Finite Spaces Handouts 1

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1 General Topology

1.1 Basic Definitions

Definition 1.1.1. A *topological space* $(X,\mathcal{U})$ consists of a set X and a collection $\mathcal{U}$ (called a *topology* of X) of subsets of X satisfying the following properties.

- $\emptyset, X \in \mathcal{U}$
- $\{U_i\}_{i\in I} \subseteq \mathcal{U} \Rightarrow \bigcup_{i\in I} U_i \in \mathcal{U}$ ($\mathcal{U}$ is closed under arbitrary unions)
- $\{U_i\}_{i=1}^{n} \subseteq \mathcal{U} \Rightarrow \bigcup_{i=1}^{n} U_i \in \mathcal{U}$ ($\mathcal{U}$ is closed under *finite* intersections)

Sets in $\mathcal{U}$ are called *open sets* in X. Sets whose complements are in $\mathcal{U}$ are called *closed sets* in X. If $x \in U \in \mathcal{U}$, we call U a *neighborhood* of x.

Exercise 1.1.2. Recall the definition of open sets in a metric space (X, d). A subset $U \subseteq X$ is open if for every $x \in U$, there exists $r > 0$ such that $B_r(x) \subseteq U$, where $B_r(x) = \{y \in X \mid d(x, y) < r\}$ is an open ball. Show that open sets defined this way satisfy the axioms above, i.e., the two notions of open sets coincide.

Fun Exercise 1.1.3. Why aren't open sets closed under arbitrary intersections? Try to find a simple counterexample? (Hint: Consider $\mathbb{R}$).

Example 1.1.4. For any set X, the discrete topology $\mathcal{U}_{dis}$ and the trivial topology $\mathcal{U}_{triv}$ are defined as

$$\mathcal{U}_{dis} = 2^X \qquad \text{(every subset of } X \text{ is open)}$$
$$\mathcal{U}_{triv} = \{\emptyset, X\}$$

In other words, the discrete topology and the trivial topology are the minimal and the maximal topology of X satisfying the axioms, respectively. We will see why these two examples are not "interesting" in Fun Exercise 1.3.3..

Definition 1.1.5. For any *infinite* set X, the *cofinite* topology is defined as

$$\mathcal{U}_{cof} = \{U \mid X \backslash U \text{ is finite }\} \cup \{\emptyset\}$$

In other words, close sets are either finite sets or X.

Exercise 1.1.6. Check that cofinite topology satisfies all the axioms. Moreover, show that no metric on X defines the cofinite topology. This gives an example of a topological space which is not a metric space.

1.2 Basis

Definition 1.2.1. A *basis* of a set X is a collection $\mathcal{B}$ of subsets of X such that
(1) For every $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.
(2) If $x \in B_1 \cap B_2$, where $B_1, B_2 \in \mathcal{B}$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

We say a topology $\mathcal{U}$ is generated by $\mathcal{B}$ if for every $x \in U \in \mathcal{U}$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Proposition 1.2.2. The topology generated by a basis $\mathcal{B}$ is just $\left\{\bigcup_{i \in I} B_i \mid B_i \in \mathcal{B}\right\}$.

Example 1.2.3. The set of all open intervals $\{(x, y)\}_{x<y}$ is a basis generating the standard (metric) topology on $\mathbb{R}$.

Fun Exercise 1.2.4. Is the set of all open intervals with rational endpoints $\{(x, y)\}_{x<y, x, y \in \mathbb{Q}}$ also a basis generating the standard topology on $\mathbb{R}$? If so, do you think this is a better choice of a basis for $\mathbb{R}$? Why?

In most situations, it suffices to check facts on a basis rather than on a topology. The following proposition gives a great example.

Proposition 1.2.5. On a set X, let $\mathcal{U}$ and $\mathcal{U}'$ be the topologies generated by the bases $\mathcal{B}$ and $\mathcal{B}'$, respectively. Then $\mathcal{U} \subseteq \mathcal{U}'$ if and only if for every $x \in B \in \mathcal{B}$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

Exercise 1.2.6. In $\mathbb{R}^2$, one might consider the set of open disks (balls) $\{B_r(x, y)\}$ as a basis, while others might prefer the set of open rectangles (boxes) $\{(x_1, y_1) \times (x_2, y_2)\}$. Show that these two bases generate the same topology on $\mathbb{R}^2$. (Hint: Show that every open ball contains an open box and vice versa.)

1.3 Continuous Maps

We have learn the epsilon-delta definition for continuous maps. However, in Exercise 1.1.6., we know that not all topological spaces can be equipped with a metric. Hence we need a more general definition for continuous maps between general topological spaces.

Definition 1.3.1. A map $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is *continuous* if for every $V \in \mathcal{V}$, its preimage $f^{-1}(V) \in \mathcal{U}$.

Proposition 1.3.2. If X, Y are metric spaces, then the definition for continuous maps above is equivalent to the epsilon-delta definition for continuous maps.

Fun Exercise 1.3.3. Recall the definition of the discrete topology $\mathcal{U}_{dis}$ and the trivial topology $\mathcal{U}_{triv}$. Let $(Y, \mathcal{V})$ be an arbitrary topological space. What do you observe from an arbitrary map $f : (X, \mathcal{U}_{dis}) \to (Y, \mathcal{V})$? How about an arbitrary map $f : (Y, \mathcal{Y}) \to (X, \mathcal{U}_{triv})$?

A continuous map $f : X \to Y$ is called a *homeomorphism* if it is a bijection and its inverse $f^{-1} : Y \to X$ is also continuous. In this case, we say X and Y are homeomorphic.

Fun Exercise 1.3.4. Classify all upper case alphabets up to homeomorpshim. (This is just for fun, really.)

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1.4 Four Kinds of Induced Topology

Definition 1.4.1. (Subspace Topology) Let $(X,\mathcal{U})$ be a topological spaces and $A \subseteq X$ be a subset. The subspace topology on A relative to X is defined as

$$\mathcal{U}_A = \{U \cap A \mid U \in \mathcal{U}\}$$

In other words, the subspace topology is the smallest topology so that the inclusion map $i : A \to X$ is continuous.

Definition 1.4.2. (Quotient Topology) Let $(X,\mathcal{U})$ be a topological spaces and $f : X \to Y$ be a *surjective* map. The quotient topology on Y is defined as

$$\mathcal{U}_Y = \left\{O \mid f^{-1}(O) \in \mathcal{U}\right\}$$

In other words, the quotient topology is the smallest topology so that f is continuous.

Exercise 1.4.3. Consider the map $f : [0,1] \to \{a,b\}$ defined as

$$f(x) = \begin{cases} a & \text{if } x = 0 \\ b & \text{if } x \neq 0 \end{cases}$$

Clearly f is surjective. What is the quotient topology on $\{a,b\}$?

Assume we have an equivalence relation $\sim$ on a topological space $(X,\mathcal{U})$. Let $X/\sim$ be the set of equivalence classes. There is a natural map $q : X \to X/\sim$, which sends each element to its equivalence class. Clearly q is surjective, so we can talk about the quotient topology on $X/\sim$.

Fun Exercise 1.4.4. Define $\sim$ on $\mathbb{R}$ as $x \sim y \Leftrightarrow x - y \in \mathbb{Z}$. The quotient $\mathbb{R}/\sim$ is also denoted as $\mathbb{R}/\mathbb{Z}$. Does $\mathbb{R}/\mathbb{Z}$ look like any topological space that we are familiar with?

Definition 1.4.5. (Product Topology) Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be two topological spaces. The product topology on $X \times Y$ is generated by the basis

$$\mathcal{B}_{prod} = \mathcal{U} \times \mathcal{V}$$

In other words, the product topology is the smallest topology so that the projection maps $p_X : X \times Y \to X, p_Y : X \times Y \to Y$ are continuous.

Exercise 1.4.6. Let X, Y be topological spaces and $A \subseteq X, B \subseteq Y$. There are two ways to define a topology on $A \times B$. We can either first take the subspace topologies on $A \subseteq X$ and $B \subseteq X$ and then take their product topology, or first take the product topology on $X \times Y$ and then take the subspace topology on $A \times B \subseteq X \times Y$. Show that these two ways define the same topology on $A \times B$.

Definition 1.4.7. (Disjoint Union (also called Coproduct) Topology) Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be two disjoint topological spaces. The coproduct topology on $X \amalg Y$ is defined as

$$\mathcal{T}_{\amalg} = \{U \cup V \mid U \in \mathcal{U}, V \in \mathcal{V}\}$$

In other words, the coproduct topology is the smallest topology so that the inclusion maps $i_X : X \to X \amalg Y, i_Y : Y \to X \amalg Y$ are continuous.

Fun Exercise 1.4.8. The assumption X, Y being disjoint is totally unnecessary. For example, we may want to take a disjoint union of two copies of X. Can you come up with a definition that does not require X and Y to be disjoint? (Hint: A disjoint union of two copies of $\mathbb{R}$ can be thought as a union of two lines $\{y = 0\}$ and $\{y = 1\}$ in $\mathbb{R}^2$.)

1.5 Separation Axioms and Alexandroff Spaces

Definition 1.5.1. A topological space $(X,\mathcal{U})$ is T_0 if for every $x \neq y$ in X, there exists $U \in \mathcal{U}$ such that U contains one of x, y but not the other.

Fun Exercise 1.5.2. Show that X is T_0 if and only if for every $x \neq y$ in X, there exists $B \in \mathcal{B}$ such that B contains one of x, y but not the other. In other words, if sets in $\mathcal{B}$ cannot distinguish x, y, then neither can sets in $\mathcal{U}$. (Hint: Every set in $\mathcal{U}$ is a union of sets in B.)

Definition 1.5.3. A topological space $(X,\mathcal{U})$ is T_1 if $\{x\}$ is a closed set for all $x \in X$. Equivalently, for every $x \neq y$ in X, there exist $U, V \in \mathcal{U}$ such that $x \in U, y \in V$ but $x \notin V, y \notin U$. (Check that these two definitions are equivalent.)

Fun Exercise 1.5.4. Show that the only topology which is both Alexandroff and T_1 is the discrete topology.

Definition 1.5.5. A topological space $(X,\mathcal{U})$ is T_2, or Hausdorff, if for every $x \neq y$ in X, there exist $U, V \in \mathcal{U}$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

Exercise 1.5.6. Show that metric spaces are Hausdorff. (In fact, metric spaces are T_4, or normal.) Check that any infinite space with the cofinite topology is T_1 but not Hausdorff.

Fun Exercise 1.5.7. For any topological space X, define the *diagonal* Δ_X as

$$\Delta_X = \{(x, x) \mid x \in X\} \subseteq X \times X$$

Show that X is Hausdorff if and only if Δ_X is closed with respect to the product topology on $X \times X$.

Definition 1.5.8. An *Alexandroff space* is a topological space where open sets are closed under *arbitrary* intersections. In particular, all finite spaces are Alexandroff spaces.

Exercise 1.5.9. Let $\mathcal{U}$ be the topology generated by a basis $\mathcal{B}$ on X. Show that $(X,\mathcal{U})$ is Alexandroff if and only if arbitrary intersections of sets in B are open. (Hint: Every open set is a union of sets in B.)

2 Partially-Ordered Sets

2.1 Basic Definitions

Definition 2.1.1. A space $(X, \leq)$ is *pre-ordered* if it satisfies the following axioms.

- $x \leq y, y \leq z \Rightarrow x \leq z$ (Transitivity)
- $x \leq x$ (Reflexivity)

Moreover, it is *partially ordered* (also called a *poset*) if

- $x \leq y, y \leq x \Rightarrow x = y$ (Antisymmetricity)

Definition 2.1.2. A map $f : X \to Y$ is order-preserving if $x \leq y \Rightarrow f(x) \leq f(y)$.

Lemma 2.1.3. An Alexandroff space $(X,\mathcal{U})$ admits a pre-order structure $\leq$. Moreover, $\mathcal{U}$ is T_0 if and only if $\leq$ is partially ordered.

Proof. For every $x \in X$, define U_x to be the intersection of all open sets containing x. Note that U_x is open since X is Alexandroff. Define $x \leq y$ if $U_x \subseteq U_y$. Clearly $\leq$ is reflexive and transitive. Moreover, $x \leq y, y \leq x$ if and only if $U_x = U_y$, so $\mathcal{U}$ it T_0 if and only if $\leq$ is partially ordered. $\square$

On the other hand, given a pre-ordered set $(X, \leq)$. Define

$$U_x = \{y \mid y \leq x\}$$

Claim 2.1.4. $\{U_x\}_{x \in X}$ forms a basis on X.

Proof. Clearly $x \in U_x$ since $x \leq x$. If $x \in U_y \cap U_z$, i.e., $x \leq y$ and $x \leq z$, then $U_x \subseteq U_y$ and $U_x \subseteq U_z$ by transitivity. We have $x \in U_x \subseteq U_y \cap U_z$. Hence $\{U_x\}_{x \in X}$ is a basis. □

Define $\mathcal{U}$ to be the topology generated by the basis $\{U_x\}_{x \in X}$.

Lemma 2.1.5. $\mathcal{U}$ is Alexandroff.

Proof. For every collection of open set $\{U_{y_i}\}_{i \in I}$. Let $U = \bigcap_{i \in I} U_{y_i}$. For every $x \in U$, i.e., $x \in U_{y_i}$ for all $i \in I$. We have $x \leq y_i$ for all i, i.e., $U_x \subseteq U_{y_i}$. Thus $U_x \subseteq \bigcap_{i \in I} U_{y_i} = U$. Hence $U = \bigcup_{x \in U} U_x$ is open. Therefore $\mathcal{U}$ is Alexandroff. □

Next, we prove the following claim.

Claim 2.1.6. For every $U \in \mathcal{U}$, we have

$$U_x = \bigcap_{x \in U \in \mathcal{U}} U$$

In other words, U_x is the smallest open set containing x. In this case, $\{U_x\}_{x \in X}$ is called the *minimal basis* for $\mathcal{U}$.

Proof. For every $U \in \mathcal{U}$ containing x, express U as a union of basic sets $\bigcup_{i \in I} U_{y_i}$. There exists i such that $x \in U_{y_i}$, i.e., $x \leq y_i$. The transitivity tells us that $U_x \subseteq U_{y_i} \subseteq U$. Also, clearly $x \in U_x$. Hence $U_x = \bigcap_{x \in U \in \mathcal{U}} U$. □

Lemma 2.1.7. $\leq$ is partially ordered if and only if $\mathcal{U}$ is T_0.

Proof. We know that $x \leq y, y \leq x$ if and only if $U_x = U_y$. By the previous claim, U_x and U_y are the smallest open sets containing x, y, respectively. The result follows. □

Now that we have obtained a bijection between pre-ordered structures and Alexandroff topologies, and a bijection between partially ordered structures and Alexandroff T_0 topologies. Moreover, we can say more about maps between spaces.

Proposition 2.1.8. Let X, Y be two Alexandroff spaces, i.e., pre-ordered sets, a map $f : X \to Y$ is continuous if and only if it is order-preserving.

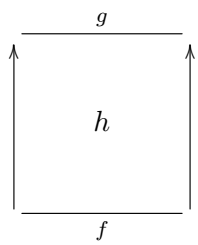
Proof. • ($\Rightarrow$): Assume f is continuous, for every $x \leq y$ in X, we have $x \in U_u \subseteq f^{-1}\left(U_{f(y)}\right)$. Thus $f(x) \in U_{f(y)}$ and therefore $f(x) \leq f(y)$.

• ($\Leftarrow$): Assume f is order-preserving, for every open set V in Y. If $f(y) \in V$, then $U_{f(y)} \subseteq V$. If $x \in U_y$, i.e., $\leq y$, we have $f(x) \leq f(y)$. Hence $f(x) \in U_{f(y)} \subseteq V$ and that $x \in f^{-1}(V)$. Therefore $f^{-1}(V) = \bigcup_{y, f(y) \in V} U_y$ is open. □

3 Algebraic Topology

3.1 Homotopy

Definition 3.1.1. Let $f, g : X \to Y$ be two continuous maps. We say f is *homotopic* to g, denoted by $f \simeq g$, if there exists a continuous map $h : X \times [0,1] \to Y$ such that $h(x,0) = f(x), h(x,1) = g(x)$. In this case h is called a *homotopy* between f and g. We also use the following cartoon to illustrate a homotopy. (Imagine $X = [0,1]$.)



Exercise 3.1.2. Show that $\simeq$ is an equivalence relation. More explicitly, show that

$$\begin{aligned} &f \simeq f && \text{(Reflexivity)} \\ &f \simeq g \Rightarrow g \simeq f && \text{(Symmetricity)} \\ &f \simeq g, g \simeq k \Rightarrow f \simeq k && \text{(Transitivity)} \end{aligned}$$

You can either define the homotopies explicitly, or play with the cartoons above.

A map $f : X \to Y$ is a *homotopy equivalence* if there exists another map $g : Y \to X$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$. In this case we say that X and Y are *homotopy equivalent*, denoted by $X \simeq Y$. (Check that homotopy equivalence is an equivalent relation.) We say a space X is *contractible* if $X \simeq *$, where $*$ is any one point space.

Exercise 3.1.3. Show that $\mathbb{R}^n$ is contractible. Note that $\mathbb{R}^n$ and $*$ are not homeomorphic (why?). Also, show that $\mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$, where $S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$ is the unit sphere in $\mathbb{R}^n$.

Fun Exercise 3.1.4. Classify all upper case alphabets up to homotopy equivalence.

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Compare this to Fun Exercise 1.3.2..

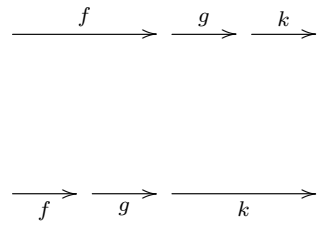
Exercise 3.1.5. Show that if $f, g : X \to Y$ are two continuous maps and Y is contractible, then $f \simeq g$.

A *based* topological space is a topological space X with a basepoint $x \in X$. A *path* in X is a continuous map $f : [0,1] \to X$, and it is called a *loop* if $f(0) = f(1)$. A *based loop* is a continuous map $f : [0,1] \to X$ such that $f(0) = f(1) = x$. A homotopy $h : X \times [0,1] \to Y$ is *basepoint preserving* if $h(x,t) = y$.

For two based loops f, g on X. their product $f \cdot g$ is defined as the concatenation of f and g. More precisely,

$$(f \cdot g)(x) = \begin{cases} f(2x) & \text{if } x < \frac{1}{2} \\ g(2x-1) & \text{if } x \geq \frac{1}{2} \end{cases}$$

Fun Exercise 3.1.6. Show that this operation is associative up to homotopy. That is, $(f \cdot g) \cdot k \simeq f \cdot (g \cdot k)$. (Hint: Play with the cartoon introduced above.)



Definition 3.1.7. (Fundamental Group) The fundamental group of a based space (X, x), denoted $\pi_1(X, x)$ is the set of homotopy equivalence classes of loops on X.

Fun Exercise 3.1.2. tells us that the operation $\cdot$ on $\pi_1(X, x)$ is associative. The following exercise checks that $\pi_1(X, X)$ satisfies the rest of the group axioms.

Exercise 3.1.8. Let $c_x : [0, 1] \to X$ be the constant map at x. Check that c_x is the identity element of the fundamental group. For every $f \in \pi_1(X, x)$, try to guess what the inverse of f looks like. (Hint: More Cartoons.)