Measure Zero

Henry Y. Chan

July 1, 2013

1 Measure Zero

Lebesgue measure gives a concrete way to measure the volume (or area) of subsets of $\mathbb{R}^n$. For simplicity, we will only discuss the special case about sets which have Lebesgue measure zero. These sets are "small" in some senses, but they can behave surprisingly.

Let n be a natural number and $\mathbb{R}^n$ be the standard Euclidean space.

Definition 1.1. An *open cube* in $\mathbb{R}^n$ is a product of open intervals

$$U = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$$

and the *volume* of this cube is defined to be

$$\text{vol}\,(U) = (b_1 - a_1)\,(b_2 - a_2) \cdots (b_n - a_n)\,.$$

Definition 1.2. A subset X of $\mathbb{R}^n$ is said to have *measure zero* if for every $\epsilon > 0$, there exist open cubes $U_1, U_2, \cdots$ such that $A \subseteq \bigcup_{i=1}^{\infty} U_i$, and moreover

$$\sum_{i=1}^{\infty} \text{vol}\,(U_i) < \epsilon.$$

Essentially, it means that the "volume" of U is less than ϵ for all $\epsilon > 0$.

Proposition 1.3. *If $X \subset \mathbb{R}^n$ is countable or finite, then it has measure zero. (In particular, $\mathbb{Q}$ has measure zero in $\mathbb{R}$.)*

Proof. For simplicity, assume $n = 1$. (The proof for $n > 1$ is similar.) Enumerate the elements of X as $x_1, x_2, \cdots$. For every $\epsilon > 0$, for each n, define

$$U_i = \left(x_i - \frac{\epsilon}{2^{i+2}}, x_i + \frac{\epsilon}{2^{i+2}}\right).$$

Note that $\text{vol}\,(U_i) = \frac{\epsilon}{2^{i+1}}$, so

$$\sum_{i=1}^{\infty} \text{vol}\,(U_i) = \sum_{i=1}^{\infty} \frac{\epsilon}{2^{i+1}} = \frac{\epsilon}{2} < \epsilon$$

Therefore X has measure zero. $\square$

Exercise 1.4. Show that every k-dimensional subspace of $\mathbb{R}^n$ has measure zero if $k < n$. In other words, lines have no area, and planes have no volume.

In particular, faces of an n-dimensional polyhedron have measure zero in $\mathbb{R}^n$.

Exercise 1.5. Prove the following statements.

1. A subset of a set of measure zero also has measure zero.
2. A countable union of sets of measure zero also has measure zero.

Definition 1.6. We say something happens *almost everywhere* if it happens everywhere except on a set of measure zero.

The following result is an important theorem to determine whether a function is Riemann integrable.

Theorem 1.7. *(Lebesgue's Theorem) A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if it is continuous almost everywhere on $[a, b]$.*

Measure is not preserved by bijections. The most famous example would be the Cantor set C (look it up if you don't know what the Cantor set is, it is really interesting). One can show that C has measure zero, yet there exists a bijection between C and $[0, 1]$, which does not have measure zero. (This is not hard to prove. In fact, $[0, 1]$ has measure 1.) Let's end with an interesting example showing that measure is not "additive".

Exercise 1.8. Define

$$C + C = \{a + b \mid a, b \in C\} .$$

Show that $C + C = [0, 2]$. Hence we have a sum of two measure zero sets which has positive measure.