|O_{24}||G| = 24n.$$

Step 4. Again, we can repair the punctures in the natural way and put a Riemann surface structure on our manifolds to obtain a ramified covering $\Pi : S_K \to S_{0,8}$, where $S_{0,8}$ is the orbifold with signature $(0; n, n, n, n, n, n, n, n)$.

Step 5. The covering $\Pi : S_K \to S_{0,8}$ has a branching number of n at each of the 8 points of the cube, since there are n distinct lifts of a path from a basepoint $p_0 \in S'_{0,8}$ to one of the ramification points – one lifted path for each lift $\Pi^{-1}(p_0)$. Therefore the ramification of Π at each ramified point is $n-1$, and the total order of ramification is $8(n-1)$. Let g denote the genus of S_K. Then by the Riemann-Hurwitz formula, we have that

$$2g-2 = n(-2) + 8(n-1),$$

which implies $g = 3n-3$. Thus S_K, the surface of genus g, has an automorphism group of order

$$24n = 24\left(\frac{g+3}{3}\right) = 8(g+3).$$

Step 6. Allowing n in the above construction to range over integers greater than 1, by Lemma 3.3 (Metric Symmetrization), the hyperbolic Riemann surface Y_g admits an isometry group of order $8(g+3)$, for all $g \geq 2$ such that g is a multiple of 3.

□

Theorem 3.6 and Theorem 3.7 give lower bounds on $N(g)$ for all $g \geq 2$ and for $g \geq 2$ divisible by 3, respectively. In fact, these bounds are sharp, in the sense that no tighter bounds hold in general.

Theorem 3.8. *For infinitely many $g \geq 2$, $N(g) = 8(g+1)$; that is, there is no hyperbolic Riemann surface of genus g that admits a group of isometries of order larger than $8(g+1)$. For infinitely many $g \geq 2$ such that g is divisible by 3, $N(g) = 8(g+3)$; that is, there is no hyperbolic Riemann surface of genus g that admits a group of isometries of order larger than $8(g+3)$.*

This theorem is Theorem 3 and Theorem 4 in Accola [1]. Sharpness is proved for each bound by directly constructing an infinite family of surfaces Y_g of genus g for which all possible isometry groups larger than $8(g+1)$ or $8(g+3)$, acting on Y_g, can be ruled out using group theoretic computations and the Riemann-Hurwitz formula.

3.3. **Dodecahedral symmetry.** It is natural to wonder whether the construction used in the proofs of Theorem 3.6 and Theorem 3.7 can be applied to the isometries of the sphere induced by the symmetry group of the regular dodecahedron, or equivalently, of the icosahedron. There are hints of the requisite symmetry in the corresponding punctured sphere.

To see this symmetry, take $S'_{0,30}$, the sphere punctured at the points fixed by the rotations of order two that fix a pair of edges, so that there is one puncture for each of the 30 edges of the dodecahedron. We can inscribe five cubes in the sphere so that they share all their vertices with the vertices of the dodecahedron. There are then five sets of imprimitivity of punctures, each of which consists of the six punctures opposite the faces of one of the inscribed cubes. Elements of the dodecahedral group will permute these sets but will not mix them. (See Figure 6.)

It would be nice if we could find a family of homomorphisms from $\pi_1(S'_{0,30})$ to a family of groups that grows with the genus of the resulting cover; then we might find a tighter lower bound on $N(g)$ for a subset of $g \geq 2$. The construction lifting the octahedral group O_{24} works because although the homomorphism $\phi : \pi_1(S'_{0,8}) \to G$ must send $x_1 x_2 \dots x_8$ to the identity in G, the order of G can still grow. We accomplished this by having the images of generators of $\pi_1(S'_{0,8})$ cancel in G, in such a way that the entire kernel was preserved under the action of O_{24}; this was made possible by the sets of imprimitivity of punctures.

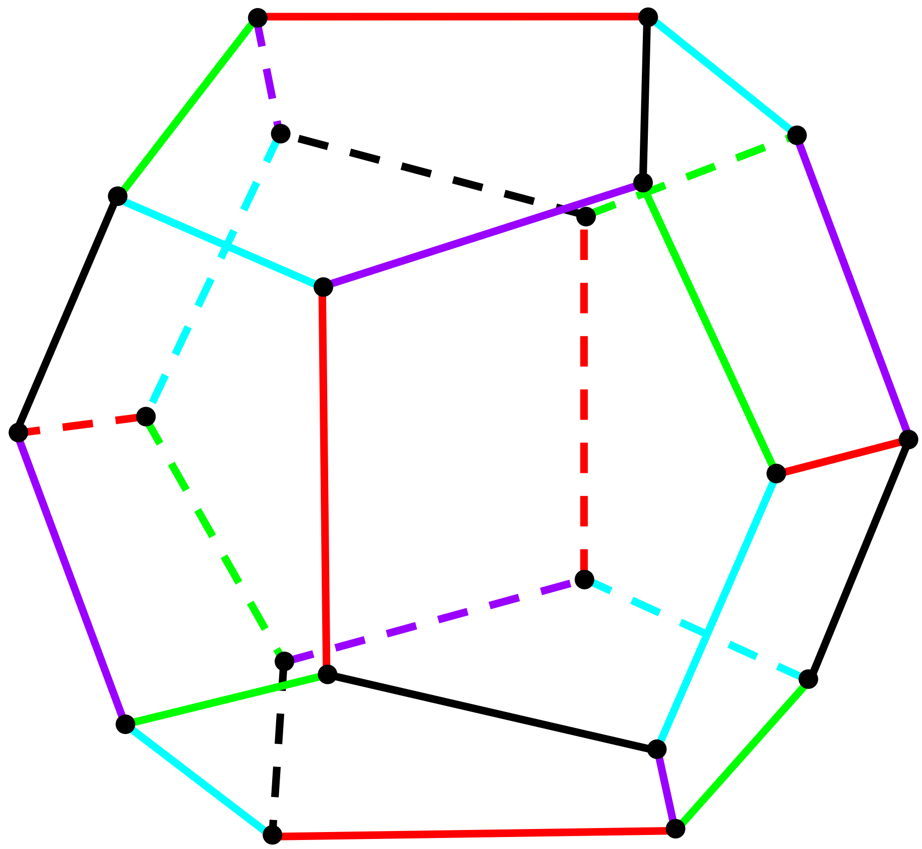

FIGURE 6. The sets of imprimitivity of edges under the action of the dodecahedral group, each set in a different color. The punctures of $S'_{0,30}$ are opposite the midpoints of each edge.

By analogy, we might hope to use the sets of imprimitivity described above for the dodecahedral symmetry of $S'_{0,30}$ to insure that (a) the element $x_1 x_2 \ldots x_{30}$ is sent to the identity in the image groups, and (b) the kernel of each homomorphism is preserved under the action of the dodecahedral automorphism group. Unfortunately, there does not appear to be a straightforward way to construct such a family of homomorphisms that interacts nicely with the dodecahedral automorphism group A_5. Indeed, no such homomorphism exists with a cyclic group of order greater than 30 as the target group.

Proposition 3.9. *Suppose that $\phi : \pi_1(S'_{0,30}) \to G, 30 < |G| < \infty$, is a surjective group homomorphism such that $\ker(\phi)$ is preserved by the action of the dodecahedral group A_5 (condition (b)). Suppose further that for a given set of imprimitivity, each generator of $\pi_1(S'_{0,30})$ circling a point in the given set is taken by ϕ to the same element of G. Then G cannot be cyclic.*

Proof. Suppose for contradiction that G is cyclic of order $n > 30$. We use additive notation for $G \cong \mathbb{Z}/n\mathbb{Z}$. Denote the five sets of imprimitivity by I_1, I_2, I_3, I_4, I_5, and denote the generators around points in I_j by $x_{j+5m}, 1 \leq m \leq 6$. Denote by $a_1, a_2, a_3, a_4, a_5 \in G$ the five images under ϕ of the generators of $\pi_1(S'_{0,30})$, one for each of the five sets of imprimitivity, so that $a_i = \phi(x_{i+5m})$.

The order in G of all the a_i must be the same. Indeed, suppose that $ka_i \equiv 0 \mod n$, for some $0 < k < n$. Then x^k_{i+5m} is in the kernel of ϕ. For each $1 \leq j \leq 5$, there is an automorphism that takes I_i to I_j (see Figure 6; the rotations of order 5 that fix a pair of faces of the inscribed dodecahedron suffice). By condition (b), $\ker(\phi)$ is preserved by each of these automorphisms, so we have in particular that x^k_{j+5m} is in the kernel of ϕ for each $1 \leq j \leq 5$. Thus we have $a^k_j = \phi(x^k_{j+5m}) = 0$, so that the order of a_j is less than or equal to k. Therefore the orders of the a_i are all less than or equal to each other, and so they are all equal.

Since ϕ is surjective, each a_i generates G. So without loss of generality, assume $a_1 = 1$. Then for any of the a_i, we have

$$-a_i \cdot 1 + 1 \cdot a_i \equiv 0 \mod n \tag{3.10}$$
$$-a_i \cdot a_i + 1 \cdot 1 \equiv 0 \mod n \tag{3.11}$$
$$a_i^2 \equiv 1 \mod n. \tag{3.12}$$

Here we are viewing G as a $\mathbb{Z}$ module, so that e.g. "$-a_i \cdot 1$" means "the sum of a_i copies of $-1 \in G$", and also as the ring $\mathbb{Z}/n\mathbb{Z}$ in Equation 3.12. The second equality holds by condition (b) and because for each $1 \leq i \leq 5$ and $1 \leq j \leq 5$, there is an automorphism that takes I_i to I_j and I_j to I_i. See Figure 6; the rotations of order 2 that fix a pair of edges of the inscribed dodecahedron suffice.

In fact, A_5 acts as the alternating group on the five sets of imprimitivity (hence the notation); the rotations fixing a pair of vertices are the 3-cycles, the rotations fixing a pair of faces are the 5-cycles, and the rotations fixing a pair of edges are the disjoint pairs of transpositions. This allows us to derive, for any a_i, a_j,

$$(-a_i - a_j) \cdot 1 + 1 \cdot a_i + 1 \cdot a_j \equiv 0 \mod n \tag{3.13}$$
$$(-a_i - a_j) \cdot a_i + 1 \cdot 1 + 1 \cdot a_j \equiv 0 \mod n \tag{3.14}$$
$$-a_i^2 - a_j a_i + 1 + a_j \equiv 0 \mod n \tag{3.15}$$
$$a_j \equiv a_j a_i \mod n. \tag{3.16}$$

Equation 3.14 follows from condition (b) (A_5 preserves $\ker(\phi)$) using the automorphism that switches I_1 and I_2 while fixing I_3, corresponding to the even permutation (12)(45). Equation 3.16 follows by substitution with Equation 3.12.

By symmetry, we also have

$$a_i \equiv a_i a_j \mod n. \tag{3.17}$$

Since the ring $\mathbb{Z}/n\mathbb{Z}$ is commutative, Equations 3.16 and 3.17 give that $a_i = a_j$. Therefore, all of the a_i are equal to $1 \in G$.

By condition (a), the well-definedness of ϕ, we have $x_1 x_2 \ldots x_{30} = 1 \in \pi_1(S'_{0,30})$, so

$$\begin{aligned} \phi(x_1 x_2 \ldots x_{30}) &\equiv \phi(1) \mod n \\ 1 + 1 + \cdots + 1 &\equiv 0 \mod n \\ 30 &\equiv 0 \mod n. \end{aligned}$$

In other words, 30 is divisible by $n = |G|$, contradicting that $|G| > 30$.

$\square$

Proposition 3.9 implies that there is no infinite family of homomorphisms to cyclic groups respecting the sets of imprimitivity I_i, although we haven't ruled out the existence of an infinite family of more complicated homomorphisms that would allow us to lift the dodecahedral group. Such homomorphisms would each have to satisfy conditions (a) and (b) given above, and would also need to have the size n of the target group G grow linearly with the order k of $\phi(x_i)$. Otherwise, the Riemann-Hurwitz formula

$$2g - 2 = n(-2) + 30(k - 1)$$

would give a contradiction.

4. Upper bound on the size of $|\mathrm{Isom}^+(Y_g)|$

In Section 3 we proved lower bounds on $N(g)$. The proofs relied on the fact that isometries of a hyperbolic Riemann surface can be obtained by finding orientation-preserving automorphisms of the underlying topological space. The Riemann surface structure was only used to calculate the genus of the covering spaces we constructed for the two different orbifolds of genus 0. To prove an upper bound on the size of the isometry groups of Riemann surfaces, we will make more use of the conformal structure and the hyperbolic metric on the surfaces. As we saw in Section 3, bounding the size of hyperbolic isometry groups is sufficient to bound $N(g)$.

4.1. **Upper bound on** $N(g)$**.**

From the facts in Section 2, we know that if Y_g is a hyperbolic Riemann surface of genus g and a group G acts on Y_g by isometries, then Y_g/G is a hyperbolic orbifold with area $\mathrm{Area}(Y_g)/|G|$. By the Gauss-Bonnet formula (Theorem 2.7), we have that $\mathrm{Area}(Y_g) = \tau(2g - 2)$. Thus the quotient space Y_g/G has area

$$\mathrm{Area}(Y_g/G) = \mathrm{Area}(Y_g)/|G| = \tau(2g - 2)/|G|.$$

Equivalently, $|G| = \tau(2g - 2)/\mathrm{Area}(Y_g/G)$, so that if we can lower bound the area $\mathrm{Area}(Y_g/G)$ of the quotient orbifold, we will have an upper bound on $|G|$. With this motivation, we prove the existence of a unique smallest hyperbolic orbifold by repeatedly applying the orbifold Gauss-Bonnet formula.

Lemma 4.1. *The hyperbolic orbifold with signature* $(0; 2, 3, 7)$ *has area* $\tau/42$*. All other hyperbolic orbifolds have area greater than* $\tau/42$*.*

Proof. By the orbifold Gauss-Bonnet formula (Theorem 2.9), the signature $(g; r_1, r_2, \ldots, r_m)$ of an orbifold $S_{g,m}$ completely determines its area:

$$\mathrm{Area}(S_{g,m})/\tau = 2g - 2 + \sum_{i=1}^{m} \left(1 - \frac{1}{r_m}\right).$$

Denote the quantity $\sum_{i=1}^{m}(1 - \frac{1}{r_m})$, the **total ramification**, by v. Note that v is always non-negative.

Now, assume that $S_{g,m}$ is a hyperbolic orbifold with signature $(g; r_1, r_2, \ldots, r_m)$ such that

$$\mathrm{Area}(S_{g,m})/\tau \leq 1/42.$$

Case 1: $g \geq 2$. Since $v \geq 0$ and $2g-2 \geq 2$, Gauss-Bonnet gives us $\text{Area}(S_{g,m})/\tau \geq 2$, a contradiction.

Case 2: $g = 1$. If $m = 0$, i.e. $S_{g,m}$ has no ramification points, then $S_{g,m}$ is not hyperbolic, as $S_{1,0}$ is just the torus. But each ramification point has branching number at least 2 (by definition), and so contributes at least $1 - 1/2 = 1/2$ to $v = \sum_{i=1}^{m}(1 - \frac{1}{r_m})$. Then $v \geq 1/2$, implying that $\text{Area}(S_{g,m})/\tau \geq 2 \cdot 1 - 2 + 1/2 > 1/42$, a contradiction.

Case 3a: $g = 0$, $m \geq 5$. We have $v \geq 5 \cdot 1/2$, so that $\text{Area}(S_{g,m})/\tau \geq -2 + 5/2 > 1/42$, a contradiction.

Case 3b: $g = 0$, $m = 4$. If the signature of $S_{g,m}$ is $(0; 2, 2, 2, 2)$ then $v = 4 \cdot 1/2$, so that $\text{Area}(S_{g,m})/\tau = -2 + 2 = 0$, which is ridiculous. However, if any of the four ramification points has branching number greater than 2, then it will contribute at least $1 - 1/3$ to v. Then $\text{Area}(S_{g,m})/\tau \geq -2 + 3/2 + 2/3 = 1/6 > 1/42$, a contradiction.

Case 3d: $g = 0$, $m \leq 2$. Each ramification point contributes less than 1 to v, so $v < 2$. Then $\text{Area}(S_{g,m})/\tau < -2 + 2 = 0$, which is very ridiculous.

Case 3c: $g = 0$, $m = 3$. The remaining calculations are carried out the same way as in the above cases. The table below summarizes the calculations, so that for example the first row reads "If the signature is $(0; 3, 3, 3)$ then $v = 3 \cdot 2/3 = 2$, giving $\text{Area}(S_{g,m})/\tau = 0$".

Signature of $S_{g,m}$	v	$\text{Area}(S_{g,m})/\tau$
$(0; 3, 3, 3)$	$= 3 \cdot 2/3 = 2$	$= -2 + 2 = 0$
$(0; x, y, z)$ with $x, y \geq 3$, $z \geq 4$	$\geq 4/3 + 3/4 = 25/12$	$\geq -2 + 25/12 = 1/12$
$(0; 2, 2, x)$ with $x \geq 2$	$< 2 \cdot 1/2 + 1 = 2$	$< -2 + 2 = 0$
$(0; 2, 4, 4)$	$= 1/2 + 2 \cdot 3/4 = 2$	$= -2 + 2 = 0$
$(0; 2, y, z)$ with $y \geq 4$, $z \geq 5$	$\geq 1/2 + 3/4 + 4/5$	$\geq -2 + 41/20 = 1/20$
$(0; 2, 3, z)$ with $z \leq 6$	$\leq 1/2 + 2/3 + 5/6$	$\leq -2 + 2 = 0$
$(0; 2, 3, z)$ with $z \geq 8$	$\geq 1/2 + 2/3 + 7/8$	$\geq -2 + 49/24 = 1/24$

Therefore the only possible orbifold satisfying the hypotheses of the theorem is $S_{0,3}$ with signature $(0; 2, 3, 7)$. The total ramification is $1/2 + 2/3 + 6/7 = 85/42$, so that the area of this orbifold is indeed $\tau/42$.

□

The fact that a quotient of the hyperbolic plane by a Fuchsian group can never have an area smaller than $\tau/42$ contrasts strongly with the situation in $\mathbb{C}$ with the flat metric. Indeed, the group of isometries generated by vertical and horizontal translations by $\epsilon > 0$ gives a quotient of $\mathbb{C}$ of area ϵ^2, i.e. as small as we want. The "incompressibility" of the hyperbolic plane lets us prove the following theorem.

Theorem 4.2 (Hurwitz). *Let Y_g be any Riemann surface of genus $g \geq 2$, equipped with any conformal metric. Then we have $|Isom^+(Y_g)| \leq 84(g-1)$. In other words, $N(g) \leq 84(g-1)$ for all $g \geq 2$.*

Proof. Suppose Y_g admits a group of isometries G with $|G| > 84(g-1)$. By the discussion in Section 3, we can uniformize the metric on Y_g. Conjugating the isometries in G by the uniformizing map, we obtain isometries of a hyperbolic Riemann surface X_g of genus g. By the Gauss-Bonnet formula, $\text{Area}(X_g) = \tau(2g-2)$.

The quotient orbifold X_g/G has area $\mathrm{Area}(X_g)/|G| < \tau(2g-2)/84(g-1) = \tau/42$. But this contradicts Lemma 4.1, which says that there *is* no hyperbolic orbifold that small!

4.2. **Hurwitz surfaces and Klein's quartic.** Despite the apparently arbitrary appearance of the number 42 in the bound $N(g) \leq 42(2g-2)$, this bound is sharp: there are infinitely many $g \geq 2$ for which there is a hyperbolic Riemann surface Y_g of genus g such that $|\mathrm{Isom}^+(Y_g)| = 84(g-1)$. This is discussed in Farb and Margalit [3], Chapter 7, and one proof is given by Macbeath [10]. The idea is to construct a surface $\mathbb{H}/G'$ of genus g with an isometry group of size $84(g-1)$, and then take an infinite family of normal, finite-sheeted covering surfaces of $\mathbb{H}/G'$. These covering surfaces inherit the lifted isometries of $\mathbb{H}/G'$.

It is not obvious that such surfaces $\mathbb{H}/G'$ with maximal symmetry, known as Hurwitz surfaces, even exist. If there is such a surface, we know that if we take the quotient of that surface by its isometry group, we will have the orbifold $(0; 2, 3, 7)$. So, the first step is to realize the orbifold $(0; 2, 3, 7)$ as the quotient space $\mathbb{H}/G$ for some isometry group G. This is done by taking the triangle group $(2, 3, 7)$, generated by reflections over the edges of a triangle in $\mathbb{H}$ with angles $\tau/4$, $\tau/6$, and $\tau/14$. This group of hyperbolic isometries does not preserve orientation (that is, it contains elements in Isom^-), which explains why the fundamental triangle has area $\tau/84$ rather than $\tau/42$.

Taking the index two subgroup G of $(2, 3, 7)$ consisting of maps that preserve orientation, we obtain the isometry group generated by the rotations through $\tau/2$ about the vertex with internal angle $\tau/4$, the rotations through $\tau/3$ about the vertex with internal angle $\tau/6$, and the rotations through $\tau/7$ about the vertex with internal angle $\tau/14$. Now a pair of the fundamental triangles of $(2, 3, 7)$ forms a fundamental region for the group G, and the quotient space $\mathbb{H}/G$ is the orbifold with signature $(0; 2, 3, 7)$ and hyperbolic area $\tau/42$. See Figure 7.

It turns out that the algebraic curve of genus 3 known as Klein's quartic is realized as the quotient of $\mathbb{H}$ by a normal subgroup K of G. This normal subgroup K is such that the quotient group G/K is $PSL(2, \mathbb{Z}/7\mathbb{Z})$.

A single sheet of the covering of Klein's quartic by the Poincaré disk is depicted in Figure 8, outlined in cyan. The sheet is composed of 24 heptagons, whose centers are marked. The isometry group G acts on the marked points transitively. Thus the heptagon marked purple can be sent to one of 24 heptagons (that is, 24 equivalence classes of heptagons under the action of K). There are seven rotations about the purple point, giving a total of $7 \times 24 = 168 = 84(3-1)$ isometries; thus Klein's quartic has maximal symmetry.

Put another way, the group G acts on the quotient surface, through the covering, as $G/K = PSL(2, \mathbb{Z}/7\mathbb{Z})$, which has order 168. Taking normal finite-index subgroups of G that are contained in K gives normal finite-sheeted coverings of Klein's quartic, forming an infinite family of Riemann surfaces with maximal symmetry. For beautiful pictures and discussion of Klein's quartic, see Baez [2].

4.3. **A note on computing isometry groups.** As explained in Kuribayashi [7], isometry groups of a Riemann surface Y_g of genus g can be represented as subgroups of $GL(g, \mathbb{C})$. This is achieved by associating to an isometry α of Y_g the matrix that encodes the action of α on the $2g$-dimensional space of holomorphic 1-forms on Y_g.

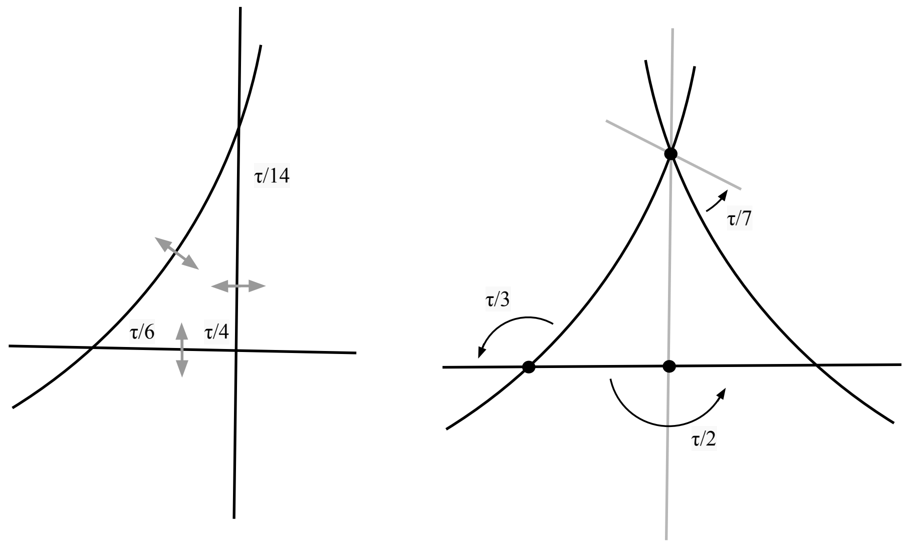

FIGURE 7. Left: the three reflections over the edges of the triangle with the indicated internal angles, generating $(2, 3, 7)$. Right: the three rotations through the indicated angles, generating G.

Then one can formulate an algebraic analog of the Riemann-Hurwitz formula and find algebraic conditions that characterize which subgroups of $GL(g, \mathbb{C})$ can act as isometries on Y_g.

Once the geometric question of isometries of Riemann surfaces is translated into an algebraic question, the possible isometry groups of Y_g can be computed directly. Kuribayashi [7], Kuribayashi and Kuribayashi [8], and Kuribayashi and Kimura [9] used this method to compute classifications of the possible isometry groups acting on Riemann surfaces of genus 2, 3, 4, and 5.

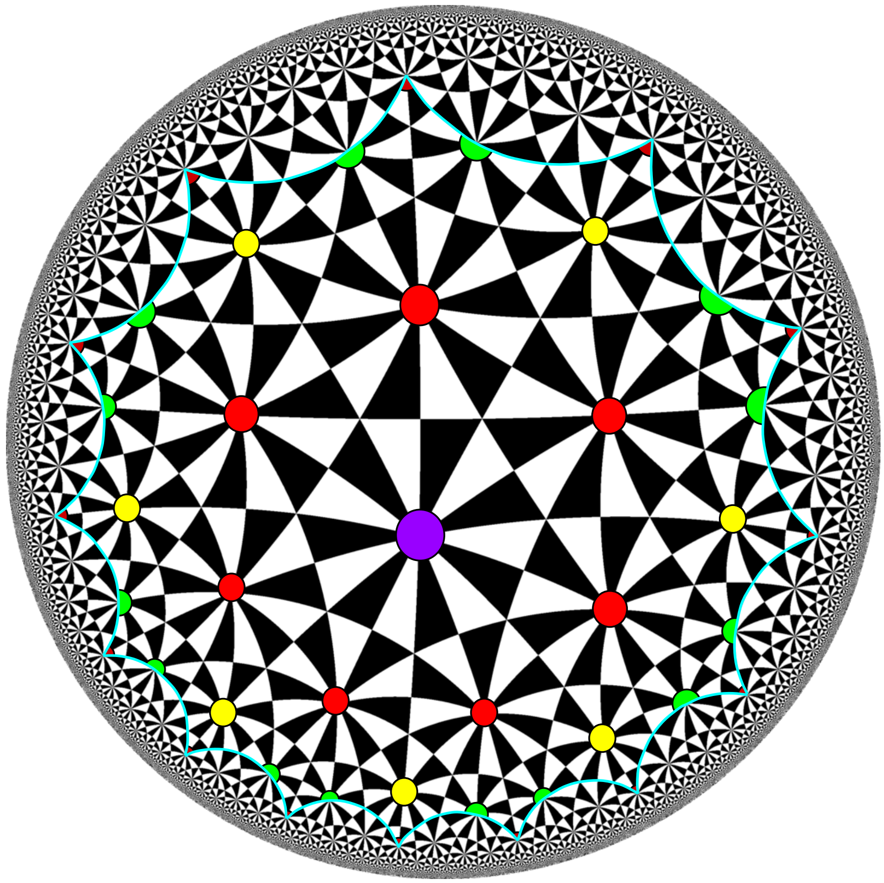

FIGURE 8. One sheet of the universal covering of Klein's quartic by the Poincaré disk, outlined in cyan.

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List of Figures

The covering of Klein's quartic was modified from the tiling picture, made by Wikipedia user Tamfang and released into the public domain. All other figures were made by the author with Google Draw, and are hereby in the public domain (CC0).

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