BROWNIAN MOTION AND LIOUVILLE'S THEOREM

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Abstract. Probability theory has many deep and surprising connections with the theory of partial differential equations. We explore one such connection, namely the proof of Liouvilles theorem by means of stochastic integration.

Contents

1. Introduction

In this paper, we prove Liouville's theorem from complex analysis by using results about Brownian motion and stochastic integration. While it is certainly not the easiest proof of Liouville's theorem, it demonstrates the intricate connections between probability and partial differential equations.

We assume the reader is familiar with basic abstract measure theory at the level of Kolmogorov [2]. We also assume the reader is familiar with basic results from measure-theoretic probability theory. A reader unfamiliar with measure-theoretic probability theory can consult Williams [6] up to the section on conditional expectation. Finally, throughout this paper we shall use a common probabilists shorthand "a.s" in place of "almost surely".

The proof of Liouville's theorem via Brownian motion will proceed by showing the following facts:
(1) 2-dimensional Brownian motion is neighborhood recurrent, and therefore dense

Date: August 29, 2012.

in the plane.
(2) If f is an analytic function, then f composed with Brownian motion is a time change of Brownian motion.

Section 2 addresses the necessary background to show that 2-dimensional Brownian motion is dense in the plane. This section begins with the definition and basic properties of Brownian motion, then discusses the solution to Dirichlet's problem using Brownian motion, which then leads directly into the density of Brownian motion.

Section 3 develops the machinery of Stochastic calculus, and in particular, Ito's formula, which is used in the proof of statement (2).

Section 4 addresses local martingales and Levy's theorem. Statement (2) is a corollary of Levy's theorem. Finally, section 4 concludes by summarizing the results from the preceding sections to prove Liouville's theorem from complex analysis.

2. Brownian Motion

2.1. **Definition and basic properties.**

Definition 2.1. A d-dimensional Brownian motion is a stochastic process $B_t : \Omega \to \mathbb{R}^d$ from the probability space $(\Omega, \mathscr{F}, P)$ to $\mathbb{R}^d$ such that the following properties hold:
(1) [Independent Increments] For any finite sequence of times $t_0 < t_1 < \cdots < t_n$, the distributions $B(t_{i+1}) - B(t_i), i = 1, \ldots, n$ are independent.
(2) [Stationary] For any pair $s, t \geq 0$,

$$P(B(s+t) - B(s) \in A) = \int_A \frac{1}{(2\pi t)^{d/2}} e^{-|x|^2/2t} dx.$$

(3) The parameterization function $t \mapsto B_t$ is continuous a.s.
Standard Brownian motion is Brownian motion where $B_0(\omega) = 0$.

Property (1) is known as the property of independent increments. Note that it is the increments rather than the values of Brownian motion itself that are independent. Property (2) states that the increments are distributed normally with mean 0 and covariance tI, that is, the covariance matrix $\left(E(X_i - \mu_i)(X_j - \mu_i)\right)_{i,j}$ equals tI, which implies the increments in each dimension are normally distributed with mean 0 and variance t. Property (3) is the desired continuity property. Intuitively, we can think of Brownian motion as "random" motion of a particle in liquid, where the future motion of the particle at any given time is not dependent on the past, but its position is certainly dependent on its past and current information.

There are two ways of thinking about Brownian motion. The first is as a stochastic process. Under this interpretation, one thinks of Brownian motion as a sequence of random variables $B_t(\omega)$ indexed by time from the probability space Ω to $\mathbb{R}^d$. The second interpretation is of a path of Brownian motion. Under this interpretation, one fixes an ω and considers the function $B_\omega(t)$, more commonly denoted $B(t)$, from R_+ to $\mathbb{R}^d$. Probabilist use both B_t and $B(t)$ to denote Brownian motion often interchanging the two. We adopt this common practice depending on which interpretation we wish to emphasize.

For completeness, we define bounded variation and quadratic variation, which are properties that guarantee that a function does not vary too wildly.

Definition 2.2. A right-continuous function $f : [0,t] \to \mathbb{R}$ is a function of bounded variation if

$$V_f^{(1)}(t) := \sup_{k\in\mathbb{N}} \sum_{j=1}^{k} |f(t_j) - f(t_{j-1})| < \infty,$$

where we take the supremum over all $k \in \mathbb{N}$ and partitions $0 = t_0 \le t_1 \le \cdots \le t_k = t$.

Definition 2.3. The quadratic variation of a stochastic process $X(t)$ is given by

$$[X]_t := \lim_{||P||\to 0} \sum_{k=1}^{n} (X(t_k) - X(t_{k-1}))^2$$

where $0 = t_0 \le t_1 \le \cdots \le t_n = t$ is a partition of $[0,t]$ and $||P||$ denotes the length of the largest partition $[t_{k-1}, t_k]$.

Theorem 2.4. *Brownian motion exists.*

A constructive proof of this non-trivial fact may be found in Peres and Morters in [4] as well as in Durrett [1].

Proposition 2.5. *The following are some of the basic properties of Brownian motion:*
(1) Brownian motion is nowhere differentiable.
(2) Brownian motion is Holder continuous of order $\alpha < 1/2$.
(3) Brownian motion has unbounded variation.
(4) Brownian motion has finite quadratic variation.

Brownian motion also possesses two important shift properties, which we list in the following proposition.

Proposition 2.6. *Shift Properties of standard Brownian motion.*
(1) [Scale Invariance] Suppose $\{B(t) : t \ge 0\}$ is a standard Brownian motion and let $a > 0$, then $\{\frac{1}{a}B(a^2t) : t \ge 0\}$ is also a standard Brownian motion.
(2) [Time inversion] Suppose $\{B(t) : t \ge 0\}$ is a standard Brownian motion, then the process $\{tB(\frac{1}{t}) : t \ge 0\}$ is also a standard Brownian motion.

Finally, we state without proof a two basic fact about Brownian motion that will be used later.

Lemma 2.7. *If $f : [0,1] \to \mathbb{R}$ is a continuous function such that $f(0) = 0$, then for standard Brownian motion $B(t)$ and any $\epsilon > 0$,*

$$\mathbb{P}\left\{\sup_{0\le t\le 1} |B(t) - f(t)| < \epsilon\right\} > 0.$$

Lemma 2.8. *Almost surely,*

$$\limsup_{n\to\infty} \frac{B(n)}{\sqrt{n}} = +\infty$$

and

$$\liminf_{n\to\infty} \frac{B(n)}{\sqrt{n}} = -\infty.$$

2.2. **Markov property.** Brownian motion possesses a particularly important property of random processes called the Markov property. Intuitively, the Markov property is simply the idea that the future state of a process depends only on the present time t and not on any previous time $s < t$. The simplest example of the Markov property is in the discrete case of a random walk. Brownian motion also satisfies the Markov property:

Theorem 2.9 (Markov property)**.** *Suppose that* $\{B(t) : t \geq 0\}$ *is a Brownian motion started at* $x \in \mathbb{R}^d$*. Fix* $s > 0$*, then the process* $\{B(t+s) - B(s) : t \geq 0\}$ *is a Brownian motion starting at the origin and independent of* $\{B(t) : 0 \leq t \leq s\}$*.*

Proof. This follows immediately from the the definition of Brownian motion. □

Brownian motion also satisfies the strong Markov property, which is the Markov property relative to the underlying structure on a space. A probability space is equipped with a time-sensitive structure called a filtration.

Definition 2.10. A filtration on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is a collection of sub σ-algebras $\{\mathscr{F}(t) : t \geq 0\}$ such that $\mathscr{F}(s) \subset \mathscr{F}(t) \subset \mathscr{F}$ for $s < t$.

Definition 2.11. A stochastic process $\{X(t) : t \geq 0\}$ from a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ equipped with a filtration $\mathscr{F}(t)$ is adapted if each $X(t)$ is measurable with respect to $\mathscr{F}(t)$.

Definition 2.12. We now define notation for several filtrations that will be used frequently:

- $\mathscr{F}^0(t) = \sigma(B(s) : 0 \leq s \leq t)$ is the sigma algebra such that all Brownian motions less than t are measurable.
- $\mathscr{F}^+(t) = \bigcap_{s>t} \mathscr{F}^0(s)$.

The first filtration is the smallest filtration that makes Brownian motion adapted. The second filtration is the first filtration plus an infinitesimal extension into the future. Observe that Brownian motion is also adapted to the second filtration since $\mathscr{F}^0(t) \subset \mathscr{F}^+(t)$ for all t and no significant sets were added. In general, we prefer to work with the second filtration, since it is right-continuous.

The Markov property also holds for a particularly important class of random times called stopping times. A stopping time T is a random time such that we can decide whether $T \leq t$ simply by knowing the states of the stochastic process until time t. The simplest example of a stopping time is the first occasion of a given event.

Definition 2.13. A random variable $T : \Omega \to [0, \infty]$ defined on a filtered probability space is called a stopping time with respect to the filtration $\mathscr{F}(t)$ if the set $\{x \in \Omega : T(x) \leq t\} \in \mathscr{F}(t)$ for all t.

Proposition 2.14. *The following are some basic facts about and examples of stopping times.*

1. *Every time* $t \geq 0$ *is a stopping time with respect to every filtration* $\mathscr{F}(t)$*.*
2. *Stopping times are closed under limits, i.e., if* T_n *is an increasing sequence of stopping times with respect to a fixed filtration* $\mathscr{F}(t)$*, then* $\lim_{n\to\infty} T_n =: T$ *is also a stopping time.*

(3) *Let T be a stopping time with respect to $\mathscr{F}(t)$, then the discrete times given by*
$$T_n = \frac{m+1}{2^n}$$
if $\frac{m}{2^n} \le T < \frac{m+1}{2^n}$ are a stopping times.

(4) *Every stopping time T with respect to $\mathscr{F}^0(t)$ is also a stopping time with respect to $\mathscr{F}^+(t)$.*

(5) *The first hitting time $T = \inf\{t \ge 0 : B(t) \in H\}$ of a closed set H is a stopping time with respect to $\mathscr{F}^0(t)$ and hence $\mathscr{F}^+(t)$ too.*

(6) *The first hitting time in an open set H is a stopping time with respect to the filtration $\mathscr{F}^+(t)$ but not necessarily with respect to $\mathscr{F}^0(t)$.*

(7) *Suppose a random time T satisfies $\{x \in \Omega : T(x) < t\} \in \mathscr{F}(t)$ and $\mathscr{F}(t)$ is a right-continuous filtration, then T is a stopping time with respect to $\mathscr{F}(t)$.*

Proof. All of these facts follow immediately from the definitions by expanding out the sets in terms of unions and intersections. We prove the last statement as illustration. Let T be a random time with the appropriate properties and $\mathscr{F}(t)$ be a right-continuous filtration. Then
$$\{x \in \Omega : T(x) \le T\} = \bigcap_{k=1}^{\infty} \left\{x \in \Omega : T(x) < t + \frac{1}{k}\right\} \in \bigcap_{n=1}^{\infty} \mathscr{F}\left(t + \frac{1}{n}\right) = \mathscr{F}(t)$$
by properties of intersection and right continuity of the filtration. □

Definition 2.15. Let T be a stopping time. Then we define
$$\mathscr{F}^+(T) := \{A \in \mathscr{F} : A \cap \{x \in \Omega : T(x) \le t\} \in \mathscr{F}^+(t)\ \forall t \ge 0\}.$$

Intuitively, this is the collection of events that happened before a stopping time T.

We now state and prove the important strong Markov property of Brownian motion.

Theorem 2.16. *Let T be a stopping time with respect to the filtration $\mathscr{F}^+(t)$. Let f be a bounded measurable function, then*
$$\mathbb{E}_x[f \circ \theta_T \mid \mathscr{F}^+(T)] = \mathbb{E}_{B(T)}[f].$$

Proof. The expectation on the right hand side is the composition of functions
$$\omega \mapsto B_{T(\omega)}(\omega) \mapsto E_{B(T)}(f).$$
The first function is measurable with respect to $\mathscr{F}^+(T)$ and the second is Borel measurable, so the composition is $\mathscr{F}^+(T)$ measurable. Therefore, it is sufficient to show that $\mathbb{E}_x[h \cdot (f \circ \theta_T)] = \mathbb{E}_x[h \cdot \mathbb{E}_{B(T)}[f]]$ for every bounded $\mathscr{F}^+(T)$-measurable function h. By the monotone class theorem from probability theory, it is sufficient to prove the above for the special class of functions
$$f(\omega) = \prod_{k=1}^{n} f_k(B_{t_k}(\omega))$$
where $n \in \mathbb{N}, 0 < t_1 < t_2 < \cdots < t_n$ and each f_k is a bounded continuous function. For the class of functions with this form the expected value function $x \mapsto \mathbb{E}_x[f]$ is bounded and continuous.

We proceed by approximating the stopping time T by a sequence of dyadic stopping times T_n defined by:

$$T_n(\omega) = \begin{cases} \frac{k+1}{2^n} & : \frac{k}{2^n} < T(\omega) < \frac{k+1}{2^n} \ \forall k \\ \infty & : T(\omega) = \infty \end{cases}$$

For each $t > 0$, since T is a stopping time, the set $\{\omega : T(\omega) < \frac{k+1}{2^n}\} \in \mathscr{F}^+(\frac{k+1}{2^n})$, so we can write

$$\{T_n \leq t\} = \bigcup_{k: \frac{k+1}{2^n} \leq t} \left\{ \frac{k}{2^n} \leq T < \frac{k+1}{2^n} \right\} \in \mathscr{F}^+(t).$$

By the definition of stopping time, each T_n is a stopping time and T_n decreases to T. Since $t \mapsto B_t(\omega)$ is a.s continuous, $lim_{n\to\infty} B_{T_n} = B_T$, hence

$$\lim_{n\to\infty} \mathbb{E}_{B_{T_n}}[f] = \mathbb{E}_{B_T}[f].$$

Having established the approximation, it remains to compute the expected value $\mathbb{E}_{B_{T_n}}[f]$. Observe that $\{T < \infty\} = \{T_n < \infty\}$, so

$$\mathbb{E}_x[h \cdot \mathbb{E}_{B_T}[f] \mid T < \infty] = \mathbb{E}[h \circ \lim_{n\to\infty} \mathbb{E}_{B_{T_n}}[f] \mid T_n < \infty] = \lim_{n\to\infty} \mathbb{E}_x[h \cdot \mathbb{E}_{B_{T_n}}[f] \mid T_n < \infty].$$

By summing over disjoint discrete sets, applying the regular Markov property, and Fubini's theorem,

$$\begin{aligned} \mathbb{E}_x[h \cdot \mathbb{E}_{B(T_n)}[f] \mid T_n < \infty] &= \sum_{k=0}^{\infty} \mathbb{E}_x \left[h \cdot \mathbb{E}_{B(\frac{k+1}{2^n})}[f]; \mid \frac{k+1}{2^n} \right] \\ &= \sum_{k=0}^{\infty} \mathbb{E}_x \left[h \cdot (f \circ \theta_{(k+1)2^{-n}}) \mid \frac{k+1}{2^n} \right] \\ &= \mathbb{E}_x[h \cdot (f \circ \theta_{T_n}) \mid T_n < \infty]. \end{aligned}$$

Finally, since f is specially chosen such that the map $t \mapsto f(\theta_t \omega)$ is continuous a.s., the limit $\lim_{n\to\infty} f(\theta_{T_n}\omega) = f(\theta_T \omega)$ holds. By the dominated convergence theorem,

$$\lim_{n\to\infty} \mathbb{E}_x[h \cdot (f \circ \theta_{T_n}) \mid T_n < \infty] = \mathbb{E}_x[h \cdot (f \circ \theta_T) \mid T < \infty].$$

□

The following equivalent formulation of the strong Markov property is often convenient and illustrates the intuition better.

Theorem 2.17 (Strong Markov Property)**.** *For every almost surely finite stopping time T, the process $\{B(T+t) - B(T) : t \geq 0\}$ is a standard Brownian motion independent of $\mathscr{F}^+(T)$.*

Proof. Define $\tau_x : \Omega \to \Omega$ by $(\tau_x \omega)(t) := \omega(t) - x$. Then for each $x = B_T(\omega)$, set $\theta_{T(\omega)} = \tau_x[\theta_{T(\omega)}\omega]$. If h is a bounded $\mathscr{F}^+$ measurable function and f a bounded $\mathscr{F}$ measurable function, then

$$\mathbb{E}[h \cdot f(B_T)] = \mathbb{E}[h \cdot f(\tau_{B(T)}\theta_T)] = \mathbb{E}[h \cdot \mathbb{E}[f(\tau_{B(T)}\theta_T) \mid \mathscr{F}^+(T)] = \mathbb{E}[h \circ f(B_T)].$$

Letting $f(x) = \mathbb{E}_x[f(\tau_x)]$, and by the shift invariance of Brownian motion,

$$f(x) = \mathbb{E}_x[f(\tau_x)] = \mathbb{E}_0[f].$$

Hence,

$$\mathbb{E}[h \cdot f(B_t)] = \mathbb{E}[h \cdot \mathbb{E}_0[f]] = \mathbb{E}[h]\mathbb{E}[f].$$

□

2.3. **Harmonic functions and Dirichlet Problem.** The recurrence of Brownian motion is linked with harmonic functions and the solution to the Dirichlet problem.

Definition 2.18. Let U be a connected open set $U \subset \mathbb{R}^d$ and ∂U be its boundary. A function $u : U \to \mathbb{R}$ is harmonic if $u \in C^2$ and for any $x \in U$,

$$\Delta u(x) := \sum_{i=1}^{d} \frac{\partial^2 u}{dx_j^2}(x) = 0.$$

The following are useful equivalent formulation of harmonic functions. These formulations are well known so we state them without proof.

Theorem 2.19 (Harmonic function)**.** *Let $U \subset \mathbb{R}^d$ be a connected open set and $u : U \to \mathbb{R}$ be measurable and locally bounded. Then the following are equivalent:*
(i) u is harmonic
(ii) for any ball $B_r(x) \subset U$,

$$u(x) = \frac{1}{\lambda(B_r(x))} \int_{B_r(x)} u(y) d\lambda(y)$$

where λ is Lebesgue measure on $\mathbb{R}^d$.
(iii) for any ball $B_r(x) \subset U$,

$$u(x) = \frac{1}{\sigma_{x,r}(\partial B_r(x))} \int_{\partial B_r(x)} u(y) d\sigma_{x,r}(y)$$

where $\sigma_{x,r}$ is the surface measure on the $d-1$-dimensional boundary $\partial B_r(x)$.

We now list two of the most important properties of harmonic functions.

Theorem 2.20 (Maximum Principle)**.** *Suppose $u : U \to \mathbb{R}$ is a harmonic function on a connected open set $U \subset \mathbb{R}^d$.*
(i) If u attains its maximum on U, then it is identically the constant function.
(ii) If u is continuous on $\overline{U}$ and U is a bounded set, then $\sup_{x \in \overline{U}} u(x) = \sup_{x \in \partial U} u(x)$.

Theorem 2.21 (Continuation)**.** *Let $u_1, u_2 : U \to \mathbb{R}$ be harmonic functions on a bounded connected open set $U \subset \mathbb{R}^d$ and continuous on the closure $\overline{U}$, then suppose that $u_1 = u_2$ on ∂U, then $u_1 = u_2$ over U.*

These two theorems are familiar to anyone who has taken a class on complex analysis and hence we do not prove them here.

There are, however, surprising links between harmonic functions and hitting time of Brownian motion.

Theorem 2.22. *Suppose U is a connected open set and $\{B(t) : t \geq 0\}$ is Brownian motion that starts inside U. Define $\tau := \tau(\partial U) = \min\{t \geq 0 : B(t) \in \partial U\}$ be the first time the Brownian motion hits the boundary. Let $\varphi : \partial U \to \mathbb{R}$ be a measurable function. Suppose that a function $u : U \to \mathbb{R}$ satisfies the property that for every $x \in U$,*

$$u(x) = \mathbb{E}_x[\varphi(B(\tau)) \mathbf{1}_{\{}\tau < \infty\}]$$

is locally bounded, then u is a harmonic function.

Proof. Fix any ball $B_\delta(x) \subset U$. Define $\tilde{\tau} = \inf\{t > 0 : B(t) \notin B_\delta(x)\}$ to be the first exit time. Since φ is a measurable function, by the strong Markov property (2.16),

$$\mathbb{E}_x[\mathbb{E}_x[\varphi(B(\tau))\mathbf{1}\{\tau < \infty\} \mid \mathscr{F}^+]] = \mathbb{E}_x[u(B(\tilde{\tau}))].$$

The first expression is simply $u(x)$ by the tower property of conditional expectation. The second expression is the expected value taken over the boundary of the ball (since the first exit time must occur at the boundary of the sphere). Hence

$$u(x) = \int_{\partial B_\delta(x)} u(y)\sigma_{x,\delta}$$

where $\sigma_{x,\delta}$ is uniform distribution on the boundary $\partial B_\delta(x)$. Thus, since u is also locally bounded, u is harmonic. □

We wish to find solutions to the Dirichlet problem given a specific boundary condition. We are given a continuous function φ on a boundary ∂U of a connected open set U, we wish to find another function $u : \overline{U} \to \mathbb{R}$ such that $u(x) = \varphi(x)$ on ∂U. Such a function u is called a solution to the Dirichlet problem given the boundary condition φ. Solutions to DirichLet us problem exist for a class of open sets U that are suitably nice.

Definition 2.23. Let $U \subset \mathbb{R}^d$ be a connected open set. We say U satisfies the Poincare cone condition at $x \in \partial U$ if there exists a cone V with base at x and opening angle $\alpha > 0$ and $h > 0$ such that $V \cap B_h(x) \subset U^c$.

Finally, we arrive at the theorem which states the existence of solutions to DirichLet us problem. Before, we prove the existence theorem, we shall prove a Lemma.

Theorem 2.24 (DirichLet us Theorem)**.** *Let $U \subset \mathbb{R}^d$ be a bounded connected open set such that every boundary point satisfies the Poincare cone condition, and suppose that φ is a continuous function on ∂U. Define $\tau(\partial U) = \inf\{t > 0 : B(t) \in \partial U\}$ be the first hitting time, which is a surely finite stopping time. Then the function $u : \overline{U} \to \mathbb{R}$ given by*

$$u(x) = \mathbb{E}_x[\varphi(B(\tau(\partial U)))]$$

is the unique continuous function that is harmonic extension of φ, that is, $u(x) = \varphi(x)$ for all $x \in \partial U$.

Lemma 2.25. *Let $0 < \alpha < 2\pi$ and $C_0(\alpha) \subset \mathbb{R}^d$ be a cone based at the origin with opening angle α. Let $a = \sup_{x \in \overline{B_{1/2}(0)}} \mathbb{P}_x\{\tau(\partial B_1(0)) < \tau(C_0(\alpha))\}$. Then $a < 1$ and for any pair of positive integers k, h,*

$$\mathbb{P}_x\{\tau(\partial B_h(z)) < \tau(C_z(\alpha))\} \leq a^k$$

for all x, z such that $|x - z| < s^{-k}h$.

Proof. Suppose that $x \in B_{2^{-k}}(0)$. Then clearly, there is a nontrivial probability for Brownian motion to reach the boundary of the ball before hitting the cone, so $a < 1$. Then by the strong Markov property,

$$\mathbb{P}_x\{\tau(\partial B_0(1)) < \tau(C_0(\alpha))\} \leq \prod_{i=0}^{k-1} \sup_{x \in B_{2^{-k+i}}(0)} \mathbb{P}_x\{\tau(\partial B_{2^{-k+i+1}}(0)) < \tau(C_0(\alpha))\} = a^k.$$

For any positive integer k and h, by scaling, we get

$$\mathbb{P}_x\{\tau(\partial B_h(z)) < \tau(C_z(\alpha))\} \leq a^k$$

for all x such that $|x - z| < 2^{-k}h$. □

Proof of DirichLet us Theorem. Uniqueness follows trivially from harmonic continuation. Moreover, since the stopping time is almost surely finite, the function u is locally bounded and hence harmonic on U by Theorem 2.22.

It remains to show that the Poincare cone condition guarantees that u is continuous on the boundary. To do so, fix $z \in \partial U$. Then there is a cone $C_z(\alpha)$ based at z with angle α such that $C_z(\alpha) \cap B_h(z) \subset U^c$. By Lemma 2.25, for positive integers k, h, we have

$$\mathbb{P}_x\{\tau(B_h(z)) < \tau(C_z(\alpha))\} \leq a^k$$

for all $|x - z| < 2^{-k}h$. In particular, given $\epsilon > 0$, by fixing $\delta < h$, we find that if $|y - z| < \delta$, then $|\varphi(y) - \varphi(z)| < \epsilon$. Thus, for all $x \in \overline{U}$ such that $|z - x| < 2^{-k}\delta < 2^{-k}h$,

$$|u(x) - u(z)| = |\mathbb{E}_x\varphi(B(\tau(\partial U))) - \varphi(z)| \leq \mathbb{E}_x|\varphi(B(\tau(\partial U))) - \varphi(z)|.$$

Therefore, if the Brownian motion hits the cone $C_z(\alpha)$, which is in U^c, before the sphere $\partial B_\delta(z)$, then $|z - B(\tau(\partial U))| < \delta$ and $\varphi(B(\tau(\partial U)))$ is close to $\varphi(z)$. This implies

$$2||\varphi||_\infty \mathbb{P}_x\{\tau(\partial B_\delta(z)) < \tau(C_z(\alpha))\} + \epsilon \mathbb{P}_x\{\tau(\partial U) < \tau(\partial B_\delta(z))\} \leq 2||\varphi||_\infty a^k + \epsilon.$$

Since the bound can be made arbitrarily small, we have continuity on the boundary. □

2.4. **Recurrence.**

Definition 2.26. Brownian motion $\{B(t) : t \geq 0\}$ is:
(1) transient if $\lim_{t\to\infty}|B(t)| = \infty$ a.s.
(2) point recurrent if a.s for every $x \in \mathbb{R}^d$, there is an increasing sequence t_n such that $B(t_n) = x$ for all $n \in \mathbb{N}$.
(3) neighborhood recurrent if a.s for every $x \in \mathbb{R}^d$ and $\epsilon > 0$, there exists an increasing sequence t_n such that $B(t_n) \in B_\epsilon(x)$ for all $n \in \mathbb{N}$.

The recurrence or transience of Brownian motion is characterized up to the dimension, as evident by the next theorem. It is not hard to believe that Brownian motion is neighborhood recurrent, because if we think of Brownian motion as a random path in $\mathbb{R}^2$ with time going to infinity, then it is plausible that the set of all paths will fill $\mathbb{R}^2$. Moreover, this intuition should apply to higher dimensions; however, this is incorrect, as evidenced by the following classification theorem.

Theorem 2.27. *(1) Planar Brownian motion $B(t) : \Omega \to \mathbb{R}^2$ is neighborhood recurrent but not point recurrent.*
(2) For $d \geq 3$, Brownian motion $B(t) : \Omega \to \mathbb{R}^d$ is transient.

Remark 2.28. Linear Brownian motion is point recurrent, which follows by an argument similar to the one for the neighborhood recurrence of planar Brownian motion.

Before we prove this result, we give some useful definitions and lemmas. In our proof of the recurrence of Brownian motion, we will consider the exit probability of Brownian motion from the annulus $A = \{x \in \mathbb{R}^d \mid r < |x| < r\}$. This will be closely related to the Dirichlet problem.

For simplicity of notation, we will let $r := (r, 0, 0, \ldots, 0)$. We define stopping times

$$T_r := \tau(\partial B_r(0)) = \inf\{t > 0 \mid |B(t)| = r\}$$

for $r > 0$, which are the first time the Brownian motion hits the $d-1$ dimensional shell of radius r. Then the first exit time from the annulus A is simply $T := \min\{T_r, T_R\}$.

Lemma 2.29.

$$\mathbb{P}_x\{T_r < T_R\} = \frac{u(R) - u(x)}{u(R) - u(r)}$$

Proof. Since the annulus satisfies the Dirichlet condition, applying Theorem 2.24 to the boundary conditions defined by $u : \overline{A} \to \mathbb{R}$ restricted to ∂A, we get that

$$u(x) = \mathbb{E}_x[u(B(\tau(\partial A)))] = \mathbb{E}_x[u(B(T))] = u(r)\mathbb{P}_x\{T_r < T_R\} + u(R)(1 - \mathbb{P}_x(T_r < T_R)).$$

By elementary algebraic operations, we get that

$$\mathbb{P}_x\{T_r < T_R\} = \frac{u(R) - u(x)}{u(R) - u(r)}.$$

□

Now it remains to find explicit solutions to the boundary condition.

Lemma 2.30. *Let $u(r), u(R)$ be fixed and constant on the boundary of the annulus. Then the Dirichlet solution to this boundary condition is given by:*

$$u(x) = \begin{cases} |x| & : d = 1 \\ 2\log|x| & : d = 2 \\ |x|^{2-d} & : d \geq 3 \end{cases}$$

Proof. By the definition of u, which is spherically symmetrical, there is a function $v : [r, R] \to \mathbb{R}$ such that $u(x) = v(|x|^2)$. Expressing the first and second partial derivatives of v, we get $\frac{\partial}{\partial x} u(x) = \frac{\partial}{\partial x} v(|x|^2) = v'(|x|^2)2x_i$ and $\frac{\partial^2}{\partial x^2} u(x) = \frac{\partial^2}{\partial x^2} v(|x|^2) = v''(|x|^2)4x_i^2 + 2v(|x|^2)$. Then, the condition that $\Delta u(x) = 0$ is equivalently to the statement that

$$0 = \sum_{i=1}^{d} \frac{\partial^2}{\partial x^2} u(x) = \sum_{i=1}^{d} v''(|x|^2)4x_i^2 + 2v(|x|^2) = 4|x|^2 v''(|x|^2) + 2dv(|x|^2).$$

Simplifying and letting $y = |x|^2$, we get

$$v''(y) = \frac{-d}{2y} v'(y).$$

This differential equation is solved by v of the form $v'(y) = cy^{-d/2}$ for some constant c. Thus, $\Delta u = 0$ holds for $|x| \neq 0$ if and only if

$$u(x) = \begin{cases} |x| & : d = 1 \\ 2\log|x| & : d = 2 \\ |x|^{2-d} & : d \geq 3 \end{cases}$$

□

Corollary 2.31. *Suppose that $\{B(t) : t \geq 0\}$ is Brownian motion started at some $x \in A$ in the open annulus A, then*

$$\mathbb{P}_x\{T_r < T_R\} = \begin{cases} \frac{R-|x|}{R-r} & : d = 1 \\ \frac{\log R - \log|x|}{\log R - \log r} & : d = 2 \\ \frac{R^{2-d} - |x|^{2-d}}{R^{2-d} - r^{2-d}} & : d \geq 3 \end{cases}$$

Proof. Plug $u(x)$ into the expression of $\mathbb{P}_x$. □

Corollary 2.32. *For any $x \notin B_r(0)$, we have*

$$\mathbb{P}_x\{T_r < \infty\} = \begin{cases} 1 & : d = 1, 2 \\ \frac{r^{d-2}}{|x|^{d-2}} & : d \geq 3 \end{cases}$$

Proof. Take the limit as $R \to \infty$. □

We are now in a position to prove the recurrence theorem.

Proof of 2.27. Lets prove the case for $d = 2$. Fix $\epsilon > 0$ and $x \in \mathbb{R}^2$. Then by the shift invariance property of Brownian motion, we get the first stopping time $t_1 := \inf\{t > 0 \mid B(t) \in B_\epsilon(x)\}$. By the previous corollary, since $d = 2$, this is almost surely finite. Now consider the time $t_1 + 1$. By the strong Markov property, we can obtain another stopping time $t_2 := \{t > t_1 + 1 \mid B(t) \in B_\epsilon(x)\}$, which is again almost surely finite. Repeating in this manner, we produce an increasing sequence of stopping times $t_1 \leq t_2 \leq \ldots \leq t_n \leq \ldots$ such that $B(t_n) \in B_\epsilon(x)$.

Now consider dimension $d \geq 3$. Consider events $A_n := \{|B(t)| > n \text{ for all } t \geq T_{n^3}\}$. By 2.8, T_{n^3} is almost surely finite. However, by the strong Markov property, for n sufficiently large, i.e., $n \geq |x|^{1/3}$,

$$\mathbb{P}_x(A_n^c) = \mathbb{E}_x\left[\mathbb{P}_{B(T_{n^3})}\{T_n < \infty\}\right] = \left(\frac{1}{n^2}\right)^{d-2}.$$

Then by Borel Cantelli, only finitely many events A_n^c can occur, thus $|B(t)|$ diverges to infinity almost surely. □

Corollary 2.33. *Neighborhood recurrence implies the path of planar Brownian motion is dense in the plane.*

2.5. **Martingales.** We now discuss another important property of Brownian motion: the martingale property.

Definition 2.34. A stochastic process $X_t : \Omega \to \mathbb{R}$ is a martingale with respect to a fixed filtration $\mathscr{F}(t)$ if the following hold:
(1) $X(t)$ is adapted to $\mathscr{F}(t)$.
(2) $\mathbb{E}|X(t)| < \infty$.
(3) for any pair of times $0 \leq s \leq t$, $\mathbb{E}[X(t) \mid \mathscr{F}(s)] = X(s)$ a.s.

Definition 2.35. A stochastic process X_t is a submartingale if conditions (1) and (2) hold from the definition of a martingale, and for any pair of times $0 \leq s \leq t$, $\mathbb{E}[X(t) \mid \mathscr{F}(s)] \leq X(s)$ a.s.

Intuitively, a martingale is a process where the current state $X(s)$ is the best predictor of the future states. The classic example of a martingale is a fair game. We now give two useful facts about martingales: the optional stopping theorem and the Doob's maximal inequality. We are primarily concerned with continuous martingales.

First, we tackle the optional stopping theorem, which addresses the extension of the equation $\mathbb{E}[X(t) \mid \mathscr{F}(s)] = X(s)$ to the case when $s := S, t := T$ are stopping times.

Theorem 2.36 (Optional Stopping Theorem)**.** *Suppose $X(t)$ is a continuous martingale and $0 \leq S \leq T$ are stopping times. If the process $X(\min\{t, T\})$ is dominated by an integrable random variable Y, then $\mathbb{E}[X(T) \mid \mathscr{F}(S)] = X(S)$ a.s.*

We prove the continuous case of the Optional Stopping Theorem by approximating from the discrete case.

Lemma 2.37 ((Discrete) Optional Stopping Theorem)**.** *Let X_n be a discrete martingale which is uniformly integrable. Then for all stopping times $0 \leq S \leq T$, we have $\mathbb{E}[X_n(T) \mid \mathscr{F}(S)] = X_n(S)$ almost surely.*

Proof. Since martingales are closed under limits, $X_n(T)$ converges to $X(T)$ in L^1 with

$$\mathbb{E}[X(T) \mid \mathscr{F}_n] = X(\min\{T, n\}) = X_n(T).$$

We may assume $X(T)$ is a strictly positive function; if not, apply the standard trick from integration theory of splitting it into a sum of positive components. Therefore, each $X_n(T)$ is also positive almost surely. Then considering the conditional expectation with respect to $\mathscr{F}(\min\{S, n\})$, we get

$$\mathbb{E}[X(T) \mid \mathscr{F}(\min\{S, n\})] = X(\min\{S, n\}).$$

Now consider a set $A \in \mathscr{F}_S$. It remains to show that $\mathbb{E}[X(T)\mathbf{1}_A] = \mathbb{E}[X(S)\mathbf{1}_A]$, where $\mathbf{1}_A$ is the indicator function. First, notice that $A \cap \{S \leq n\} \in \mathscr{F}(\min\{S, n\})$. Therefore, we get

$$\mathbb{E}[X(T)\mathbf{1}_{A\cap\{S\leq n\}}] = \mathbb{E}[\mathbf{1}_{\min\{S,n\}}] = \mathbb{E}[X(S)\mathbf{1}_{A\cap\{S\leq n\}}].$$

Then letting $n \to \infty$, we get the desired result by the monotone convergence theorem. □

Proof of the Optional Stopping Theorem. We will prove this by approximation from the discrete case. Fix $N \in \mathbb{N}$ and define a discrete time martingale by $X_n = X(\min\{T, n2^{-N}\})$ and stopping times $S' = \lfloor 2^N S \rfloor + 1, T' = \lfloor S^N T \rfloor + 1$ and the corresponding discrete filtration $\mathscr{G}(n) = \mathscr{F}(n2^{-N})$. By assumption X_n is dominated by an integrable random variable, and hence the discrete time Optional Stopping theorem gives $\mathbb{E}[X_{T'} \mid \mathscr{G}(S')] = X_{S'}$. Substituting for T', S', we get $\mathbb{E}[X(T) \mid \mathscr{F}(S_n)] = X(\min\{T, S_N\})$ where $S_N = 2^{-N}(\lfloor 2^N S \rfloor + 1)$. Hence for $A \in \mathscr{F}(S)$, by the dominated convergence theorem,

$$\int_A X(T)d\mathbb{P} = \lim_{N\to\infty} \int_A \mathbb{E}[X(T) \mid \mathscr{F}(S_N)]d\mathbb{P}$$
$$= \int_A \lim_{N\to\infty} X(\min\{T, S_N\})d\mathbb{P} = \int_A X(S)d\mathbb{P}.$$

□

The next result is a useful martingale inequality.

Theorem 2.38 (Doob's maximal inequality)**.** *Suppose $X(t)$ is a continuous martingale and $p > 1$. Then for any $t \geq 0$,*

$$\mathbb{E}\left[(\sup_{0\leq s\leq t} |X(s)|)^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[|X(t)|^p].$$

As with the Optional Stopping Theorem, we state and prove the discrete case first, then extend to the continuous case by approximation. We begin with a Lemma that compares the L^p norms of two random variables.

Lemma 2.39. *Let X, Y be two nonnegative random variables such that for all $\lambda > 0$,*

$$\lambda\mathbb{P}\{Y \geq \lambda\} \leq \mathbb{E}[X\mathbf{1}_{Y\geq\lambda}].$$

Then for all $p > 1$,

$$\mathbb{E}[Y^p] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[X^p].$$

Proof. First, notice that $x^p = \int_0^x p\lambda^{p-1}d\lambda$, so applying Fubini's theorem, we get

$$\mathbb{E}[X^p] = \mathbb{E}\int_0^\infty p\lambda^{p-1}d\lambda\mathbf{1}_{\lambda\leq X}$$

$$= \int_0^\infty \int_0^\infty (\mathbf{1}_{\lambda\leq X}dP)\, p\lambda^{p-1}d\lambda = \int_0^\infty p\lambda^{p-1}\mathbb{P}\{\lambda \leq X\}d\lambda.$$

By assumption, we write $\mathbb{E}[Y^p]$,

$$\mathbb{E}[Y^p] = \int_0^\infty p\lambda^{p-1}\mathbb{P}\{\lambda \leq Y\}d\lambda \leq \int_0^\infty p\lambda^{p-2}\mathbb{E}[X\mathbf{1}_{\lambda\leq Y}]d\lambda.$$

Applying Fubini's theorem to the right hand side and integrating $d\lambda$, we get

$$\int_0^\infty p\lambda^{p-2}\left(\int_0^Y XdP\right)d\lambda = \int_0^\infty \left(\int_0^Y p\lambda^{p-2}d\lambda\right)dP$$

$$= \mathbb{E}\left[X\int_0^Y p\lambda^{p-2}d\lambda\right] = \mathbb{E}\left[X\left(\frac{p}{p-1}\right)Y^{p-1}\right].$$

By Holder's inequality with $q = p/p-1$, we get that

$$\mathbb{E}\left[X\left(\frac{p}{p-1}\right)Y^{p-1}\right] = q\mathbb{E}[XY^{p-1}] \leq q||X||_p||Y^{p-1}||_q.$$

In summary, we get $\mathbb{E}[Y^p] \leq q(\mathbb{E}[X^p])^{1/p}(\mathbb{E}[Y^p])^{1/q}$.

For the first case, suppose that $\mathbb{E}Y^p < \infty$, then the above inequality implies

$$(\mathbb{E}[Y^p])^{1/p} \leq q(\mathbb{E}[X^p])^{1/p}$$

giving the desired result.

For the second case, suppose that $\mathbb{E}[Y^p] = \infty$, then consider the sequence of random variables $Y_n := \min\{Y, n\}$. The first case holds for each Y_n and the desired result follows by letting $n \to \infty$ and applying the monotone convergence theorem. □

Lemma 2.40 (Discrete Doob's L^p maximal inequality)**.** *Suppose X_n is a discrete martingale or a nonnegative submartingale. Define $M_n := \max_{1\leq k\leq n} X_k$ and fix $p > 1$. Then*

$$\mathbb{E}[M_n^p] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[|X_n|^p].$$

Proof. Suppose that X_n is a martingale, then $|X_n|$ is a nonnegative submartingale. Therefore, it is sufficient to prove the result for nonnegative submartingales. For all $\lambda > 0$,

$$\lambda\mathbb{P}\{\lambda \leq M_n\} \leq \mathbb{E}[X_n \mathbf{1}_{\{\lambda \leq M_n\}}]$$

where $M_n := \max_{1\leq j\leq n} X_j$.

To prove this, define stopping times

$$\tau := \begin{cases} \min\{k : X_k \geq \lambda\} & : M_n \geq \lambda \\ 0n & : M_n < \lambda \end{cases}$$

Observe that $\{M_n \geq \lambda\} = \{X_\tau \geq \lambda\}$. Therefore,

$$\lambda\mathbb{P}\{M_n \geq \lambda\} = \lambda\mathbb{P}\{X_\tau \geq \lambda\} = \mathbb{E}[\lambda \mathbf{1}_{\{\tau\geq\lambda\}}] \leq \mathbb{E}[X_\tau \mathbf{1}_{X_\tau \geq \lambda}] = \mathbb{E}[X_\tau \mathbf{1}_{\{M_n\geq\lambda\}}].$$

It remains to show that $\mathbb{E}[X_\tau \mathbf{1}_{\{M_n\geq\lambda\}}] \leq \mathbb{E}[X_n \mathbf{1}_{\{M_n\geq\lambda\}}]$. This follows since τ is bounded by n and X_τ is a submartingale, so $\mathbb{E}[X_\tau] \leq \mathbb{E}[X_n]$ implies that

$$\mathbb{E}[X_\tau \mathbf{1}_{\{M_n<\tau\}}] + \mathbb{E}[X_\tau \mathbf{1}_{\{M_n\geq\lambda\}}] \leq \mathbb{E}[X_n \mathbf{1}_{\{M_n<\lambda\}}] + \mathbb{E}[X_n \mathbf{1}_{\{M_n\geq\lambda\}}].$$

By the definition of τ, $X_\tau \mathbf{1}_{\{M_n<\lambda\}} = X_n \mathbf{1}_{\{M_n<\lambda\}}$, so the above equality is equivalent to

$$\mathbb{E}[X_\tau \mathbf{1}_{\{M_n\geq\lambda\}}] \leq \mathbb{E}[X_n \mathbf{1}_{\{M_n\geq\lambda\}}]$$

as desired.

Finally, letting $X = X_n$ and $Y = M_n$ in the previous lemma, we proof follows. □

Proof of Doob's maximal inequality. Again, we approximate from the discrete case using the monotone convergence theorem. Fix $N \in \mathbb{N}$ and define the discrete martingale $X_n = X(tn2^{-N})$ with respect to the discrete filtration $\mathscr{G}(n) = \mathscr{F}(tn2^{-N})$. By the discrete version of Doob's maximal inequality,

$$\mathbb{E}\left[\left(\sup_{1\leq k\leq 2^N} |X_k|\right)^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[|X_{2^N}|^p] = \left(\frac{p}{p-1}\right)^p \mathbb{E}[|X(t)|^p].$$

Letting $N \to \infty$, by monotone convergence, we get the desired result. □

3. stochastic Calculus

3.1. **Definition and basic properties.** We will now define the stochastic integral and an important property of the integral called Ito formula. In standard Riemann or Lebesgue integration, we are often given a function v that represents velocity and we wish to determine the integral of v, which models the position of a particle. In Riemann and Lebesgue integration, the function v needs to be sufficiently nice, which is not always achievable in real life. In reality, the differential equation that we wish to integrate will be of the form:

$$\frac{dX}{dt} = v(t, X_t) + w(t, X_t) \cdot W_t,$$

where W_t is the term that adds "randomness" to the equation. To achieve "randomness", we formally desire the random process W_t to have the properties that:
(1) If $t_1 \neq t_2$, then W_{t_1}, W_{t_2} are independent.
(2) The set $W_t, t \geq 0$ is stationary.
(3) $\mathbb{E}[W_t] = 0$ for all t.

We now formally integrate the stochastic differential equation given above. To do so, we develop a new integral known as the stochastic integral. As with any other form of integration, we first develop the stochastic integral by approximating from the discrete case, which is defined in the obvious manner.

Let us consider the discrete case and take the limit as the discrete time intervals go to 0. Fix discrete times $0 = t_0 < t_1 < \cdots < t_n = t$. Then the discrete version of the differential equation is:

$$X_{k+1} - X_k = X(t_{k+1}) - X(t_k) = v(t_k, X_k)\Delta t_k + w(t_k, X_k)W_k \Delta t_k.$$

We have expressed the first differential equation as a sum of two other differential equations. The first is in terms of a difference in time. The second is $W_k \Delta t_k = W_k t_{k+1} - W_k t_k =: V_{k+1} - V_k$, which can be thought of as a increment of another stochastic process V_k. The desired properties (1), (2), (3) of W_k above translate to the desire for V_k to be a stochastic process with stationary independent increment with 0 mean. In other words, $V_k = B_k = B(t_k)$ is Brownian motion. Thus, summing over finite discrete k, we get:

$$X_k = X_0 + \sum_{j=0}^{k-1} v(t_j, X_j)\Delta t_j + \sum_{j=0}^{k-1} w(t_j, X_j)\Delta B_j.$$

To revert to the continuous case by taking the limit as $\Delta t_k \to 0$, we see that,

$$X_t = X_0 + \int_0^t u(s, X_s)ds + \int_0^t w(s, X_s)dB_s.$$

Thus, it remains to define the integral over Brownian motion B_s.

The class of functions that we ultimately wish to integrate are progressively measurable processes, which we now define.

Definition 3.1. A process is called progressively measurable if for each $t \geq 0$, the map $X : [0,t] \times \Omega \to \mathbb{R}$ is measurable with respect to the sigma algebra $\mathscr{B}[0,t] \otimes \mathscr{F}$, where $\mathscr{B}[0,t]$ is the borel sigma algebra on $[0,t]$ and $\mathscr{F}$ is any filtration on Ω such that Brownian motion has the strong Markov property (e.g., $\mathscr{F}^+(t)$).

As suggested earlier, we will want to integrate processes that respect the filtration $\mathscr{F}^+(t)$. The following theorem shows that these processes are progressively measurable.

Lemma 3.2. *Any process $\{X(t) : t \geq 0\}$ that is adapted and either left or right-continuous is progressively measurable.*

Proof. Without loss of generality, let $X(t)$ be an adapted right-continuous process. Fix $t > 0$. Then for $n \in \mathbb{N}$ and $0 \leq s \leq t$, define

$$X_n(t, \omega) = \begin{cases} X(0, \omega) & : t = 0 \\ X(\frac{(k+1)t}{2^n}, \omega) & : \frac{kt}{2^n} < s \leq \frac{(k+1)t}{2^n} \end{cases}$$

For each n, the function $X_n(s,\omega)$ is $\mathscr{B}([0,t]) \otimes \mathscr{F}(t)$ measurable. By right continuity, $X(s,\omega) = \lim_{n\to\infty} X_n(s,\omega)$, which is measurable since limits carry through measurable functions. □

The construction of the Ito integral proceeds analogous to the construction of the Riemann and Lebesgue integrals. We begin by integrating progressively measurable step functions, which will approximate all progressively measurable functions.

Definition 3.3. A progressively measurable step function is a function $[0,\infty]\times\Omega \to \mathbb{R}$ such that

$$H(t,\omega) := \sum_{i=1}^{k} e_i(\omega)\mathbf{1}_{(t_i,t_{i+1}]}(t),$$

where $0 \le t_1 \le t_2 \le \cdots \le t_k + 1$ and e_i is each $\mathscr{F}(t_i)$ measurable. Then we define the integral of the step function to be

$$\int_0^\infty H(s,\omega)dB_s(\omega) := \sum_{i=1}^{k} e_i(B_{t_{i+1}} - B_{t_i})(\omega).$$

For step functions such that $||H||_2^2 = \mathbb{E}\int_0^\infty H(s)^2 ds < \infty$, we will show that

$$\int_0^\infty H(s)dB_s = \lim_{n\to\infty}\int_0^\infty H_n(s)dB_s.$$

Thus, it remains to check that every progressively measurable process with finite L^2 norm can be approximated in the L^2 norm by progressively measurable step functions.

Lemma 3.4. *For every progressively measurable process $H(s,\omega)$ such that $\mathbb{E}\int_0^\infty H(s)^2 ds < \infty$, there exists a sequence H_n of progressively measurable step functions such that $\lim_{n\to\infty} ||H_n - H||_2 = 0$.*

Proof. Let $H(s,\omega)$ be a progressively measurable process.

Step 1: Approximate $H(s,\omega)$ by bounded progressively measurable processes $J_n(s,\omega)$.
Define

$$I_n(s,\omega) = \begin{cases} H(s,\omega) & : s \le n \\ 0 & : n < s \end{cases}$$

Then consider $J_n = \min\{I_n, n\}$. Then clearly $\lim_{n\to\infty} ||J_n - H||_2 = 0$.

Step 2: Approximate uniformly bounded $F(s,\omega)$ by bounded a.s continuous progressively measurable processes $G_n(s,\omega)$.
Let $f = 1/n$ and define

$$G_n(s,\omega) := \frac{1}{f}\int_{s-f}^{s} F(t,\omega)dt,$$

where $F(s,\omega) = F(0,\omega)$ for $s < 0$. Since we are averaging over the past values, G_n is progressively measurable and it is a.s continuous. Finally, for every $\omega \in \Omega$ and almost every $s \in [0,t]$,

$$\lim_{n\to\infty} G_n(s,\omega) = \lim_{f\to 0}\frac{1}{f}\int_{s-f}^{s} f(t,\omega)dt = F(s,\omega).$$

Thus, by the bounded convergence theorem for L^2 bounded martingales, $\lim_{n\to\infty} ||G_n - F||_2 = 0$.

Step 3: Approximate bounded a.s continuous progressive measurable functions $C(s,\omega)$ by progressively measurable step processes $D_n(s,\omega)$.
Define $D_n(s,\omega) := C(\frac{j}{n},\omega)$ for $\frac{j}{n} \leq s < \frac{j+1}{n}$. Each D_n is clearly a step function and $\lim_{n\to\infty} ||D_n - C||_2 = 0$.

By taking limits three times, we have the desired approximation. □

We have shown the desired approximation holds for functions when integrating dt instead of dB_t. The following lemma and corollary shows that is sufficient.

Lemma 3.5. *Let H be a progressively measurable step process and $\mathbb{E}\int_0^\infty H(s)^2 ds < \infty$, then*

$$\mathbb{E}\left[\left(\int_0^\infty H(s)dB(s)\right)^2\right] = \mathbb{E}\int_0^\infty H(s)^2 ds.$$

Proof. By the Markov property,

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^\infty H(s)dB(s)\right)^2\right] &= \mathbb{E}\left[\left(\int_0^\infty H(s)dB(s)\right)^2\right]\\ &= \mathbb{E}\left[\sum_{i,j=1}^k e_i i e_j (B(t_{i+1}) - B(t_i))(B(t_{j+1}) - B(t_j))\right]\\ = 2\sum_{i=1}^k \sum_{j=i+1}^k \mathbb{E}\left[e_i e_j (B(t_{i+1}) - B(t_i))\mathbb{E}[B(t_{j+1}) - B(t_j) \mid \mathscr{F}(t_j)]\right]&\\ + \sum_{i=1}^k \mathbb{E}[e_i^2 (B(t_{i+1}) - B(t_i))^2]&\\ = \sum_{i=1}^k \mathbb{E}[t_i^2](t_{i+1} - t_i) = \mathbb{E}\int_0^\infty H(s)^2 ds.&\end{aligned}$$

□

Corollary 3.6. *Suppose that H_n is a sequence of progressively measurable step processes such that $\mathbb{E}\int_0^\infty (H_n(s) - H_m(s))^2 ds \to 0$ as $n, m \to 0$, then*

$$\mathbb{E}\left[\left(\int_0^\infty H_n(s) - H_m(s)dB_s\right)^2\right] \to 0$$

as $n, m \to 0$.

Proof. Apply the Lemma to $H_n - H_m$, which is also a step process. □

We can now finally prove the desired approximation theorem for progressively measurable functions by integrating dB_t.

Theorem 3.7. *Suppose that H_n is a sequence of progressively measurable functions and H is a progressively measurable function such that*

$$\lim_{n\to\infty} \mathbb{E}\int_0^\infty (H_n(s) - H(s))^2 ds = 0,$$

then

$$\int_0^\infty H(s,\omega)dB_s := \lim_{n\to\infty}\int_0^\infty H_n(s)dB_s$$

exists as an L^2 limit and is independent of the choice of H_n and

$$\mathbb{E}\left[\left(\int_0^\infty H(s,\omega)dB_s\right)^2\right] = \mathbb{E}\int_0^\infty H(s)^2 ds.$$

Proof. Recall that $||H||_2^2 := E\int_0^\infty H(s,\omega)^2 ds$. So by the triangle inequality, $||H_n - H_m||_2 \leq ||H_n - H||_2 + ||H_m - H||_2 \to 0$ as $n, m \to \infty$. Thus, H_n is a Cauchy sequence in L^2, which is complete, so H_n converges to H in L^2. The limit is independent of the choice of approximating sequence by the previous corollary, which shows that the limit of the difference of the expected values goes to 0 for two sequence H_n, H_m. Finally, the last statement follows by carrying limits through expected values:

$$\mathbb{E}\left[\left(\int_0^\infty \lim_{n\to\infty} H_n(s,\omega)dB_s\right)^2\right] = \lim_{n\to\infty}\mathbb{E}\left[\left(\int_0^\infty H_n(s,\omega)dB_s\right)^2\right]$$

$$= \lim_{n\to\infty}\mathbb{E}\int_0^\infty H_n(s)^2 ds = \mathbb{E}\int_0^\infty \lim_{n\to\infty} H_n(s)^2 ds.$$

□

The stochastic integral defines its own stochastic process, namely the process

$$I(t,\omega) := \int_0^t H(s,\omega)dB_s = \int_0^\infty H(s,\omega)\mathbf{1}_{[0,t]}dB_s.$$

Definition 3.8. Let B_t be Brownian motion. An Ito process is a stochastic process I_t of the form:

$$I_t = I_0 + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dB_s$$

where v satisfies the following properties:
(1) The function $(t,\omega) \mapsto v(t,\omega)$ is $\mathscr{B}[0,\infty] \times \mathscr{F}^+$ measurable where $\mathscr{B}[0,\infty]$ is the Borel σ-algebra on $[0,\infty]$ and $\mathscr{F}^+$ is the right-continuous filtration defined above.
(2) There exists an increasing family of σ-algebras $H_t, t \geq 0$ such that B_t is a martingale with respect to H_t and v_t is H_t adapted.
(3)

$$\mathbb{P}\left[\int_S^T v(s,\omega)^2 ds\right] = \infty.$$

And, the function u must satisfy the following properties:
(a) u is H_t adapted.
(b)

$$\mathbb{P}\left[\int_0^t |u(s,\omega)|ds < \infty\right] = 1.$$

Since the form of the Ito process above is cumbersome to work with, we introduce a common shorthand. Moving I_0 to the left hand side, we get

$$I_t - I_0 = \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dB_s$$

Since we're considering each integral as a process in terms of t, we write the process above as

$$dI_t = udt + vdB_t.$$

We can modify the process I (i.e., find another process J that differs by a set of measure zero) such that the modification is an almost surely continuous martingale.

Theorem 3.9. *Suppose $H(s,\omega)$ is a progressively measurable process such that $\mathbb{E}\int_0^t H(s,\omega)^2 ds < \infty$ for all t. Then there exists an almost surely continuous modification of the stochastic process $I(t) = \int_0^t H(s,\omega)dB_s$ such that the process is a martingale.*

Proof. For notational simplicity, define $H^{t_0}(s) = H(s) \cdot \mathbf{1}_{\{s<t_0\}}$, where $\mathbf{1}$ is the indicator function. Fix a large integer t_0 and let H_n be a sequence of step processes such that $||H_n - H^{t_0}||_2 \to 0$ for $t < t_0$. Then

$$\mathbb{E}\left[\left(\int_0^\infty (H_n(s) - H^{t_0}(s))dB(s)\right)^2\right] \to 0$$

For any $s < t$, the random variable $\int_0^s H_n(u)dB(u)$ is clearly $\mathscr{F}(s)$ measurable and $\int_s^t H_n(u)dB(u)$ is independent of $\mathscr{F}(s)$. Therefore, the process

$$\int_0^t H_n(u)dB(u); 0 \le t \le t_0$$

is a martingale for each $n \in \mathbb{N}$. Now for any $0 \le t \le t_0$, define

$$X(t) = \mathbb{E}\left[\int_0^{t_0} H(s)dB(s) \mid \mathscr{F}(t)\right]$$

so that $\{X(t) : 0 \le t \le t_0\}$ is a martingale and $X(t_0) = \int_0^{t_0} H(s)dB(s)$. Then by Doob's maximal inequality, fixing $p = 2$, we get

$$\mathbb{E}\left[\sup_{0\le t\le t_0}\left(\int_0^t H_n(s)dB(s) - X(t)\right)^2\right] \le 4\mathbb{E}\left[\left(\int_0^{t_0} (H_n(s) - H(s))dB(s)\right)^2\right]$$

converges to 0 as $n \to \infty$. Therefore, a.s, the process $\{X(t) : 0 \le t \le t_0\}$ is the uniform limit of continuous processes and hence is continuous itself. For fixed $0 \le t \le t_0$, by taking limits, we extend the result to show that the random variables $\int_0^t H(s)dB(s)$ are $\mathscr{F}(t)$ measurable and independent with zero expectation. Hence, $\int_0^t H(s)dB(s) = \mathbb{E}[X(t_0) \mid \mathscr{F}(t)] = X(t)$ a.s. □

3.2. **Ito formula.** Like the chain rule in standard integration theorem, there exists a "chain rule" for stochastic calculus called Ito formula. We first deal with the 1-dimensional Ito formula, then extended this to functions from $\mathbb{R}^d$ by using vector notation.

Theorem 3.10 (Ito formula)**.** *Let I_t be an Ito process. Let $g(t,x) \in \mathbb{C}^2([0,\infty),\mathbb{R})$. Then the process $J_t = g(t, I_t)$ is also an Ito process and*

$$dJ_t = \frac{\partial g}{dt}(t, I_t)dt + \frac{\partial g}{dx}(t, I_t)dI_t + \frac{1}{2}\frac{\partial^2 g}{dx^2}(t, I_t)(dI_t)^2$$

where $(dI_t)^2 = dI_t \cdot dI_t$ is computed by the rules $dtdt = dtB_t = dB_t dt = 0, dB_t dB_t = dt$.

Proof of Ito formula. We first show that J_t is an Ito process.

Expanding $(dI_t)^2$ and making the appropriate substitutions, we see that

$$(dI_t)^2 = (udt + vdB_t)^2 = u^2(dt)^2 + 2uvdtdB_t + v^2(dB_t)^2 = 0 + 0 + v^2 dt.$$

Then substituting, we get

$$dJ_t = \frac{\partial g}{dt}(t, I_t)dt + \frac{\partial g}{dx}(t, I_t)dI_t + \frac{1}{2}\frac{\partial^2 g}{dx^2}(t, I_t)(dI_t)^2$$
$$= \frac{\partial g}{dt}(t, I_t)dt + \frac{\partial g}{dx}(t, I_t)(udt + vdB_t) + \frac{1}{2}\frac{\partial^2 g}{dx^2}(t, I_t)v^2 dt$$
$$= \left(\frac{\partial g}{dt}(t, I_t) + \frac{\partial g}{dx}(t, I_t)u + \frac{1}{2}\frac{\partial^2 g}{dx^2}(t, I_t)v^2\right)dt + \frac{\partial g}{dx}(t, I_t)vdB_t.$$

Now writing out the shorthand, we get

$$g(t, I_t) = g(0, I_0) + \int_0^t \left(\frac{\partial g}{ds}(s, I_s) + \frac{\partial g}{dx}(s, I_s)u + \frac{1}{2}\frac{\partial^2 g}{dx^2}(s, I_s)v^2\right)ds + \int_0^t \frac{\partial g}{\partial x}vdB_s.$$

This expression is of the form required by Ito formula. It remains to show that the initial expression is correct. We do so by approximating by step functions.

We may assume that $g, \frac{\partial g}{\partial t}, \frac{\partial g}{\partial x}, \frac{\partial^2 g}{\partial x^2}$ are bounded, because if not, we may approximate the unbounded functions by bounded ones over compact sets. Similarly, we may assume that u, v are progressively measurable step functions for the same reason. Then by Taylor's theorem approximating to order 2,

$$g(t, I_t) = g(0, I_0) + \sum_j \Delta g(t_j, I_j) = g(0, I_0) + \sum_j \frac{\partial g}{\partial t}\Delta t_j + \sum_j \frac{\partial g}{\partial x}\Delta I_j$$
$$+\frac{1}{2}\sum_j \frac{\partial^2 g}{\partial t^2}(\Delta t_j)^2 + \frac{1}{2}\sum_j \frac{\partial^2 g}{\partial t \partial x}(\Delta t_j)(\Delta I_j) + \frac{1}{2}\sum_j \frac{\partial^2 g}{\partial x^2}(\Delta I_j)^2 + \sum_j R_j,$$

where the remainder term $R_j = o(|\Delta t_j|^2 + |\Delta I_j|^2)$. Taking the limit as $\Delta t_j \to 0$, we check what each sum converges to.
The first sum:

$$\sum_j \frac{\partial g}{\partial t}(t_j, I_j)\Delta t_j \to \int_0^t \frac{\partial g}{\partial t}(s, I_s)ds = \frac{\partial g}{\partial t}dt.$$

The second sum:

$$\sum_j \frac{\partial g}{\partial x}(t_j, I_j)\Delta I_j \to \int_0^t \frac{\partial g}{\partial x}(x, I_s)dI_s = \frac{\partial g}{\partial x}dI_t.$$

The third sum:

$$\sum_j \frac{\partial^2 g}{\partial t^2}(t_j, I_j)(\Delta t_j)^2 \to \Delta t_j \int_0^t \frac{\partial^2 g}{\partial t^2}(s, I_s)ds \to 0.$$

The fourth sum:

$$\sum_j \frac{\partial^2 g}{\partial t \partial x}(t_j, I_j)(\Delta t_j)(\Delta I_j) \to \Delta t_j \int_0^t \frac{\partial^2 g}{\partial t \partial x}(s, I_s)dI_s \to 0.$$

It remains to show that the fifth sum converges as desired:

$$\frac{1}{2}\sum_j \frac{\partial^2 g}{\partial x^2}(\Delta I_j)^2 \to \frac{1}{2}\int_0^t \frac{\partial^2 g}{\partial x^2}(s, I_s)(dI_s)^2.$$

We have, however, already showed that

$$\frac{1}{2}\int_0^t \frac{\partial^2 g}{\partial x^2}(s, I_s)(dI_s)^2 = \frac{1}{2}\frac{\partial^2 g}{dx^2}(s, I_s)v^2 ds,$$

therefore it is sufficient to show that the sum converges to the expression on the right hand side. Expanding the sum, we see that

$$\frac{1}{2}\sum_j \frac{\partial^2 g}{\partial x^2}(\Delta I_j)^2 = \frac{1}{2}\left[\sum_j \frac{\partial^2 g}{\partial x^2}u_j^2(\Delta t_j)^2 + \sum_j \frac{\partial^2 g}{\partial x^2}u_j v_j(\Delta t_j)(\Delta B_j) + \sum_j \frac{\partial^2 g}{\partial x^2}v_j^2(\Delta B_j)^2\right].$$

By the same argument as above, the first two sums converge to 0, so the desired convergence is equivalent to showing that the following convergence holds:

$$\sum_j \frac{\partial^2 g}{\partial x^2}v_j^2(\Delta B_j)^2 \to \int_0^t \frac{\partial^2 g}{\partial x^2}v^2 dB_s.$$

To prove this, let $a(t) = \frac{\partial^2 g}{\partial x^2}(t, I_t)v^2(t,\omega)$. For notational simplicity, let $a_j = a(t_j)$. Then

$$\mathbb{E}\left[\left(\sum_j a_j(\Delta B_j)^2 - \sum_j a_j \Delta t_j\right)^2\right] = \sum_{i,j}\mathbb{E}[a_i a_j((\Delta B_i)^2 - \Delta t_i)((\Delta B_j)^2 - \Delta t_j)].$$

If $i < j$ or $i > j$, then the random variables $a_i a_j((\Delta B_i)^2 - \Delta t_i)$ and $(\Delta B_j)^2 - \Delta t_j$ are independent and hence the term in the sum vanishes. Therefore, the sum can be reduced to summing over $i = j$. In other words,

$$\sum_j \mathbb{E}[a_j^2((\Delta B_j)^2 - \Delta t_j)^2] = \sum_j \mathbb{E}[a_j^2]\mathbb{E}[(\Delta B_j)^4 - 2(\Delta B_j)\Delta t_j + (\Delta t_j)^2]$$

$$= \sum_j \mathbb{E}[a_j^2](3(\Delta t_j)^2 - 2(\Delta t_j)^2 + (\Delta t_j)^2) = 2\sum_j \mathbb{E}[a_j^2](\Delta t_j)^2.$$

This equivalent version of the sum converges like

$$2\sum_j \mathbb{E}[a_j^2](\Delta t_j)^2 \to \Delta t_j \int_0^t \mathbb{E}[a_j]^2 dt \to 0.$$

This implies

$$\sum_j a_j(\Delta B_j)^2 \xrightarrow{L^2} \int_0^t a(s)ds.$$

Plugging in for a, we get

$$\sum_j \frac{\partial^2 g}{\partial x^2}(t_j, I_t)v^2(t_j,\omega)(\Delta B_j)^2 \xrightarrow{L^2} \int_0^t \frac{\partial^2 g}{\partial x^2}(s, I_t)v^2(s,\omega)ds.$$

By the definition of asymptotic, the final sum converges:

$$\sum_j R_j \to 0$$

completing the proof. □

We extend this to the multidimensional Ito formula in the obvious manner: by writing it in vector notation. For completeness, we define the d-dimensional Ito process and give the d-dimensional Ito formula.

Definition 3.11. A d-dimensional Ito process I_t is a d-dimensional stochastic process given by $dI_t(\omega) = u(t,\omega)dt + v(t,\omega)dB_t$, where we use the analogous short hand from the 1-dimensional case. In this case, u is a d dimensional vector, B_t is a d-dimensional Brownian motion, and v is a $d \times d$ matrix where each u_i, v_j satisfies the requirements of a 1-dimensional Ito process.

A d-dimensional Ito process takes the form:

$$d\begin{pmatrix} I_1 \\ I_2 \\ \vdots \\ I_d \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_d \end{pmatrix} dt + \begin{pmatrix} v_{11} & \cdots & \cdots & v_{1,d} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ v_{d,1} & \cdots & \cdots & v_{d,d} \end{pmatrix} d\begin{pmatrix} B_1(t) \\ B_2(t) \\ \vdots \\ B_d(t) \end{pmatrix}$$

Theorem 3.12 (Multidimensional Ito formula)**.** *Let $dI_t = udt + vdB_t$ be a d-dimensional Ito process. Let $f(t,\omega) : [0,\infty] \times \mathbb{R}^d \to \mathbb{R}^p$ be C^2 such that $f(t,x) = (f_1(t,x), \dots, f_p(t,x))$. Then $J(t,\omega) = f(t, I(t,\omega))$ is a p-dimensional Ito process where for $1 \le k \le p$, the k-th component of $J(t,\omega)$ is given by*

$$dJ_k = \frac{\partial f_k}{\partial t}dt + \sum_{i=1}^{d} \frac{\partial f_k}{\partial x_i}dX_i + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 f_k}{\partial x_i \partial x_j}dX_i dX_j$$

following the rule that $dtdB_i = dB_j dt = 0$ and $dB_i dB_j = \delta_{i,j}dt$.

The proof proceeds similar to the 1-dimensional case presented above, by repeating the argument component wise.

4. Levy's Theorem and Liouville's Theorem

4.1. **Levy's Theorem.** We now turn to Levy's characterization, which identifies certain continuous local martingales as Brownian motion and a useful corollary due to Dubins and Schwartz that classifies all continuous local martingales as time changes of Brownian motion. To properly develop these ideas, we need the idea of a local martingale.

Definition 4.1. An adapted stochastic process $\{X(t) : 0 \le t \le T\}$ is a local martingale if there exists a sequence of stopping times T_n that increase a.s to T such that $\{X(\min\{t, T_n\}) : t \ge 0\}$ is a martingale for every n.

The following is an important lemma that will be used to prove Liouville's theorem.

Theorem 4.2. *Let $D \subset \mathbb{R}^d$ be a connected and open set and let $f : D \to \mathbb{R}$ be harmonic on D. Suppose that $\{B(t) : 0 \le t \le T\}$ is Brownian motion that starts inside D and stops at time T. Then the process $\{f(B(t)) : 0 \le t \le T\}$ is a local martingale.*

Proof. This follows immediately from Ito formula.

First note that Ito formula holds for all times $s \in [0,t]$ including stopping times bounded by t. Then, in particular, let f be C^2 function on an open set U and fix any $K \subset U$ that is compact. Then let $g : \mathbb{R}^m \to [0,1]$ be smooth with compact support inside U such that $g = 1$ on K. Then defining $f^* = fg$, we can apply Ito's formula a.s to $\min\{s, T\}$ for all $s < T$ where T is the first exit time from the compact set K. Then $f(B(s), X(s))$ where $B(s)$ is Brownian motion and $X(s)$ is a stochastic process can be expanded as follows:

$$f(B(\min\{t,T_n\}), X(\min\{t,T_n\})) - f(B(0), X(0)) = \int_0^{\min\{t,T_n\}} \nabla_x f(B(s), X(s))dB(s)$$
$$+ \int_0^{\min\{t,T_n\}} \nabla_y f(B(s), X(s))dX(s) + \frac{1}{2}\int_0^{\min\{t,T_n\}} \Delta_x f(B(s), X(s))ds.$$

Now suppose that K_n is an increasing sequence of compact sets such that $\bigcup_{n\in\mathbb{N}} K_n = D$. Let T_n be the exit time from each K_n. Letting $X(s) = 0$, $\nabla_y f(B(s)) = 0$, so the middle integral is zero. Also since f is harmonic, $\Delta_x f = 0$, so the right most integral is zero. Therefore, we are left with

$$f(B(\min\{t,T_n\})) = f(B(0)) + \int_0^{\min\{t,T_n\}} \nabla f(B(s))dB(s).$$

We have, therefore, identified an increasing sequence to stopping time T_n which converges to the final stopping time T such that $f(B(\min\{t,T_n\}))$ is a martingale and hence a local martingale. □

We now arrive at Levy's characterization of local martingales. We give the proof originally due to Watanabe and Kunita from Williams in [5]. For the general form of this theorem, we will need the notion of quadratic covariation. Intuitively, it is simply the extension of quadratic variation to two random variables.

Definition 4.3. Let X, Y be random variables. The quadratic covariation of X and Y is given by

$$[X,Y]_t = \lim_{||P||\to 0} \sum_{k=1}^{n} (X(t_k) - X(t_{k-1}))(Y(t_k) - Y(t_{k-1})).$$

Quadratic covariation can be written in terms of the quadratic variation of X, Y individually by using the polarization identity from Hilbert space theory:

$$[X,Y]_t = \frac{1}{2}([X+Y]_t - [X]_t - [Y]_t).$$

Theorem 4.4 (Levy's Theorem)**.** *Suppose that $X(t)$ is a continuous local martingale such that $X(0) = 0$ and component wise $X_i(t), X_j(t)$ has finite quadratic covariation (i.e., $[X_i(t), X_j(t)]_t = \delta_{i,j} t$). Then $X(t)$ is Brownian motion.*

Proof. Let $X(t)$ be d-dimensional local martingale with the properties detailed above. Fix a vector $\theta \in \mathbb{R}^d$, and define a smooth function

$$f := e^{i\langle\theta,x\rangle + \frac{1}{2}|\theta|^2 t}$$

where $\langle\theta, x\rangle$ is the inner product. Then by Ito formula,

$$d(f(X(t),t)) = \frac{\partial f}{\partial x_j}(X(t),t)dX_j(t) + \frac{\partial f}{\partial t}(X(t),t)dt + \frac{1}{2}\frac{\partial^2 f}{\partial x_j \partial x_k}(X(t),t)d[X_j, X_k].$$

Writing out the partial derivatives,

$$= i\theta_j f(X(t),t)dX_j(t) + \frac{1}{2}|\theta|^2 f(X(t),t)dt + \frac{1}{2}(i\theta_j)(i\theta_k)f(X(t),t)\delta_{j,k}dt$$
$$= i\theta_j f(X(t),t)dX_j(t).$$

This implies that $f(X(t),t)$ is the sum of stochastic integrals with respect to continuous local martingales $X(t)$ and hence is a local martingale itself. Moreover, since

$|f(X(t),t)| = e^{\frac{1}{2}|\theta|^2 t} < \infty$ for each t, $f(B(t),t)$ is actually a martingale. Hence for $s < t$,

$$\mathbb{E}\left[e^{i(\theta, X(t)-X(s))} \mid \mathscr{F}(s)\right] = e^{-\frac{1}{2}(t-s)|\theta|^2}.$$

Thus the stochastic process $X(t)$ is indeed normally distributed with mean 0 and covariance $(t-s)I$ and independent of $\mathscr{F}(s)$ and hence is Brownian motion. □

Finally, we get to Dubins and Schwartz's corollary of Levy's theorem.

Corollary 4.5 (Dubins and Schwarz)**.** *Let M be a continuous local martingale null at 0 such that $[M]_t$ is increasing as $t \to \infty$. For $t \geq 0$, we define stopping times $\tau_t := \inf\{u : [M]_u > t\}$ and a shifted filtration $\mathscr{G}(t) = \mathscr{F}(\tau_t)$. Then $X(t) := M(\tau_t)$ is standard Brownian motion.*

Proof. The desired result will follow from Levy's theorem if we can show that $M(\tau_t)$ is a continuous, local martingale with the appropriate quadratic variation.

It is clear that since M is $\mathscr{F}(t)$ adapted, $X(t) = M(\tau_t)$ is $\mathscr{G}(t)$ adapted. We now need to show that X is continuous. It suffices to show that for almost every ω, M is constant on each step of $[M]$. Since we're checking continuity, which is a local property, it suffices to check continuity when M is a bounded L^2 martingale.

For any rational q, we define

$$S_q := \inf\{t > q \mid [M]_t > [M]_q\}.$$

Then observe that

$$\mathbb{E}[M(S_q)^2 - [M](S_q) \mid \mathscr{F}(q)] = M(q)^2 - [M]_q.$$

However, by the definition of infimum, $[M](S_q) = [M]_q$, therefore the expectation is equal to 0 and hence M is constant on $[q, S_q)$. Since this holds for each step of $X(t) = M(\tau_t)$, X is indeed a.s continuous.

Next, we show that X and $X^2 - t$ are local martingales and hence (as we will show), the desired quadratic variation to apply Levy's theorem holds. First, we define

$$T(n) := \inf\{t \mid |M_t| > n\}, U(n) := [M](T(n))$$

Then for each t, $\tau_{\min\{t,U(n)\}} = \min\{T(n), \tau_t\}$. In other words,

$$X(\min\{t, U(n)\}) = M)(\tau(t))\mathbf{1}_{\{\tau(t) \leq T(n)\}},$$

where $\mathbf{1}$ is the indicator function.

Since $U(u)$ is a stopping time, by the optional stopping theorem applied to $M\mathbf{1}_{t \leq T(n)}$, for $s < t$,

$$\mathbb{E}[X(\min\{t, U(n)\}) \mid \mathscr{G}(s)] = \mathbb{E}[M(\tau(t))\mathbf{1}_{\tau(t) \leq T(n)} \mid \mathscr{G}(s)]$$

$$= M(\tau(t))\mathbf{1}_{\tau(t) < T(n)} = X(\min\{s, U(n)\}).$$

Therefore, by the definition of local martingale, $X(t)$ is a local martingale. Similarly, by the optional stopping theorem applied to $X(t) - t$, the same argument shows that $X(t) - t$ is a local martingale too.

By Ito formula,

$$d(X^2 - t) = 2XdX + \frac{1}{2} \cdot 2dXdX - dt.$$

Since we know that X and $X(t)^2 - t$ are local martingales, $dXdX - dt = 0$. Integrating, we see that

$$\int_0^t (dX(s))^2 = \int_0^t ds.$$

However,

$$[X]_t = \int_0^t (dX(s))^2 \text{ and } \int_0^t ds = t.$$

In other words, $[X]_t = t$. By Levy's theorem, X is a Brownian motion. Taking the inverse stopping time, we see that $M(t) = X(\tau^{-1}(t)))$, which is also a Brownian motion, hence M is a Brownian motion with time-change τ^{-1}. □

4.2. **Liouville's Theorem.** In this section, we summarize the results we have gathered to prove Liouville's theorem.

Theorem 4.6 (Liouville's Theorem)**.** *Suppose f is a complex valued function that is entire and bounded, then it is constant.*

Proof. Suppose that f is entire and non-constant. By theorem 4.2, $f(B(t))$ is a continuous local martingale. By Dubin and Schwartz, $f(B(t))$ is a time change of Brownian motion. By the recurrence of 2-dimensional Brownian motion, $f(B(t))$ is dense in the complex plane and hence unbounded. □

Acknowledgments. It is my pleasure to thank Peter May for organizing the REU, and my mentor, Marcelo Alvisio, for his willingness to meet with me and help edit this paper.

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