AN EXPLORATION OF KHINCHIN'S CONSTANT

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Abstract. Every real number can be expressed as a continued fraction in the following form, with $n \in \mathbb{Z}$ and $a_i \in \mathbb{N}$ for all i.

$$x = n + \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}},$$

For this paper, we only consider numbers in the set $[0, 1] \setminus \mathbb{Q}$, so we write x as a sequence $x = [a_0, a_1, a_2, \cdots]$. Khinchin showed that the limit of the geometric mean of the first n "terms" as n grows arbitrarily large of almost any real number x both exists and is independent of x (given that x is expressed in our sequence form). The rational numbers and certain irrationals would not have this property, but Khinchin showed that these numbers exist only on a set of measure zero. This paper follows an alternate proof, originally by C. Ryll-Nardzewski, to arrive at Khinchin's result using ergodic theory, ending with a conjecture that would provide an alternate argument for a step in Ryll-Nardzewski's proof.

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1. Continued Fractions

Let us now adopt a new notation of sorts for dealing with continued fractions. Suppose that x is irrational and we are only concerned with the first n terms of the continued fraction. Then, we say $x = [a_0, a_1, a_2, \cdots, a_{n-1} + x_n]$ or $x = [a_0, a_1, a_2, \cdots, a_{n-1} + x']$ depending on the situation. The number $x = [a_1, a_2 + x_2]$ would yield the fraction

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + x_2}}.$$

The following proofs will make it immediately apparent how this notation is useful.

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Theorem 1.1. *A number $x \in (0,1)$ is rational if and only if x has a finite continued fraction.*

Proof. One direction is trivial, if x has a finite continued fraction then it is rational. Since $a_i \in \mathbb{N}$ for all i and the rationals are closed under finite division and addition, x is certainly a rational number.

Suppose that $x_0 = \frac{p_0}{q_0}$ in the interval $(0,1)$ with p_0, q_0 coprime natural numbers. Then we can say that

$$x_0 = \frac{p_0}{q_0} = \frac{1}{\frac{q_0}{p_0}} = \frac{1}{n + \frac{q_0}{p_0} - n\frac{p_0}{p_0}} = \frac{1}{n + \frac{q_0 - np_0}{p_0}}.$$

Let us define $a_0 \in \mathbb{N}$ such that $0 < q_0 - a_0 p_0 < p_0$ (since p_0 and q_0 are coprime, there is no n such that $q_0 - np_0 = 0$ or p_0 so we know that a_0 exists). Set $q_1 = p_0 < q_0$, $p_1 = q_0 - a_0 p_0 < p_0$. Note that p_1, q_1 are still coprime with $x_0 = [a_0 + x_1]$ where $x_1 = \frac{p_1}{q_1}$.

Now, we apply the same argument to x_1 and we will find that $x_1 = [a_1 + x_2]$ with $x_2 = \frac{p_2}{q_2}$ and $p_2 < p_1$,$q_2 < q_1$. Our algorithm always yields that $p_{i+1} < p_i$,$q_{i+1} < q_i$ where defined, so in at most p_0 steps, we will have found the finite continued fraction for x_0. □

The following proof is partially constructive in nature, demonstrating how to find the continued fraction of an irrational number. The argument would also work for finding the continued fraction of a rational number (as opposed to the method above), however this method does not provide a (relatively) nice manner of showing that a rational number would have a finite continued fraction.

Proposition 1.2. *Every $x \in [0,1] \setminus \mathbb{Q}$ has a unique continued fraction.*

Proof. Consider $x \in [0,1] \setminus \mathbb{Q}$. Set $x = x_0$ and $a_0 = \lfloor \frac{1}{x} \rfloor$ and $x_1 = \frac{1}{x_0} - a_0$. Note that $x_1 \in [0,1] \setminus \mathbb{Q}$ and $x_0 = \frac{1}{a_0 + x_1}$ (or $x_0 = [a_0, x_1]$). We say that $a_1 = \lfloor \frac{1}{x_1} \rfloor$ and $x_2 = \frac{1}{x_1} - a_1$, and $x_2 \in [0,1] \setminus \mathbb{Q}$ and $x_1 = [a_1, x_2]$. We continue in this manner saying that $a_{i+1} = \lfloor \frac{1}{x_i} \rfloor$ and $x_{i+1} = \frac{1}{x_i} - a_i$. Note that we can telescope backwards to find that $x = [a_0, a_1, a_2, \cdots]$. Thus, we are able to find the continued fraction for any irrational number up to as many digits as we like.

Now suppose $x = [a_0, a_1, a_2, \cdots]$ and $x = [b_0, b_1, b_2, \cdots]$. Consider $\frac{1}{x} = a_0 + [a_1, a_2, \cdots] = b_0 + [b_1, b_2, \cdots]$. Since $a_i, b_i \in \mathbb{N}$ for all i, we know that $[a_1, a_2, \cdots], [b_1, b_2, \cdots] \in (0,1)$. Now, consider $\frac{1}{x} - a_0 = [a_1, a_2, \cdots] = (b_0 - a_0) + [b_1, b_2, \cdots] \in [0,1]$. Thus, $a_0 = b_0$. Continue to apply this to $[a_1, a_2, \cdots], [b_1, b_2, \cdots]$ and we find that $a_i = b_i$ for all i. □

Proposition 1.3. *A number $x = [a_0, a_1, \cdots]$ is in $(\frac{1}{n+1}, \frac{1}{n})$ if and only if $a_0 = n$.*

Proof. Suppose $x \in (\frac{1}{n+1}, \frac{1}{n})$. Then, let $x = \frac{1}{a_0 + y}$. Since $\frac{1}{n+1} < x < \frac{1}{n}$, we can say that $n < a_0 + y < n+1$. Since $y \in (0,1)$, then $a_0 = n$. Suppose $x = \frac{1}{n+y}$ for $y \in (0,1)$. Then $n < n + y < n + 1$, so taking multiplicative inverses gives us $x \in (\frac{1}{n+1}, \frac{1}{n})$. □

We introduce here a transformation that will be very important throughout this paper and its strictly decreasing property over certain subsets of $[0,1] \setminus \mathbb{Q}$ make it very useful for the next propositions.

Definition 1.4 (Gauss-Kuzmin Operator)**.** We define the Gauss-Kuzmin Operator $T\colon [0,1]\setminus\mathbb{Q} \to [0,1]\setminus\mathbb{Q}$, as $T(x) = \frac{1}{x} - \lfloor\frac{1}{x}\rfloor$.

Quickly, note that $T([a_1, a_2, a_3, \cdots]) = [a_2, a_3, \cdots]$. This is since $\frac{1}{x} = a_1 + [a_2, a_3, \cdots]$ and $\lfloor\frac{1}{x}\rfloor = a_1$.

Lemma 1.5. *The Gauss-Kuzmin Operator is strictly decreasing over* $[\frac{1}{n+1}, \frac{1}{n}]\setminus\mathbb{Q}$.

Proof. Suppose $x < y \in [\frac{1}{n+1}, \frac{1}{n}]\setminus\mathbb{Q}$. Then, by the previous proposition, the first term of each continued fraction is n. Thus, for all $a \in [\frac{1}{n+1}, \frac{1}{n}]\setminus\mathbb{Q}$, $T(a) = \frac{1}{a} - n$. If we invert x, y, we reverse order and subtracting n from each side of the inequality does not affect this, thus $T(y) < T(x)$. Thus, T is strictly-decreasing over $[\frac{1}{n+1}, \frac{1}{n}]\setminus\mathbb{Q}$. □

Note that over these specific subsets of $[0,1]$, T also preserves 'between-ness'; that is if x is between $y, z \in (\frac{1}{n+1}, \frac{1}{n})\setminus\mathbb{Q})$, then $T(x)$ is between $T(y), T(z)$.

Proposition 1.6. *Suppose x, y are numbers such that $x = [a_1, a_2, \cdots, a_n + x']$, $y = [a_1, a_2, \cdots, a_n + y']$. Then $z = [c_1, c_2, \cdots, c_n + z']$ is between x and y if and only if $c_i = a_i$ for all $i \leq n$ and z' is between x' and y'.*

Proof. Since $x, y \in [\frac{1}{a_1+1}, \frac{1}{a_1}]\setminus\mathbb{Q}$ by Proposition 1.3, we know that necessarily z is as well. Thus, $c_1 = a_1$. Now, we apply T and find that $T(x), T(y) \in [\frac{1}{a_2+1}, \frac{1}{a_2}]\setminus\mathbb{Q}$, so again $T(z)$ is as well. We continue in this manner to find that $T^i(z)$ is between $T^i(x), T^i(y)$ for $i \leq n$ since $T^i(x), T^i(y) \in [\frac{1}{a_{i+1}+1}, \frac{1}{a_{i+1}}]\setminus\mathbb{Q}$ for $i < n$. Thus, $c_i = a_i$ for all $i \leq n$. Since $T^n(x) = x'$ (and similarly for y, z), z' is between x' and y'.

The reverse direction is easier. We have that $z = [a_1, a_2, \cdots, a_n + z']$ by our hypothesis. Without loss of generality, suppose that $x' < z' < y'$. Then, clearly $a_n + x' < a_n + z' < a_n + y'$, so $\frac{1}{a_n+x'} > \frac{1}{a_n+z'} > \frac{1}{a_n+y'}$. We can now consider these new numbers as x', y', z' with a different value of n and continue, so between-ness is clearly preserved. □

2. Gauss-Kuzmin Measure

Definition 2.1 (Gauss-Kuzmin Measure)**.** We define the Gauss-Kuzmin Measure, μ, over the interval $[0,1]$ of a set A as

$$\mu(A) = \frac{1}{\log 2}\int_A \frac{d\lambda}{1+x}.$$

with $d\lambda$ indicating we integrate with respect to Lebesgue measure.

We now show that the Gauss-Kuzmin Measure satisfies the properties of a measure. It is non-negative, the empty set has measure zero, and satisfies countable additivity over disjoint sets.

Theorem 2.2. *Gauss-Kuzmin Measure is a measure.*

Proof. Since we are integrating over only positive numbers, the integral given is positive. Since the empty set has no points, integrating over the empty set yields zero, thus the empty set has measure zero. Summing countably many integrals over disjoint sets is the same as integrating over the countable union of those sets, thus Gauss-Kuzmin measure is, in fact, a measure. □

Theorem 2.3. *Gauss-Kuzmin Measure, μ, is equivalent to Lebesgue measure, λ, over* $[0,1]$.

Proof. Suppose $\lambda(A)=0$. Then, $\mu(A)=\frac{1}{\log 2}\int_A \frac{1}{1+x}=0$, since integrals over sets of measure zero are zero. Suppose $\mu(A)=0$. Since $0<x<1$ for all $x\in A$, $\int_A \frac{1}{2}\ dx \leq \int_A \frac{dx}{1+x}$. Since $\mu(A)=0$, then $\int_A \frac{1}{2}\ dx=0$, so $\int_A 1\ dx=0$, so A must have Lebesgue measure zero. □

We now begin the steps required to show that the Gauss-Kuzmin Operator is ergodic by showing that it is measure-preserving with respect to Gauss-Kuzmin measure. With some help from Silva's text [1], it will be sufficient to show that T is measure preserving over an arbitrary interval.

Definition 2.4. A semi-ring $\mathscr{C}$ of measurable subsets of X of finite measure is said to be a *sufficient semi-ring* for $(X,\mathscr{S},\mu)$ if it satisfies the following approximation property:

$$\mu(A)=\inf\left\{\sum_{j=1}^{\infty}\mu(I_j)\colon A\subset\bigcup_{j=1}^{\infty}I_j \text{and} I_j\in\mathcal{C} for j\geq 1\right\}$$

Theorem 2.5. *Let $(X,\mathscr{S},\mu)$ be a σ-finite measure space with a sufficient semi-ring $\mathscr{C}$. If for all $I\in\mathscr{C}$, $T^{-1}(I)$ is a measurable set and $\mu(T^{-1}(A))=\mu(A)$, then T is a measure-preserving transformation.*

The above theorem is from [1] and will be cited in the next proof. Also from that text is the fact that intervals form a sufficient semi-ring over the reals, so for our purposes intervals without rationals will serve as a sufficient semi-ring over Lebesgue measurable sets.

Definition 2.6. A set A is an *almost-interval* if there exists a null set N such that $A\cup N$ is an interval in the reals.

The above definition is simply used to make talking about elements of our sufficient semi-ring easier.

Theorem 2.7. *The Gauss-Kuzmin Operator is measure-preserving with respect to Gauss-Kuzmin Measure.*

Proof. We consider only a sufficient semi-ring of almost-intervals on $[0,1]\setminus\mathbb{Q}$ to show that T is measure-preserving by the theorem from Silva's text. When one computes the integral $\frac{1}{\log 2}\int_a^b \frac{dx}{1+x}$, the result is that

$$\mu([a,b])=\frac{1}{\log 2}\log\frac{b+1}{a+1}.$$

Since the preimage of $[a,b]$ is equal to the countable disjoint union of intervals $\coprod_{n=1}^{\infty}[\frac{1}{n+b},\frac{1}{n+a}]$ for all n, we can say that $T^{-1}([a,b])$ is measurable since intervals are measurable. We know by countable subadditivity that $\mu(T^{-1}([a,b]))=\sum_{n=1}^{\infty}\mu([\frac{1}{n+b},\frac{1}{n+a}])$. Thus, we seek to show that

$$\sum_{n=1}^{\infty}\frac{1}{\log 2}\cdot\log\frac{\frac{1}{n+a}+1}{\frac{1}{n+b}+1}=\frac{1}{\log 2}\cdot\log\frac{b+1}{a+1}.$$

We immediately multiply out the coefficient of $\log 2^{-1}$ and seek to show equality of the left hand side, simplified below, with $\log \frac{b+1}{a+1}$.

$$\sum_{n=1}^{\infty} \log \frac{n+1+a}{n+a} - \sum_{n=1}^{\infty} \log \frac{n+1+b}{n+b}$$

Which is equal to

$$\lim_{m\to\infty} \log\left(\prod_{n=1}^{m} \frac{(n+1)+a}{n+a}\right) - \log\left(\prod_{n=1}^{m} \frac{(n+1)+b}{n+b}\right).$$

The products telescope, leaving us with

$$\lim_{m\to\infty} \log\left(\frac{m+1+a}{1+a}\right) - \log\left(\frac{m+1+b}{1+b}\right) = \lim_{m\to\infty} \log\frac{1+b}{1+a} + \log\frac{m+1+a}{m+1+b}.$$

This, in turn, becomes the result we desired.

$$\log\frac{1+b}{1+a} + \lim_{m\to\infty} \log\frac{m+1+a}{m+1+b} = \log\frac{b+1}{a+1}.$$

Thus, every almost-interval in the sufficient semi-ring of almost-intervals over $[0,1]\setminus\mathbb{Q}$ has a measurable preimage under T and the measure of the preimage is equal to the measure of the set, so T is measure-preserving with respect to Gauss-Kuzmin Measure. □

3. Khinchin's Constant

In this section we explore the final steps of Ryll-Nardzewski's proof of Khinchin's constant, which uses the rather major (and as of yet unsubstantiated claim) that the Gauss-Kuzmin Operator is ergodic and applies the Ergodic Theorem, cited here:

Theorem 3.1 (The Ergodic Theorem)**.** *If F is an ergodic transformation over X and f is a measurable function with respect to μ, then*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(F^k(x)) = \frac{1}{\mu(X)}\int_X f d\mu \quad \text{for } \mu\text{-almost-all } x \in X.$$

Crucial to implementing this theorem will be the following definition. However, showing that the Gauss-Kuzmin Operator is ergodic will be reserved for the next section. We do provide a quick lemma to show that the function we will use to finish Ryll-Nardzewski's proof is measurable.

Definition 3.2. A transformation $T\colon X \to X$ is *ergodic* with respect to μ if it is measure-preserving with respect to μ and for all A such that $A = T^{-1}(A)$, $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

Lemma 3.3. *The function $f(x) = \log\lfloor \frac{1}{x} \rfloor$ is measurable over $[0,1]$ with respect to μ.*

Proof. By nature of the floor function, the only outputs will be of the form $\log n$ for $n \in \mathbb{N}$. By work in our section on continued fractions, we know that $\lfloor \frac{1}{x} \rfloor = a_0$ if $x = [a_0, a_1, a_2, \cdots]$. Thus, again from the first section we know that if $x \in (\frac{1}{n+1}, \frac{1}{n}]$, then $f(x) = \log n$. Thus, in order to determine if f is measurable, we only need to examine whether or not the set $\{x \mid f(x) \leq \log n\}$ is measurable for all n. Note, that if $f(x) \in X$, then for $x = [a_0, a_1, a_2, \cdots]$ we have that $a_0 \leq n$, so $x \in (\frac{1}{n+1}, 1)$, which is a measurable set with respect to μ. $\square$

As noted in the proof, the function $f(x)$ returns the first term of the continued fraction of x. Now, we apply the Erogodic Theorem and assuming (for now) that the Gauss-Kuzmin Operator is ergodic, Khinchin's constant falls out of the calculation. Without further ado, we apply the Ergodic Theorem, giving us that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(T^k(x)) = \frac{1}{\mu([0,1])} \int_0^1 f(x)\ d\mu.$$

Primarily simplifying the left-hand side, we see that $f(T^k(x)) = \log a_k$ if $x = [a_0, a_1, a_2, \cdots]$, leaving us with the right hand side only dropping $\frac{1}{\mu([0,1]\setminus\mathbb{Q})} = 1$.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \log a_k = \lim_{n\to\infty} \log \left(\sqrt[n]{\prod_{k=1}^{n} a_k} \right) = \int_0^1 f(x)\ d\mu.$$

That is to say that the left hand side is the log of the limit as n grows large of the geometric mean of the first n terms of the continued fraction of x is a constant for μ-almost-all x. Given the equivalence to Lebesgue measure, this translates into almost-all x in the unit interval. Now, we will simplify the right-hand side. Due to the nature of f, we can say that

$$\int_0^1 f(x)\ d\mu = \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{n} \log n\ d\mu = \sum_{n=1}^{\infty} \log n \cdot \mu\left(\left[\frac{1}{n+1}, \frac{1}{n} \right] \right).$$

This follows as the measure of $[\frac{1}{n+1}, \frac{1}{n}]$ is $(\log 2)^{-1} \log \frac{\frac{1}{n}+1}{\frac{1}{n+1}+1}$. Further simplified, we plug in and get

$$\lim_{n\to\infty} \log \left(\sqrt[n]{\prod_{k=1}^{n} a_k} \right) = \sum_{n=1}^{\infty} \log n \cdot \frac{\log\left(\frac{1}{n(n+2)} + 1\right)}{\log 2}.$$

Wolfram Alpha assures me that the series converges by the comparison test, but I have not found a suitable partner for comparison. Rearranging the right hand side, we get the form presented in Ryll-Nardzewski's paper, where the convergence of the product is also unsubstantiated.

$$\log \left(\prod_{n=1}^{\infty} \left(1 + \frac{1}{n(n+2)} \right)^{\frac{\log n}{\log 2}} \right).$$

Thus, if $x = [a_0, a_1, a_2, \cdots]$ then for almost all x, the "geometric mean" of all the terms of x's continued fraction is a constant,

$$\lim_{n\to\infty} \sqrt[n]{\prod_{k=1}^{n} a_k} = \prod_{n=1}^{\infty} \left(1 + \frac{1}{n(n+2)} \right)^{\frac{\log n}{\log 2}}.$$

4. Ergodics

First, I provide the proof by Knopp, cited in Ryll-Nardzewski's paper, that will give us that the Gauss-Kuzmin Operator is ergodic. Then, I walk through work done during the summer that leads to a claim that is probably true, but I was unsuccessful in my attempts to prove which would also give the result that the Gauss-Kuzmin Operator is ergodic.

The following theorem is necessary for Knopp's proof that the Gauss-Kuzmin Operator is ergodic.

Theorem 4.1 (Lebesgue's Density Theorem)**.** *The density of A at x, $d_A(x) = \lim_{\epsilon\to 0} d_{A,\epsilon}(x)$, is equal to 1 for almost all $x \in A$, where $d_{A,\epsilon}(x) = \frac{\lambda(A\cap B_\epsilon(x))}{\lambda(B_\epsilon(x))}$.*

Theorem 4.2 (Knopp)**.** *Let T be The Gauss-Kuzmin Operator. For all A such that $A = T^{-1}(A)$, either $\lambda(A) = 0$ or $\lambda(X \setminus A) = 0$ where λ is Lebesgue measure and X is $[0,1] \setminus \mathbb{Q}$.*

Proof. Suppose $A = T^{-1}(A)$ and $\lambda(A) < 1$, equivalently $\lambda(X \setminus A) > 0$. Let $\chi(x)$ be the indicator function of A (1 if $x \in A$, 0 otherwise). Now, choose an arbitary $\xi \in X$ and suppose $\xi = [a_1, a_2, a_3, \cdots]$. Fixing some positive integer n, suppose $y = [c_0, c_1, \cdots, c_{2n}, x + c_{2n+1}]$. If $\frac{p}{q}$ is the $2n-1$th approximate of ξ and $\frac{p'}{q'}$ is the $2n$th, then $y = \frac{px+p'}{qx+q'}$.

Thus, $T^{2n+1}(y(x)) = x$, so $\chi(y) = \chi(x)$, since $T(x) \in A$ if and only if $x \in A$. Now, set $\sigma = \frac{p'}{q'}$ and $\tau = \frac{p+p'}{q+q'}$ and calculate $\tau - \sigma = \frac{1}{q'(q+q')}$. Now, the density of A at ξ is estimated in the interval (σ, τ) in order to show that it is less than 1.

$$\frac{\lambda(A \cap (\sigma,\tau))}{\lambda((\sigma,\tau))} = q'(q+q') \int_\sigma^\tau \chi(y)\ dx = q'(q+q') \int_0^1 \chi(\frac{px+p'}{qx+q'})\ \frac{dx}{(qx+q')^2}.$$

This in turn is equal to

$$q'(q+q') \int_0^1 \chi(x)\ \frac{dx}{(qx+q')^2}.$$

Since $\lambda(A) < 1$, we know that

$$\frac{\lambda(A \cap (\sigma,\tau))}{\lambda((\sigma,\tau))} = q'(q+q') \int_0^1 \chi(x)\ \frac{dx}{(qx+q')^2} < q'(q+q') \int_0^1 \frac{dx}{(qx+q')^2},$$

Thus since $\frac{1}{q'(q+q')}$ is strictly positive, there exists at least one point $d < 1$ such that

$$\frac{\lambda(A \cap (\sigma,\tau))}{\lambda((\sigma,\tau))} \le q'(q+q') \int_0^d \frac{dx}{(qx+q')^2} = 1 - \frac{q(1-d)}{qd+q'} \le 1 - \frac{1-d}{1+d} < 1.$$

Thus, for any ξ and all corresponding intervals (σ, τ) generated by having n increase, the approximate density of A is less than 1 over all such intervals at all such ξ so the set $B = \{x \mid d_A(x) = 1\}$ is empty. We note how the intervals (σ, τ) approximate the closed balls centered at ξ referenced in the statement of Lebesque's Density Theorem. Since $\tau - \sigma = \frac{1}{q'(q+q')}$, and q' is the denominator of an approximate of an irrational number, the closer the approximate the larger q' must be, so we see how within any $B_\epsilon(\xi)$ we can select n such that (σ, τ) is a subset of that ball.

Thus, by Lebesgue's Density Theorem, if $\lambda(A) > 0$, then for almost all $x \in A$, $d(x) = 1$, that is to say that if B is as defined above, then $\lambda(B) = \lambda(A)$, so we have demonstrated that if $A = T^{-1}(A)$ and $\lambda(A) < 1$, then $\lambda(A) = \lambda(\emptyset) = 0$. $\square$

Corollary 4.3. *The Gauss-Kuzmin Operator is ergodic.*

Proof. We know that T is measure-preserving with respect to μ, and we know that if $A = T^{-1}(A)$, then $\lambda(A) = 0$ or $\lambda X \setminus A = 0$. Since μ is equivalent to λ, if $\lambda(A) = 0$, then $\mu(A) = 0$. By our theorem, if $\lambda(A) \neq 0$, we know that $\lambda(X \setminus A) = 0$ so by equivalence $\mu(X \setminus A) = 0$. Thus, T is ergodic. $\square$

The following statements and proofs are those that I worked out in an effort to prove Knopp's theorem in a different manner. For many of these, T is no longer the Gauss-Kuzmin Operator but simply a general transformation and μ is a relevant measure.

Proposition 4.4. *If $T: X \to X$ is measure-preserving and for all A with $\mu(A) > 0$, $\mu(X \setminus \cup_{n=1}^{\infty} T^{-n}(A)) = 0$, then T is ergodic.*

Proof. Note that the following alternate condition has an if and only if relation with ergodicity: if $\mu(A) > 0$ and $T^{-1}(A) = A$, then $\mu(X \setminus A) = 0$. Consider any set A such that $\mu(A) > 0$ and $T^{-1}(A) = A$. By our hypothesis, $\mu(X \setminus \cup_{n=1}^{\infty} T^{-n}(A)) = 0$. Note, however, that since $T^{-1}(A) = A$, we can simply induct and say that for all n, $T^{-n}(A) = A$. Thus, $\mu(X \setminus A) = 0$, so T is ergodic by our alternate condition. $\square$

Proposition 4.5. *If $T: X \to X$ is measure-preserving and for any two sets A, B with positive measure there exists a natural number n such that $\mu(T^{-n}(A) \cap B) > 0$, then for all A with $\mu(A) > 0$, $\mu(X \setminus \cup_{n=1}^{\infty} T^{-n}(A)) = 0$.*

Proof. Assume there exists A with $\mu(A) > 0$ such that $\mu(X \setminus \cup_{n=1}^{\infty} T^{-n}(A)) > 0$. Set $B = X \setminus \cup_{n=1}^{\infty} T^{-n}(A)$. Thus, for all n $T^{-n}(A) \cap B = \emptyset$, so for all n $\mu(T^{-n}(A) \cap B) = 0$, a contradiction of our hypothesis. $\square$

Now, the goal is to show that the Gauss-Kuzmin Operator satisfies the latter of these two (implying that it is ergodic, since we already have measure-preserving). What we can show is that for any interval, and thus any non-empty open set, B and any measurable set A with $\mu(A) > 0$, there exists n such that $\mu(T^{-n}(A) \cap B) > 0$. This leaves us with the following conjecture:

Conjecture 4.6. *Suppose $T: X \to X$ is measure-preserving. If for any measurable set A with positive measure and any non-empty open set B there exists n such that $\mu(T^{-n}(A) \cap B) > 0$, then for any two measurable sets of positive measure A and B', there exists m such that $\mu(T^{-m}(A) \cap B') > 0$.*

If proven, this conjecture would show that T is ergodic by 4.5. Intuitively, this conjecture makes sense as measurable sets can be nicely approximated above by open sets (by definition), however I was unable to conceive a proof. Mentioned in the conjecture, however, is an unsubstantiated claim that I can prove.

Lemma 4.7. *Every interval contains at least one sub-interval of the form*

$$([a_0, a_1, \cdots, a_{n-1}, a_n], [a_0, a_1, \cdots, a_{n-1}, a_n \pm 1])$$

.

Proof. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we know that every interval contains a subinterval that approximates it but has rational end points. Thus, we consider only intervals (a, b) with $a, b \in \mathbb{Q}$. Suppose that $a = [a_0, a_2, \cdots, a_n]$. Then consider the region $([a_0, a_2, \cdots, a_n - 1], [a_0, a_2, \cdots, a_n + 1])$. If b is not in that region, then we are done. If b is in that region, then we know by work in our continued fraction section that $b = [a_0, a_1, \cdots, a_n, \cdots, b_m]$. Thus, both $[a_0, a_1, \cdots, b_m \pm 1]$ are in that region, and since one is less than b and the other is greater than b, one must necessarily be in (a, b), so we have a region of the desired form. Note that if $b_m = 1$, then we can simply have $b_{m-1} + 1$ be the new b_{m-1} and disregard b_m. □

Theorem 4.8. *If A is a set of positive measure, and B is a non-empty open set, then there exists n such that $B \cap T^{-n}(A)$ has positive measure, where T is the Gauss-Kuzmin Operator.*

Proof. We simply show that for every set of the form

$$([a_0, a_1, \cdots, a_{n-1}, a_n], [a_0, a_1, \cdots, a_{n-1}, a_n + 1]) = (a, b)$$

(the upper and lower bounds may not be in that order, depending on n), $T^{-n}(A) \cap (a, b)$ has positive measure. What we will show is that for any set A of positive measure and every n, the set $\{\frac{1}{n+x} \mid x \in A\}$ has positive measure. Thus, we can start with A and a_n, then take the resulting set of positive measure as our new A and use a_{n-1}, etc., again yielding a set of positive measure until we have performed this operation n times.

Let us say that $A_n = \{\frac{1}{n+x} \mid x \in A\}$. Since $\mu(A) = \frac{1}{\log 2} \int_A \frac{d\lambda}{1+x}$ is positive, we know that $\lambda(A)$ is also positive, where λ denotes Lebesgue measure. Now, let us consider $\frac{1}{\log 2}\mu(A_n) = \int_{A_n} \frac{d\lambda}{1+y}$. Using a change of variables, we find that

$$\log 2 \cdot \mu(A_n) = \int_A \frac{1}{1 + \frac{1}{n+x}} \left| \frac{-1}{(n+x)^2} \right| \, d\lambda.$$

This in turn yields

$$\log 2 \cdot \mu(A_n) = \int_A \frac{d\lambda}{(n+x)^2 + n + x}.$$

Thus, as a lowerbound for the measure of $\log 2 \cdot \mu(A_n)$, we have

$$\frac{1}{(n+1)^2 + (n+1)} \int_A 1 \, d\lambda = \frac{\lambda(A)}{(n+1)(n+2)} > 0.$$

In terms of Gauss-Kuzmin Measure, we know that $\int_A \frac{d\lambda}{1+x} < \int_A 1 \, d\lambda$, so we have that the following is also a lowerbound

$$\frac{\log 2 \cdot \mu(A)}{(n+1)(n+2)} < \mu(A_n).$$

Thus the condition we need has been shown. □

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