ABSTRACTS: 2013 SUMMER VIGRE REU

All concepts mentioned in the abstracts will be carefully defined

APPRENTICE PROGRAM, weeks 1–5

Madhur Tulsiani and Laci Babai

Linear algebra and discrete structures

The course will develop the usual topics of linear algebra and illustrate them on (often striking) applications to discrete structures. Emphasis will be on creative problem solving and discovery. The basic topics include determinants, linear transformations, the characteristic polynomial, Euclidean spaces, orthogonalization, the Spectral Theorem, Singular Value Decomposition. Application areas to be highlighted include spectral graph theory (expansion, quasirandom graphs, Shannon capacity), random walks, clustering high-dimensional data, extremal set theory, and more.

FULL PROGRAM, weeks 1–8

Probability and Analysis

Gregory Lawler, weeks 1-2

Random Walk and the Heat Equation

Two closely related topics are random walks and heat flow. One sees this by considering heat as consisting of a large (infinite number?) of "heat particles" all moving randomly and independently. I will present some of the mathematics that makes this rigorous. I start with random walk in the integer lattice and then use this to give a model for discrete heat flow. Analysis of this flow will use tools from linear algebra. In the second week I will consider the continuous analogues: the random walk become Brownian motion; the discrete heat equation because a partial differential equation (called, amazingly, the heat equation!), and the linear algebra argument is replaced with one using Fourier series.

This course has been given before and the notes have been published in a book "Random Walk and the Heat Equation" published in the Student Mathematical Library series by the American Mathematical Society.

Antonio Auffinger, weeks 3-4

Card Shuffling, Branching and beyond

The goal of this two week course is to present some topics which are accessible to advanced undergraduates yet are areas of research in probability. We will first discuss the problem of shuffling a deck of cards. We will start by studying random permutations and the notion of a random walk on a symmetric group. We will sketch the argument that 7 shuffles are enough to get a "fair" deck. Then we will discuss a few problems related to random graphs, networks and branching processes. We will study the Galton-Watson Branching Process, the Erdös-Renyi phase transition and beyond.

Adina Ciomaga and Jessica Lin, weeks 5-6

Viscosity Solutions for Hamilton–Jacbi equations and Optimal Control

The goal of these lectures is to introduce the theory of viscosity solutions for first-order Hamilton-Jacobi equations of the form

$$u_t + H(x, u, Du) = 0 \quad \text{in} \quad \Omega \times (0, \infty)$$

where Ω is an open subset of $\mathbf{R}^n$, $u : \mathbf{R}^n \times (0, \infty) \to \mathbf{R}$ is the unknown function, $u_t = \partial u/\partial t$ is the time derivative, $Du = (\partial u/\partial x_1, \ldots, \partial u/\partial x_n)$ represents the spatial gradient of u, and $H : \Omega \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ is a continuous function referred to as the *Hamiltonian.*

Viscosity solutions were introduced in 1981 by Crandall and Lions to analyze Hamilton-Jacobi equations, but they have become a standard tool used to study fully nonlinear partial differential equations. They are particularly relevant for optimal control problems in economics, physics, and modern computer science.

In this course, we will investigate several properties of viscosity solutions such as existence, uniqueness, and stability. We will focus on specific examples which take on the structure of the equation above, and illustrate their applications in control theory and image processing.

Nota bene: We recommend that students are comfortable with the concepts of uniform convergence and calculus in higher dimensions, although all are welcome.

Matthew Emerton, Paul Herman, and Davide Reduzzi, weeks 1-5

Number Theory: Primes in Arithmetic Progressions and The Class Number Formula

Dirichlet proved in the mid nineteenth century that there are infinitely many primes of the form a+bn with fixed coprime numbers a and b. We aim to prove this result. This will require results from both algebra (ideal class groups) and analysis (generalizations of the Riemann zeta function). The class number formula ties these ideas together. If time permits, we will talk about the Birch Swinnerton-Dyer conjecture as an analog of the class number formula.

Danny Calegari and Alden Walker, weeks 1-4

Words, words, words

A word in a group means an element of a certain form; for example the word x^2 denotes an element which is a square, the word $xyx^{-1}y^{-1}$ denotes an element which is a commutator, and so on. By substituting different group elements for the quantities x, y etc. we find a subset of a group representing a given word.

If w is a word, then for any w-word g, the power g^n can be written as a product of n w-words in the obvious way ($g^n = ggg...g$). But it could happen that (for big enough n) any g^n can be written as a product of *strictly fewer* w-words. For example, in any group,

$$[a, b]^3 = [aba^{-1}, b^{-1}aba^{-2}][b^{-1}ab, b^2]$$

where $[x, y]$ means $xyx^{-1}y^{-1}$ for any x, y. (Check this out!) Thus, the cube of a commutator is always a product of two commutators.

In this project we will try to determine the extent to which this phenomenon - that "stable w-length" is smaller than w-length - holds for other words w. A reference is the paper "Stable W-length", available on the arXiv at:
http://arxiv.org/abs/1008.2219

While this is in principle a purely algebraic problem, we shall explain that it is motivated by questions in geometry.

Maryanthe Malliaris, week 1

Ultraproducts and Ultrapowers

The ultraproduct construction is a subtle and powerful method of constructing large objects of a given kind (fields, groups, graphs...) from infinite families of such objects, in such a way as to reflect or reveal the "average" structure across the family. I will explain the basics of how the construction works, give some examples of how it is used, and state some open problems.

The course has no formal prerequisites other than some mathematical maturity. Previous experience with logic or infinite combinatorics may be helpful for the open problems, but is not otherwise necessary.

Peter May and Inna Zakharevich, weeks 1-8

A pot pourri of algebra, topology, and other topics

We will explore a number of topics, giving relatively few talks in the first few weeks but lots later on. We will explore interrelated topics in algebra and topology, and maybe some category theory too, depending on your interests and curiosity. Here are a few.

Hilbert's third problem asks the following question: given two polyhedra with the same volume (the first invariant of a polyhedron), is it always possible to dissect them into finitely many pairwise congruent pieces? It turns out that the answer is "no"; this was shown by Dehn in 1901 by constructing a second invariant (called, appropriately, the Dehn invariant) and showing that a cube and a regular tetrahedron have different Dehn invariants. The obvious next question, then, is if this can be done for two polyhedra with the same volume and Dehn invariant. It turns out that the answer is "yes". The original proof was long, hard, and geometric.

That brings us to our second topic. A quite different and much shorter and easier proof comes from the algebraic theory of group homology. Suppose that a group G contains a normal subgroup N with quotient group H. Does this uniquely determine G? If not, how much more information do we need in order to determine G, and can we classify all such "extension of K by H". It turns out that the answer to the first question is "no", and that even for small K and H the situation becomes quite complicated. Consider, for example, the case when G is abelian, $K = \mathbf{Z}/m\mathbf{Z}$, and $H = \mathbf{Z}/n\mathbf{Z}$. If m and n are relatively prime then the Chinese Remainder Theorem tells us that $G = \mathbf{Z}/mn\mathbf{Z}$; however, when they are not relatively prime, things get more complicated. For example, if $m = n = 2$, there are two possibilities for G, namely $G = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ and $G = \mathbf{Z}/4\mathbf{Z}$. The abstract tool for classifying such extensions is the "first homology group of H with coefficients in K." We shall explore such homology groups and explain their use in the solution of Hilbert's third problem.

Here is a third topic. Let G be a finite group and let p be a prime that divides the order of G. When does G contain a normal subgroup N_p of order a power of p? A remarkable conjecture of the Field medalist Daniel Quillen gives a surprising answer in terms of *topological* properties of the partially ordered set (poset) $\mathcal{S}_p$ of subgroups of G of order p^n for some $n \geq 1$. The conjecture is still open, but explaining it gives a beautiful excuse for exploring the fascinating and little known theory of finite topological spaces. We will present the basic theory and how it relates to posets, classical simplicial complexes, finite groups, and maybe even categories. For example, we shall show how to describe a space with $2n + 2$ points that to the eyes of algebraic topology is "just the same" as the n-sphere S^n. Quillen's conjecture says that if $\mathcal{S}_p$ is "just the same" as a point, with the same meaning of "just the same", then G does contain such a normal subgroup N_p.

A book in progress is available on line:
http://math.uchicago.edu/ may/FINITE/FINITEBOOK/FiniteAugBOOK.pdf
(You are offered a $1.00 reward for each typo you find.)