Finite Spaces Handouts 2

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1 General Topology Revisited

1.1 Connectivity/Path Connectivity

Definition 1.1.1. A topological space \( X \) is connected if the only sets which are both open and closed are \( \emptyset, X \). The connected component of \( x \) is the largest connected subset containing \( x \). Check that connected components are both open and closed.

Definition 1.1.2. A topological space \( X \) is path-connected if for every \( x, y \in X \), there is a path \( f : [0, 1] \to X \) such that \( f(0) = x, f(1) = y \). The path-connected component of \( x \) is the largest connected subset containing \( x \).

Path-connectedness always implies connectedness. However, the converse is not always true. See the following example.

Fun Exercise 1.1.3. The topologist’s sine curve is defined as

\[
S = \left\{ \left( x, \sin \left( \frac{1}{x} \right) \right) \mid x > 0 \right\} \cup \{ \{0\} \times [-1, 1] \}
\]

Show that \( S \) is connected but not path-connected.

Now let \( X \) be an Alexandroff space. Recall that \( X \) have a pre-ordered structure compatible with its topology.

Lemma 1.1.4. If \( x \leq y \), then \( x \) and \( y \) lie in the same path-connected component. In particular, \( U_x \) is path-connected.

Proof. Define \( f : [0, 1] \to X \) as \( f(t) = \begin{cases} x & \text{if } t < 1 \\ y & \text{if } t = 1 \end{cases} \). For every open set \( U \) in \( X \), note that \( U_x \subseteq U_y \), so \( y \in U \) implies that \( x \in U \). We have the following three cases.

\[
f^{-1}(V) = \begin{cases} 
\emptyset & \text{if } x, y \in U \\
[0, 1) & \text{if } x \in U, y \notin U \\
[0, 1] & \text{if } x, y \in U
\end{cases}
\]

Hence \( f \) is continuous, i.e., \( f \) is a path from \( x \) to \( y \).

Lemma 1.1.5. For every \( x \in X \), \( U_x \) is connected.

Proof. Assume \( U_x = A \cup B \), where \( A, B \) are both open and \( A \cap B = \emptyset \). Note that \( x \in U_x = A \cup B \), so either \( x \in A \) or \( x \in B \). Without loss of generality, assume \( x \in A \), then \( U_x \subseteq A \) thus \( A = U_x \) and \( B = \emptyset \). Hence \( U_x \) is connected.

Lemma 1.1.6. If \( X \) is connected, then it is also path-connected.
Proof. For every \( x \in X \), let \( A \) be the path-connected component of \( x \). Note that for every \( y \in A \), \( U_y \) is path-connected by Lemma 1.1.4., so \( A \) is open. Similarly, for every \( y \in X \setminus A \), \( U_y \subseteq X \setminus A \), so \( X \setminus A \) is also open. Hence \( A = X \), i.e., \( X \) is path-connected.

We get the following result.

**Proposition 1.1.7.** \( X \) is connected if and only if \( X \) is path-connected.

## 2 Algebraic Topology

### 2.1 Fun Applications of the Four Constructions of Induced Topologies

Recall the definitions of the subspace topology, the quotient topology, the product topology, and the coproduct topology. Almost all topological spaces come from these four constructions.

**Example 2.1.1.** Let \( R = [0, 1] \times [0, 1] \) be the unit square. Consider the following four equivalence relations on \( R \).

\[
\begin{align*}
(x, 0) \sim_1 (x, 1), & \quad (0, y) \sim_1 (1, y) \\
(x, 0) \sim_2 (1 - x, 1) \\
(x, 0) \sim_3 (1 - x, 1), & \quad (0, y) \sim_3 (1, y) \\
(x, 0) \sim_4 (1 - x, 1), & \quad (0, y) \sim_4 (1, 1 - y)
\end{align*}
\]

Let’s use the following cartoons to illustrate the identifications.

\[
\begin{array}{c}
\text{T}_1 \quad \text{T}_2 \quad \text{T}_3 \quad \text{T}_4
\end{array}
\]

\( T_1 \) is the 2-torus, \( T_2 \) is the Mobius strip, \( T_3 \) is the Klein bottle, and \( T_4 \) is the real projective plane.

**Definition 2.1.2.** Let \( X \) be a topological space and \( A \subseteq X \) be a subspace. Define an equivalence relation \( \sim \) as \( x \sim y \iff x, y \in A \). The quotient space \( X / \sim \), also denoted \( X / A \), can be viewed as collapsing \( A \) to a point.

**Fun Exercise 2.1.3.** Let \( X = [0, 1] \) and \( A = [0, 1) \). What is \( X/A \)? Is it Hausdorff?

**Definition 2.1.4.** For two based topological spaces \((X, x)\) and \((Y, y)\), their *wedge* product, denoted \( X \vee Y \), is defined as

\[
X \vee Y = (X \times \{y\}) \cup (\{x\} \times Y) \subseteq X \times Y
\]

equipped with the subspace topology relative to the product topology on \( X \times Y \). Their *smash* product, denoted \( X \wedge Y \), is defined as \( X \wedge Y = (X \times Y) / (X \vee Y) \), equipped with the quotient topology.

**Fun Exercise 2.1.5.** Show that \( X \vee Y \cong (X \amalg Y) / (x \sim y) \).
2.2 Cones/Suspensions

**Definition 2.2.1.** Let \( X \) be a Hausdorff topological space. The *cone* of \( X \), denoted \( CX \), is defined as

\[ CX = (X \times [0,1]) / (X \times \{0\}) \]

The *suspension* of \( X \), denoted \( SX \), is defined as

\[ SX = (X \times [0,1]) / (X \times \{0,1\}) \]

**Exercise 2.2.2.** Show that \( CX \) is contractible for all \( X \).

**Fun Exercise 2.2.3.** Try to draw \( SS^1 \), does it look like some familiar spaces? What about \( SS^n \)?

For non-Hausdorff spaces, we have analogous definitions for cones and suspensions.

**Definition 2.2.4.** Let \( (X, U) \) be a non-Hausdorff topological space. The *non-Hausdorff cone* of \( X \) and the *non-Hausdorff suspension* of \( X \), denoted \( C_X \) and \( S_X \), respectively, are defined as

\[ C_X = X \cup \{\ast\}, \quad U_C = U \cup \{X \cup \{\ast\}\} \]
\[ S_X = X \cup \{\ast_+, \ast_-\}, \quad U_S = U \cup \{X \cup \{\ast_+\}, X \cup \{\ast_-\}, X \cup \{\ast_+, \ast_-\}\} \]

**Proposition 2.2.5.** Let \( X \) be a topological space, if there exists \( y \in X \) such that the only open set containing \( y \) is \( X \), (i.e., \( U_y = X \)), then \( X \) is contractible.

**Proof.** Define two maps \( r : X \to \ast, i : \ast \to X \), as \( r(x) = \ast, i(\ast) = y \). Clearly \( r \circ i = id_\ast \). Define a homotopy \( h : X \times [0,1] \to X \) as

\[ h(x, t) = \begin{cases} x & \text{if } t < 1 \\ y & \text{if } t = 1 \end{cases} \]

Clearly \( h(x, 0) = x = id_X(x) \) and \( h(x, 1) = y = (i \circ r)(x) \). If remains to check that \( h \) is continuous. For every open set \( U \subseteq X \). Note that if \( y \in U \) then \( U = X \). We have

\[ h^{-1}(U) = \begin{cases} U \times [0,1] & \text{if } y \notin U \\ U \times [0,1] & \text{if } y \in U \end{cases} \]

Hence \( h \) is continuous, so \( i \circ r \simeq id_X \). Therefore \( X \) is contractible. \( \square \)

**Exercise 2.2.6.** Show that \( C_X \) is contractible for all \( X \).