00:00

We're going to play some games between finite group theory and finite topological spaces. Let G be a finite group, we have a notion of G-actions

$$\begin{aligned} G \times X &\to X \\ (g, x) &\mapsto gx \\ g(hx) &= (gh)x \\ ex &= x \end{aligned}$$

Consider a regular polygon with n-vertices. We have a D_{2n}-action on it. Let $a, b \in G$ be the rotation and the reflection, respectively. We have $a^n = e = b^2$ and $ba = a^{-1}b$.

Consider a finite G-space X. It is remarkable that X is contractible implies that X is G-contractible. Sketch of the proof: Remove an orbit at a time and the leftover points will be the core of X. This is a special case of the proposition, proved last time, that a homotopy equivalence between finite G-spaces is a G-homotopy equivalence. A corollary of this proposition is that every contractible finite G-space has a G-fixed point.

Let p be a prime, define $\mathscr{S}_p(G)$ to be the set of non-trivial p-subgroup of G. Define a G-action on $\mathscr{S}_p(S)$ as

$$g \cdot P = P^g = \left\{ gpg^{-1} \mid p \in P \right\}$$

A fixed point P is a normal p-subgroup.

Similarly, define $\mathscr{A}_p(G)$ to be the set of non-trivial elementary abelian p-subgroups of G. We have an inclusion map $i : \mathscr{A}_p(G) \hookrightarrow \mathscr{S}_p(G)$. We claim that i is a weak homotopy equivalence but not necessary a homotopy equivalence. First, we need to keep the following diagram in mind.

$$\begin{array}{ccc} |\mathcal{K}\mathscr{A}_p(G)| & \xrightarrow{|\mathcal{K}(i)|} & |\mathcal{K}\mathscr{S}_p(G)| \\ \psi \downarrow \text{WHE} & & \text{WHE} \downarrow \psi \\ \mathscr{A}_p(G) & \xrightarrow[i]{} & \mathscr{S}_p(G) \end{array}$$

Note that i is a WHE implies that $|\mathcal{K}(i)|$ is also a WHE.

First, we prove the following proposition.

Proposition 1. $\mathscr{A}_p(G)$ and $\mathscr{S}_p(G)$ are contractible if G is a p-group.

Proof. Let B be a central p-subgroup of order p. (Peter explained why such a subgroup exists.) For any $A \in \mathscr{A}_p(G)$, we have $A \subseteq AB \supseteq B$. Observe that $id \leq f \geq c_B$, i.e., $id \simeq f \simeq c_B$.

$$\begin{array}{ccc} \mathscr{A}_p(G) & \longrightarrow & \mathscr{A}_p(G) \\ & {\scriptstyle r} \searrow \quad \nearrow {\scriptstyle i} & \\ & \{B\} & \end{array}$$

For every $P \in \mathscr{S}_p(G)$, define $U_P = \{Q \mid Q \subseteq P\} \subseteq \mathscr{S}_p(P)$. Observe that $i^{-1}(U_P) = \{$ elementary abelian $Q \mid Q \subseteq P\}$ $\mathscr{A}_p(P)$. By the proposition we just proved, we get $U_P = \mathscr{S}_p(P)$ and $i^{-1}(U_P) = \mathscr{A}_p(P)$

□

45:00

Theorem 1. If G has a non-trivial normal p-subgroup P, then $\mathscr{S}_p(G)$ is conically contractible, hence contractible.

Proof. For every $H \in \mathscr{S}_p(G)$, we have $H \subseteq HP \supseteq P$. The rest of the proof is exactly the same argument as in Proposition 1.. □

Conversely, if $\mathscr{S}_p(G)$ is contractible, then G has a non-trivial normal p-subgroup. The conjecture is, if $\mathscr{S}_p(G)$ is weakly contractible, then G has a non-trivial normal p-subgroup.

Define $\mathscr{Q}_p(G)$ as the set of non-trivial intersections of p-Sylow subgroup of G.

Proposition 2. $j : \mathscr{Q}_p(G) \hookrightarrow \mathscr{S}_p(G)$ is the inclusion of a deformation retract.

Proof. For every $P \in \mathscr{S}_p(G)$, let $f(P)$ be the intersection of those p-Sylow subgroups that contain P. We have

$$\mathscr{S}_p(G) \xrightarrow{f} \mathscr{Q}_p(G) \xrightarrow{j} \mathscr{S}_p(G)$$

Note that f is continuous since $P \subseteq f(P)$. Moreover, we have $id \leq jf$, i.e., $id \simeq jf$. □

Let $G = \Sigma_5, p = 2$. We claim that $\mathscr{A}_2(G)$ and $\mathscr{S}_2(G)$ are hot homotopy equivalent. If suffices to show that $\mathscr{A}_2(G)$ and $\mathscr{Q}_2(G)$ are not homotopy equivalent. We'll se that both $\mathscr{Q}_2(G)$ and $\mathscr{A}_2(G)$ are minimal spaces and $\mathscr{Q}_2(G)$ is smaller. Quoting some algebraic fact, we know there are 6 conjugacy classes of 2-subgroups of Σ_5, which are $D_8, C_4, C_2 \times C_2$ generated by (ab) and (cd), $C_2 \times C_2$ generated by $(ab)(cd)$ and $(ac)(bd)$, C_2 generated by (ab), C_2 generated by $(ab)(cd)$.

Quillen's conjecture says if $\mathscr{A}_p(G)$ or $\mathscr{S}_p(G)$ is weakly contractible, then it is contractible, i.e., there exists a normal p-subgroup.

Next time we will prove that $|\mathcal{K}\mathscr{A}_p(G \times H)| \simeq |\mathcal{K}\mathscr{A}_p(G)| * |\mathcal{K}\mathscr{A}_p(H)|$. The joint product is defined as $X * Y = (X \times I \times Y) / \sim$ where $(x, 0, y) \sim (x', 0, y), (x, 1, y) \sim (x, 1, y')$.

Define a category Δ whose objects are $\underline{n} = \{0, 1, \cdots, n\}$ and morphisms are non-decreasing maps. A simplicial set is a functor $\Delta^{op} \to Set$. We have two maps

$$\delta_i : \underline{n-1} \to \underline{n} \qquad \delta_i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}$$

$$\sigma_i : \underline{n+1} \to \underline{n} \qquad \sigma_i(j) = \begin{cases} j & \text{if } j < i \\ j-1 & \text{if } j \geq i \end{cases}$$

We have the corresponding map $d_i : X_n \to X_{n-1}, s_i : X_n \to X_{n+1}$, satisfying some conditions.

Define chain complexes, we have the following composition of functors.

$$OrSimp \longrightarrow sSet \xrightarrow{F} sAb \xrightarrow{Ch} Ch \xrightarrow{H_\bullet} Ab$$

At this point Peter only had two minutes left, so he was kind of rushing this diagram.