Laplacian Eigenvalues of Simplicial Complexes

Let G be a finite, simple, undirected graph on n vertices. In spectral graph theory, given such a graph, there is an interest in studying how large its Laplacian eigenvalues can get. In 2008, Brouwer conjectured that if G has $e(G)$ edges and Laplacian eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0$, then

$$\sum_{i=1}^{t} \lambda_i \leq e(G) + \binom{t+1}{2}, \forall t \in \mathbb{N}.$$

This conjecture is phenomenal in that it uses very little data to give a tight upper-bound for the partial sum of the Laplacian spectrum of the graph. While this conjecture is not completely resolved for all graphs, it has thus far been shown to hold for $t = 1, 2,$ and $n-1$, trees, threshold graphs, split graphs, regular graphs, co-graphs, and all graphs with at most 10 vertices. In this talk, we will present and discuss some of these proofs as they relate to Laplacian matrices of simplicial complexes.

In my research, I generalize this conjecture to the case of abstract simplicial complexes of any dimension k. (Simple graphs are dimension 1 abstract simplicial complexes.) In particular, I come up with a generalized bound,

$$\sum_{i=1}^{t} \lambda_i \leq (k-1) f_{k-1} + \binom{t+k-1}{k},$$

where f_{k-1} is the number of $k-1$-dimensional faces of the simplicial complex and the Laplacian eigenvalues are listed in non-increasing order.

Similar to the case of graphs, this conjecture is not completely known to hold for all simplicial complexes. However, during my research I was able to show that it holds for the first and last partial sum. I will also present the proof that this bound holds for all shifted simplicial complexes, which generalize threshold graphs and have interesting combinatorial and topological properties. We will also see that it is satisfied for all simplicial trees in the sense of Faridi.[1] As time permits, we will also discuss the progress made to show that the tth partial sum holds for all $k-1$-dimensional complexes where $k > t$ that have matching number greater than t.

[1] S. Faridi: *Simplicial Trees: Properties and Applications.*