A SKELETON IN THE CATEGORY: THE SECRET THEORY OF COVERING SPACES

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Abstract. In this paper, we try to give as comprehensive an account of covering spaces as possible. We cover the usual material on classification and deck transformations, and also show how to perceive the subject from a more abstract categorical view point. The reader is assumed to possess a working knowledge of basic topology and category theory.

Contents

1. Introduction

The theory of covering spaces is an old subject that is canonical material for most introductions to algebraic topology. It is, however, sometimes seen to be not terribly interesting in its own right, but instead useful mostly as a tool for proving a motley of results, such as that the fundamental group of the circle is isomorphic to the integers and that every subgroup of a free group is free.

Part of the point of this paper is to argue otherwise. I seek to show that there is value in attaining a deep understanding of covering spaces not just so that one can better use it as a tool, but also because the theory embodies the following recurrent theme in algebraic topology: The same subject material can often be presented at different levels of abstraction, and it is shuttling between these that helps us to understand and contextualize the subject by revealing its similarities and differences with other mathematical theories.

To present this picture, I have organized this paper into two parts. In the first, I set down the foundations of the subject as they are traditionally presented as in [7] and [3]. After a brief interlude, I take up the more abstract approach of [6] in the second part. This chronology of peeling layers off the subject to reveal an inner "skeleton" can be taken as a way of interpreting the title to this paper.

There is an element of covering space theory that has the air of folklore, and that is the oft-cited analogy between the classification of covering spaces and the correspondence between subfields and subgroups in Galois theory. I have not managed to find a single source that deals rigorously with what this analogy actually is, and it turns out that there are in fact two distinct ways in which authors have drawn analogies between the two subjects. As such, the second aim of this paper is to demystify this confusion by discussing and distinguishing these two threads of argument.

This rather unfortunate state of affairs is also the second way of interpreting the title. There is, in fact, a third way, which we shall reveal at the end of this paper.

2. Preliminaries

Definition 2.1. Let $p : E \to B$ be a surjective map. The open set U of B is said to be *evenly covered* by p if the inverse image $p^{-1}(U)$ can be written as the union of disjoint open sets V_α in E such that for each α, the restriction of p to V_α is a homeomorphism of V_α onto U. The collection $\{V_\alpha\}$ is called a partition of $p^{-1}(U)$ into *slices*.

Definition 2.2. Let $p : E \to B$ be a surjective map. If every point b of B has a neighborhood U that is evenly covered by p, then p is called a *covering map*, and E is said to be a *covering space* of B. We shall follow the convention of the extant literature and sometimes shorten these two terms to *covering* and *cover* respectively. Given a point $b \in B$, we denote its preimage by F_b and call it the *fiber* of p above b.

Note that a cover is inseparable from its associated covering. We hence often consider both as a single object and simply call it a covering.

Example 2.3. Let $p : \mathbb{R} \to S^1$ be defined by $p(t) = (\cos 2\pi it, \sin 2\pi it)$. Then every open arc spanned by an angle of radian less than π is an evenly covered by p, so p is a covering map.

Example 2.4. Now define $p : S^1 \to S^1$ to be the n-th power map $z \mapsto z^n$, where this time we view S^1 as being embedded in $\mathbb{C}$. Once can check that neighborhoods of the form $S^1 - \{z\}$ for some z are evenly covered.

Letting n run over the natural numbers, these two examples exhaust all possibilities for covering spaces of S^1, up to some suitable notion of isomorphism. This fact may seem surprising at first, but will be made apparent after we classify covering spaces, which gives a correspondence between the basepoint-preserving isomorphism classes of covering spaces of a given space, and the subgroups of the fundamental group.

The following two easy observations allow us to enlarge our pool of examples for covering spaces.

Proposition 2.5. *Let $p : E \to B$ be a covering map, let B_0 be a subspace of B, and write $E_0 = p^{-1}(B_0)$. Then $p_0 : E_0 \to B_0$ obtained by restricting p is a covering map.*

Proposition 2.6. *Let $p : E_1 \to B_1$, $q : E_2 \to B_2$ be covering maps, then their product $p \times q : E_1 \times E_2 \to B_1 \times B_2$ is also covering map.*

Corollary 2.7. *The usual projection $p : \mathbb{R}^2 \to \mathbb{T}$ is a covering map for the torus.*

3. Lifting Properties

Given a covering space $p : E \to B$, we know that paths and homotopies in E project down to paths and homotopies in B. More generally, any map $f : X \to E$ from another space X into E induces a map $pf : X \to B$. But what can we say when we start from the opposite direction, i.e. when we are given a map $g : X \to B$ instead?

It turns out that paths and homotopies can always be lifted, but general maps require additional conditions on the domain space.

Lemma 3.1. *Let $p : E \to B$ be a covering map and $\gamma : I \to B$ a path starting at a point b_0. Then for every point $e_0 \in p^{-1}(b_0)$, there is a unique lift $\tilde{\gamma} : I \to E$ such that $\tilde{\gamma}(0) = e_0$.*

Proof. Pick an open cover $\mathcal{U}$ for B consisting of evenly covered neighborhoods. Using the Lebesgue number lemma, we can pick a subdivision of $[0, 1]$, $s_0, \dots, s_n$ such that each subinterval $[s_i, s_{i+1}]$ is mapped by γ into a single open set $U_i \in \mathcal{U}$. We now proceed by induction: Suppose that we have defined a lift for γ restricted to $[0, s_j]$, for $0 \leq j \leq n-1$, we claim that we can extend this to a lift for γ defined over $[0, s_{j+1}]$.

Consider the set U_j in which $[s_j, s_{j+1}]$ lies. Its preimage consists of a collection of disjoint open sets. Now, we have already defined $\tilde{\gamma}(s_j)$ to lie in one of these, call it V. Since $p|_V : V \to U_i$ is a homeomorphism, we can then define a continuous function $g : [s_j, s_{j+1}] \to E$ by setting $g(s) = (p|_{V_\alpha})^{-1}(f(s))$. By the pasting lemma, combining g and $\tilde{\gamma}$ yields a continuous function on $[0, s_{j+1}]$. Relabelling the entire function $\tilde{\gamma}$, we clearly have $\gamma = p\tilde{\gamma}$.

We now prove uniqueness via the same step by step argument. Suppose we have two lifts of γ, call them $\tilde{\gamma}$ and $\tilde{\gamma}'$ that agree on $[0, s_j]$, with V the unique slice of $p^{-1}(U_j)$ containing $\tilde{\gamma}(s_j) = \tilde{\gamma}'(s_j)$. Then since $\tilde{\gamma}([s_j, s_{j+1}])$ and $\tilde{\gamma}'([s_j, s_{j+1}])$ are each connected, they must both lie in V. By the local homeomorphism property of p, each point $\gamma(t)$ for $s_j \leq t \leq s_{j+1}$ must have a unique preimage in V, which implies that $\tilde{\gamma}$ and $\tilde{\gamma}'$ agree over $[0, s_{j+1}]$. □

Lemma 3.2. *Let $p : E \to B$ be a covering map, and $H : I \times I \to B$ a map with $H(0, 0) = b_0$. Then for every point $e_0 \in p^{-1}(b_0)$, there is a unique lift $\tilde{H} : I \times I \to E$ such that $\tilde{H}(0, 0) = e_0$.*

Proof. We use the exact same argument as in the previous proposition, noting this time that we can subdivide the unit square into subsquares such that the image of each square under H is contained in an evenly covered neighborhood of B. □

Now, note that we have proved the proposition for general maps from the unit square into the base space B. In the special case of path homotopies, $H|_{\{0\} \times I}$ is a constant path in B, which lifts to the constant path in E at e_0. Since $\tilde{H}|_{\{0\} \times I}$

is also a lift, by the uniqueness of path liftings, it must be the constant path. By the same argument, $\tilde{H}|_{\{1\}\times I}$ is the constant path at e_0. Hence, $\tilde{H}$ is a homotopy of paths.

This fact has important implications. In particular, it says that if two paths are homotopic in the base space, their lifts at a given starting point are also homotopic and in particular must have the same end point. As a special case, if a loop in the base space lifts to a loop in the covering space, then its entire homotopy class does as well.

We can now prove the following important theorem, which I shall follow [3] in calling the *Lifting Criterion*.

Theorem 3.3 (Lifting Criterion)**.** *Let $p : (E, e_0) \to (B, b_0)$ be a covering map, and $f : (X, x_0) \to (B, b_0)$ any map with X path-connected and locally path-connected. Then a lift $\tilde{f} : (X, x_0) \to (E, e_0)$ exists iff $f_*(\pi_1(X, x_0)) \subset p_*(\pi_1(E, e_0))$. Furthermore, if such a lift exists, it is unique.*

Proof. Suppose there is a lift $\tilde{f}$ such that $f = p\tilde{f}$. Then applying the fundamental group functor gives $f_* = p_*\tilde{f}_*$, so the only if statement follows.

Given $x \in X$, choose a path γ in X from x_0 to x. Then $f\gamma$ is a path in B beginning at b_0, and we can lift it to a path $\widetilde{f\gamma}$ in E beginning at e_0. We also have that $\tilde{f}\gamma$ is a path in E beginning at e_0, but since lifts of paths are unique, we must have $\tilde{f}\gamma = \widetilde{f\gamma}$, and in particular $\tilde{f}(x) = \tilde{f}\gamma(1) = \widetilde{f\gamma}(1)$, which is independent of the choice of $\tilde{f}$, which tells us that $\tilde{f}$ is unique.

We now prove the if statement. Given $x \in X$, choose a path γ in X from x_0 to x and define $\tilde{f}(x)$ to be $\widetilde{f\gamma}(1)$ where $\widetilde{f\gamma}$ is the unique lift of $f\gamma$ to E beginning at e_0. To show that this is well-defined, let γ' be another path from x_0 to x in X. Then $\gamma * \overline{\gamma'}$ is a loop in X based at x_0, so $f\gamma * \overline{f\gamma'} = f\gamma * f\overline{\gamma'} = f(\gamma * \overline{\gamma'})$ is a loop in B based at b_0. In particular, its homotopy class is in the image of f_*, and hence the image of p_* by assumption.

This implies that $f\gamma * \overline{f\gamma'}$ is homotopic to a loop in B that lifts to a loop in E based at e_0, but since path homotopies lift to path homotopies, we see that $f\gamma * \overline{f\gamma'}$ itself lifts to a loop α in E based at e_0. Note that we can write $\tilde{\alpha} = \beta * \beta'$ where β and β' are lifts of $f\gamma$ and $\overline{f\gamma'}$ respectively. By the uniqueness of lifted paths, we need to have $\beta = \widetilde{f\gamma}$, and similarly, since $\overline{\beta'}$ is a lift of $f\gamma'$ starting at e_0, we must have $\overline{\beta'} = \widetilde{f\gamma'}$. In particular, this means that $\widetilde{f\gamma}$ and $\widetilde{f\gamma'}$ have the same end point.

To prove continuity of $\tilde{f}$ at $x \in X$, choose an evenly covered neighborhood U of $f(x)$ in B, and let V be the slice of $p^{-1}(U)$ containing $\tilde{f}(x)$. It suffices to show that we can pick a neighborhood W of x that is mapped by $\tilde{f}$ into V. Let $p_0 : V \to U$ be the homeomorphism obtained by restricting p. Since f is continuous, we can pick W such that $f(W) \subset U$. Then given $x \in W$, we have $\tilde{f}(x) = p_0^{-1}f(x) \in V$ by unique lifting, which completes the proof. □

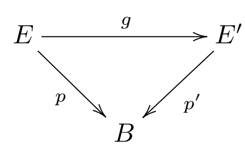
Local path-connectedness is a necessary condition in the Lifting Criterion. To see why, consider the following counterexample.

Example 3.4. Let W be the *quasi-circle*. It is the subspace of $\mathbb{R}^2$ comprising the closed topologist's sine curve (the graph of $y = \sin(1/x)$ defined for $0 \le x \le 1$ with the line segment $[-1, 1]$ in the y-axis attached) with its two ends connected via an arc. This space is path-connected but not locally path-connected. Now let q be the quotient map that collapses $[-1, 1]$ to a point. Note that q can be viewed as a map from W to S^1 because it takes the compactification of $\mathbb{R}$ with $[-1,1]$ to its one-point compactification. But while $\pi_1(W) = 0$, one can show that q does not lift to the covering space $p : \mathbb{R} \to S^1$ by arguing that any lift must map $[-1,1]$ to a copy of the integers in $\mathbb{R}$, which causes problems with injectivity on the rest of W.

4. Classification of Covering Spaces - Isomorphism Conditions

Given a base space B, we want to classify all its covering spaces and the maps between them. To do this, we need to establish a suitable notion for a *map* between covering spaces. Since classification in mathematics is always done up to isomorphism type, we also need to define what it means for two covering spaces to be *isomorphic*.

Definition 4.1. Let $p\colon E \to B$ and $p'\colon E' \to B$ be covering maps, and let $g\colon E \to E'$ be a map such that the following diagram commutes:



We then say that g is a *map of coverings over* B. If g is in addition a homeomorphism, we say that it is an *isomorphism* of coverings.

Remark 4.2. Many popular texts, in particular [3] and [7], omit the notion of a map of coverings, focusing solely on the cases where there is an isomorphism. This is unfortunate because doing so loses a lot of the richness of covering space theory.

It turns out that the existence and uniqueness statements in the Lifting Criterion are precisely the tools we need to determine when maps of coverings exist: Taking the domain space X to be not just any space, but another covering space for the base space B, we can classify maps between covering spaces, and taking X to be E allows us to classify endomorphisms.

Example 3.4 tells us, however, that this analysis is only possible when the covering space E is locally path-connected. By the local homeomorphism property, this means that we also require B to be locally path-connected. If this is the case, B can be decomposed into disjoint open sets B_α that are its path components, and Proposition 2.5 tells us that the maps $p^{-1}(B_\alpha) \to B_\alpha$ obtained by restricting p are also covering maps. This means that, for a given base space, we lose no information by considering its path components individually, so we may go ahead and assume that B is path-connected. Finally, we make the following easy observation:

Proposition 4.3. *Let B be path-connected and locally path-connected. Let $p : E \to B$ be a covering map. If E_0 is a path-component of E, then $p_0 : E_0 \to B$ obtained by restricting p is still a covering map.*

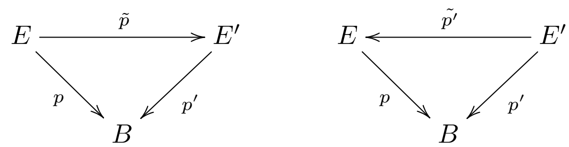
Once again, we lose no information by considering the path-components of the covering space individually. With these observations in mind, therefore, we shall make the following:

Convention 4.4. Unless stated otherwise, all spaces are assumed to be both path-connected and locally path-connected.

For the remainder of this section, we will answer the question of when two covering spaces are isomorphic. As one might suspect from the Lifting Criterion, it turns out that this has everything to do with the fundamental groups of the covering spaces, and their images under the homomorphisms induced by covering maps.

Theorem 4.5. *Let $p : (E, e_0) \to (B, b_0)$ and $p' : (E', e_0) \to (B, b_0)$ be covering maps. There is an isomorphism of coverings $h : (E, e_0) \to (E', e_0)$ if and only if the subgroups $H = p_*(\pi_1(E, e_0))$ and $H' = p_*(\pi_1(E', e_0'))$ of $\pi_1(B, b_0)$ are equal. If h exists, it is unique.*

Proof. Suppose such an isomorphism h exists. Then we have $p_* = p'_* h_*$ and by taking inverses, $p'_* = p_* h^{-1}{}_*$. These respectively imply that $H \subset H'$ and $H' \subset H$, so the two groups are equal. To prove the converse statement, assume that $H = H'$. We shall apply the Lifting Criterion four times. First, p lifts to $\tilde{p} : (E, e_0) \to (E', e_0')$, and p' lifts to $\tilde{p'} : (E', e_0') \to (E, e_0)$, which gives the following two commutative diagrams:



This shows that $\tilde{p}$ and $\tilde{p}'$ are maps of coverings. We also have $p = p'\tilde{p} = p\tilde{p}'\tilde{p}$, so $\tilde{p}'\tilde{p}$ is an isomorphism of E with itself taking e_0 to e_0. Since the identity is also such an isomorphism, the uniqueness part of the Lifting Criterion tells us that $\tilde{p}'\tilde{p} = id_E$. Analogously, we get $\tilde{p}\tilde{p}' = id_{E'}$. Hence, $\tilde{p}$ and $\tilde{p}'$ are inverse isomorphisms. □

This theorem tells us that two based covering spaces have a base-point preserving isomorphism between them if and only if they are associated with the same subgroup of the $\pi_1(B, b_0)$. It is important to note, however, that we have not determined when two covering spaces are isomorphic in general, i.e. when we do not specify base points. We consider this problem now.

Lemma 4.6. *Let $p : E \to B$ be a covering map. Let e_0 and e_1 be points of $p^{-1}(b_0)$, and let $H_i = p_*(\pi_1(E, e_i))$ for $i = 0, 1$. Then H_0 and H_1 are conjugate. Conversely, given e_0, and given a subgroup H of $\pi_1(B, b_0)$ conjugate to H_0, there exists a point e_1 of $p^{-1}(b_0)$ such that $H_1 = H$.*

Proof. We prove the first statement. Recall that we have assumed E to be path-connected, so we can find a loop γ in B based at b_0 lifting to a path $\tilde{\gamma}$ going from e_0 to e_1. Let $[\alpha]$ be an element of H_1 and take a representative α that lifts to a loop $\tilde{\alpha}$ based at e_1. Then $\tilde{\gamma} * \tilde{\alpha} * \overline{\tilde{\gamma}}$ is a loop based at e_0. Projecting down gives $[\gamma][\alpha][\gamma]^{-1} \in H_0$, so we have $[\gamma]H_1[\gamma]^{-1} \subset H_0$. By swapping the places of H_0 and H_1 in the above argument, we also get $[\gamma]^{-1}H_0[\gamma] \subset H_1$, so equality follows.

To prove the converse, suppose H is a conjugate subgroup of H_0. We can write $H_0 = [\gamma]H[\gamma]^{-1}$ for some element $[\gamma] \in \pi_1(B, b_0)$. Lift γ to a path $\tilde{\gamma}$ in E starting at e_0, and let e_1 be its end-point. By the above argument, we have that $H_1 = [\gamma]^{-1}H_0[\gamma] = H$ □

Theorem 4.7. *Let $p : E \to B$ and $p' : E' \to B$ be covering maps, and let $p(e_0) = p'(e_0) = b_0$. There is an isomorphism of coverings $h : E \to E'$ if and only if the subgroups $H = p_*(\pi_1(E, e_0))$ and $H' = p'_*(\pi_1(E', e_0'))$ of $\pi_1(B, b_0)$ are conjugate.*

Proof. Given an isomorphism $h : E \to E'$, let $e'_1 = h(e_0)$, and denote $H'_1 = p_*(\pi_1(E', e'_1))$. Then by Theorem 4.5, we have $H_0 = H'_1$, and by the Lemma 4.6, we have that H'_1 is conjugate to H'_0.

Conversely, suppose H_0 and H'_0 are conjugate. Then by Lemma 4.6, we can pick a point $e_1 \in E$ such that $H_1 = p_*(\pi_1(E, e_1))$ is equal to H'_0. By Theorem 4.5, there is an isomorphism h that takes e_1 to e'_0. □

5. Classification of Covering Spaces - Existence of Covers

Before discussing the main theoretical framework in this section, let us first consider an illustrative example.

Example 5.1. Consider the circle S^1. We stated earlier that its only covering spaces are $\mathbb{R}$ and S^1 with covering maps the usual projection and $f_n : z \mapsto z^n$ respectively. One question that naturally arises is: What does the induced homomorphism of a covering map do to the fundamental group of a covering space? Let $\gamma : I \to S^1$ be a generator for $\pi_1(S^1, s_0)$ (where s_0 is the canonical base point.) Then $f_n\gamma$ wraps around the circle n-times, so $(f_n)_*$ is clearly the n-th power map.

There are three things to notice about this. First, $(f_n)_*$ is injective so it embeds the fundamental group of the covering space as a subgroup of the fundamental group of the base space. Moreover, under the isomorphism that takes $\pi_1(S_1, s_0)$ to $\mathbb{Z}$, $(f_n)_*(\pi_1(S^1, s_0))$ is taken to $n\mathbb{Z}$. Finally, there are n preimages of any given point x under f_n, and this number is precisely the index of $(f_n)_*(\pi_1(S^1, s_0))$ in $\pi_1(S^1)$.

Letting n run over all the natural numbers, we see that every subgroup of $\pi_1(S^1, s_0)$ is the image of the fundamental group of a covering space under its induced homomorphism. Furthermore, in accordance with the results of Section 4, for each subgroup of $\pi_1(S^1, s_0)$, there is only one isomorphism type of covering space associated to it. (Here, isomorphism type and base-point preserving isomorphism type are equivalent notions since $\pi_1(S^1, s_0)$ is abelian.) In other words, there is a bijection between isomorphism types of covering spaces of S^1 and the subgroups of its fundamental group.

The various facets of the above phenomenon happen much more generally. In particular, we have:

Proposition 5.2. *Let $p : (E, e_0) \to (B, b_0)$ be a covering map, and let $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ be the induced homomorphism. This map is injective. Furthermore, the image subgroup $p_*(\pi_1(E, e_0)) \subset \pi_1(B, b_0)$ consists of the homotopy classes of loops in B based at b_0 whose lifts to E starting at e_0 are loops.*

Proof. This follows immediately from the path and homotopy lifting lemmas. □

We now see the results of the previous section in a new light: There we compared subgroups of $\pi_1(B, b_0)$ to solve the problem of when two covering spaces were isomorphic. But the observation can be turned on its head to say that each subgroup of $\pi_1(B, b_0)$ has at most one basepoint preserving isomorphism type of covering space realizing it, and each conjugacy class of subgroups, at most one isomorphism type.

In fact, given a certain "local niceness" condition on the base space that is in particular satisfied by S^1, we can establish the existence of a covering space realizing each subgroup of $\pi_1(B, b_0)$, and the association then becomes a correspondence that is reminiscent of that between subfields of a given field and the subgroups of its Galois group in Galois theory.

The true picture is trickier than it seems. We shall deal with it gradually over the rest of this paper, and here content ourselves to prove the setwise bijection. The first step is to find a covering space realizing the trivial subgroup. We define it as follows:

Definition 5.3. Let $p : E \to B$ be a covering map. We say that E is a *universal covering space* or simply a *universal cover* if it is simply connected.

Now, not every space affords a universal cover. We mentioned earlier that some "local niceness" condition needs to be imposed. We explain what this condition is now.

Definition 5.4. A space B is said to be *semilocally simply connected* if for each $b \in B$, there is a neighborhood U of b such that the homomorphism $i_* : \pi_1(U, b) \to \pi_1(B, b)$ induced by inclusion is trivial.

Lemma 5.5. *If the space B has a universal cover, then it is semilocally simply connected.*

Proof. Let $p : E \to B$ be a covering map from a universal cover. Let b be a point in B, and pick a neighborhood U of b that is evenly covered by p. Fix a point $e \in p^{-1}(b)$, and let V be the slice of $p^{-1}(U)$ containing e. Let γ be a loop in U based at b. Using the local homeomorphism property, we can lift it to a loop $\tilde{\gamma}$ in V based at e. But since E is simply connected, there is a homotopy H in E between $\tilde{\gamma}$ and the constant loop, and this projects down to a homotopy between γ and the constant loop in B. In other words, $i_*([\gamma])$ is trivial. □

It turns out that this condition characterizes the class of spaces that possess universal covers. In other words, being semilocally simply connected is not only necessary but also sufficient, and we shall prove this by constructing a universal cover for an arbitrary semilocally simply connected space. To motivate the construction, we first make the following observation, whose proof we leave as an easy exercise.

Proposition 5.6. *A space E is simply connected if and only if there is a unique homotopy class of paths connecting any two points in E.*

Suppose we have a simply connected space E and pick a base point e_0. Then this proposition tells us precisely that there is a bijection between the homotopy classes of paths starting at e_0 and the points in E. On the other hand, if E were a covering space for a base space B under a map p that sends e_0 to $b_0 \in B$, then the path and homotopy lifting properties tell us that there is a bijection between homotopy classes of paths in E starting at e_0 and homotopy classes of paths in B starting at b_0. This means that we could try to construct a universal E_0 for a base space (B, b_0) by thinking of homotopy classes of paths starting at b_0 as the points of E_0, which has the advantage of allowing us to describe E_0 purely in terms of B. We shall use this idea to prove:

Theorem 5.7. *The space B has a universal cover if and only if it is semilocally simply connected.*

Proof. Fix a basepoint b_0 of B, and define a space E_0 by setting its underlying point set to be

$$E_0 = \{[\gamma] \mid \gamma \text{ is a path in } B \text{ starting at } b_0\}$$

where, $[\gamma]$ denotes the (end-point preserving) homotopy class of γ. Let $p\colon E_0 \to B$ send $[\gamma]$ to $\gamma(1)$. Then p is surjective, and we shall see that defining the right topology on E_0 will make p into a covering map.

Let $\mathcal{U}$ be the collection of path-connected open sets $U \subset B$ such that the homomorphism induced by inclusion, $\pi_1(U) \to \pi_1(B)$ is trivial. Note that this does not depend on our choice of basepoint in U. We can find such a neighborhood for every point in B because it is both simply connected and locally path-connected. Furthermore, if $U \in \mathcal{U}$ and V is a path-connected open subset of U, then the homomorphism induced by inclusion of V in B factors as $\pi_1(V) \to \pi_1(U) \to \pi_1(B)$, and is thus also trivial. This implies that $\mathcal{U}$ is a basis for the topology on B.

We now define a topology on E_0. Given $U \in \mathcal{U}$, and a path γ in B from b_0 to a point in U, let

$$U_{[\gamma]} = \{[\gamma * \eta] \mid \eta \text{ is a path in } U \text{ with } \eta(0) = \gamma(1)\}.$$

Since the operation $*$ is well defined on homotopy classes, we see that the definition is independent of our choice of representative γ. We have that $p\colon U_{[\gamma]} \to U$ is surjective because U is path-connected, and it is injective because different choices of η with the same endpoint are homotopic in B by assumption.

Moreover, if $[\gamma'] \in U_{[\gamma]}$, then we can write $\gamma' = \gamma * \eta$ for some η in U, and elements of $U_{[\gamma']}$ are of the form $[\gamma * \eta * \mu]$, where μ is a path in U. Since $\eta * \mu$ is a path in U, we have $U_{[\gamma']} \subset U_{[\gamma]}$. Analogously, we get the opposite inclusion, so $U_{[\gamma']} = U_{[\gamma]}$.

We claim that the sets $U_{[\gamma]}$ form a basis for a topology on E_0. To prove this, suppose we pick an element $[\gamma'']$ in the intersection of two sets $U_{[\gamma]}$ and $V_{[\gamma']}$. Pick a neighborhood W of $\gamma''(1)$ that is contained in $U \cap V$. Then we clearly have $W_{[\gamma'']} \subset U_{[\gamma'']} = U_{[\gamma]}$, and $W_{[\gamma'']} \subset V_{[\gamma'']} = V_{[\gamma']}$, so $W_{[\gamma'']}$ is a neighborhood of $[\gamma'']$ that lies in the intersection. This proves the claim, and we let the topology on E_0 be that generated by this basis.

To show that p is a covering map, we first verify that each neighborhood $U \in \mathcal{U}$ is evenly covered by p. It is clear that $p^{-1}(U)$ is the union of the sets $U_{[\gamma]}$. Our remarks earlier tell us that these sets are disjoint, and if we fix one of these, we have already proved that the restriction $p|_{U_{[\gamma]}}$ gives a bijection with U. Furthermore, one can check that $p|_{U_{[\gamma]}}$ induces a bijection between the open subsets contained in $U_{[\gamma]}$ and those contained in U, and is thus a homeomorphism. The fact that p is a local homeomorphism automatically implies that it is continuous.

Finally, we show that E_0 is simply connected. Let $[b_0]$ denote the constant path at b_0 and choose it as our basepoint. Observe that given $[\gamma] \in E_0$, we can define a family of paths, $\{\gamma_s | s \in I\}$ by $\gamma_s(t) = \gamma(st)$. The map $I \to E_0$ given by $s \mapsto [\gamma_s]$ can easily be checked to be continious, and thus defines a path between $[b_0]$ and $[\gamma]$. This proves that E_0 is path-connected.

Let $f\colon I \to E_0$ be a loop based at $[b_0]$. Then pf is a loop γ based at b_0. Define the map $f'\colon s \mapsto [\gamma_s]$ as in the above paragraph. Then f' is a lift of γ based at $[b_0]$, so by the unique lifting of paths, we have $f = f'$. In particular, $[b_0] = f(1) = f'(1) = [\gamma_1] = [\gamma]$, so γ is nullhomotopic in B, and by the homotopy lifting property, f must also be nullhomotopic. In other words, $\pi_1(E_0, [b_0]) = 0$. □

Since universal covers are unique up to isomorphism, we often talk about the universal cover of a given base space. The reason for the adjective *universal* is because it is also a covering space for the covering spaces of B realizing the intermediate subgroups of $\pi_1(B, b_0)$. We shall consider this in a later section, but we note here that the existence of a universal cover is sufficient to guarantee the existence of covering spaces representing all of the subgroups of $\pi_1(B, b_0)$.

Proposition 5.8. *Let B be semilocally simply connected. Then for every subgroup $H \leq \pi_1(B, b_0)$, there is a covering space $p\colon E_H \to B$ such that $p_*(\pi_1(E_H, e_0)) = H$ for a suitably chosen basepoint $e_0 \in E_H$.*

Proof. Construct E_0, the universal cover of B with respect to the basepoint b_0, and define an equivalence relation on the set of points as follows: We set $[\gamma] \sim [\gamma']$ if $\gamma(1) = \gamma'(1)$ and $[\gamma * \overline{\gamma'}] \in H$. One can check that this relation is well-defined. We then let E_H be the quotient space $E_0/\sim$ generated by this equivalence relation.

Now, recall that the map $p\colon E_0 \to B$ sends $[\gamma]$ to $\gamma(1)$. Our definition of $\sim$ tells us that p factors through the quotient map and induces a continuous map $p'\colon E_H \to B$. We want to show that p' is a covering map, so let q denote the quotient map and let U be a neighborhood that is evenly covered by p.

First, observe that since $U_{[\gamma]}$ is locally homeomorphic with U, q is injective on each $U_{[\gamma]}$. Now, suppose we have $\gamma(1) = \gamma'(1) \in U$, and $[\gamma] \sim [\gamma']$. Then for any path η with $\eta(0) = \gamma(1)$, we have $[\gamma * \eta] \sim [\gamma' * \eta]$, so $q(U_{[\gamma]}) = q(U_{[\gamma']})$. Furthermore, for each open subset $V \subset U$, pick $[\mu], [\mu']$ such that $\mu(1) = \mu'(1) \in V$, and $V_{[\mu]} \subset U_{[\gamma]}, V_{[\mu']} \subset U_{[\gamma']}$. It is clear that we have $q(V_{[\mu]}) = q(V_{[\mu']})$. As such, the restriction $q|_{U_{[\gamma]}}\colon U_{[\gamma]} \to q(U_{[\gamma]})$ is in fact a homeomorphism. We can then write $p'|_{q(U_{[\gamma]})} = p|_{U_{[\gamma]}}(q|_{U_{[\gamma]}})^{-1}$ so p' is a local homeomorphism on each set $q(U_{[\gamma]})$.

This proves that E_H is a covering space for B, and it remains to show that $p_*(\pi_1(E_H, [b_0])) = H$. But given a loop γ in B based at b_0, we see that $[\gamma]$ is in the image of p'_* if and only if γ lifts to a loop in E_H. But using the same argument from the proof of Theorem 5.7, we see that the lift of γ starts at $[b_0]$ and ends at $[\gamma]$, and so is a loop in E_H if and only if $[\gamma] \sim [b_0]$, or equivalently, $[\gamma] \in H$. □

The end result of the previous two sections is the following classification theorem:

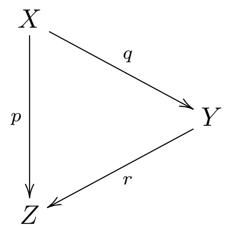
Theorem 5.9 (Classification of connected coverings)**.** *Let B be semilocally simply connected. Fix a basepoint b_0. Then*

(1) There is a bijection between the set of isomorphism classes of based covering spaces $p\colon (E, e_0) \to (B, b_0)$ and the set of subgroups of $\pi_1(B, b_0)$ obtained by associating the covering space $p\colon E \to B$ with the subgroup $p_*(\pi_1(E, e_0))$.
(2) This association also induces a bijection between the set of isomorphism classes of covering spaces $p\colon E \to B$ and conjugacy classes of subgroups of $\pi_1(B, b_0)$.

6. Classification of Covering Spaces - A Lattice Isomorphism

It is common in the literature (c.f. [3], [7]) to stop an exposition into the classification of covering spaces with Theorem 5.9. We realize, however, that we can easily prove something deeper based on the following observation:

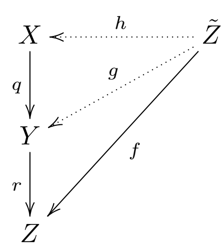
Proposition 6.1. *Let X, Y and Z be spaces. Let p, q and r be continuous maps such that the following the diagram commutes:*



(1) *If p and r are covering maps, so is q.*
(2) *If p and q are covering maps, so is r.*
(3) *Suppose Z has a universal cover, then if q and r are covering maps, so is p.*

Proof. The proof for (1) and (2) is a rather tedious and unenlightening affair in point-set topology. We hence refer the reader to [7] for a complete proof.

To prove (3), let $f\colon \tilde{Z} \to Z$ be the universal cover for Z. By the Lifting Criterion, there is a lift of f to $g\colon \tilde{Z} \to Y$. Since $r\colon Y \to Z$ and $f\colon \tilde{Z} \to Z$ are both coverings of Z, by (1), g is also a covering. We then repeat the argument one more: Applying the Lifting Criterion, we obtain a lift of g to $h\colon \tilde{Z} \to X$, and observe that since $q\colon X \to Y$ and $g\colon \tilde{Z} \to Y$ are both coverings of Y, h is also a covering by (1). One can then check that we have the following string of equalities: $ph = rqh = rg = f$, so we can apply (2) to conclude that p is a covering. The argument is encapsulated in the following diagram:



□

This proposition proves our assertion in the last section that all intermediate covering spaces of a given base space are themselves covered by the universal cover. In fact, more is true: It tells us that in the context of a space with a universal cover, maps of coverings and covering maps are one and the same thing, so the erstwhile confusing similarity between the two terms may be excused on this account.

Furthermore, the proposition tells us that the following category embeds in $\mathbf{Top}_*$, the category of based spaces, as a full subcategory.

Definition 6.2. Let (B, b_0) have a universal cover. We define a category $\mathrm{Cov}(B, b_0)$ by setting its objects to be based covers of (B, b_0) and its morphisms to be the based coverings $p\colon (E, e_0) \to (E', e_0')$.

Our goal now is to state part (1) of the classification theorem of the last section in terms of an isomorphism of categories. Unfortunately, $\mathrm{Cov}(B, b_0)$ does not quite fit our purposes because the objects described in the classification theorem are isomorphism types of based covers rather than individual spaces. We get around this by constructing a second category $\overline{\mathrm{Cov}}(B, b_0)$ as follows:

Pick one representative from each isomorphism type of based cover. We define $\overline{\mathrm{Cov}}(B, b_0)$ to be the full subcategory spanned by these objects in $\mathrm{Cov}(B, b_0)$. Notice that this actually defines a skeleton of $\mathrm{Cov}(B, b_0)$, and as with skeletons in general, our choice of $\overline{\mathrm{Cov}}(B, b_0)$ is non-unique.

Now, a poset $(\mathcal{P}, \leq)$ can be interpreted as a category, with objects the set elements of $\mathcal{P}$ and morphisms $a \to b$ if $a \leq b$ in $\mathcal{P}$. We can then turn this observation on its head and define a poset to be a category with at most one morphism between any two objects.

Proposition 6.3. *The category $\overline{\mathrm{Cov}}(B, b_0)$ is a poset.*

Proof. Let (E, e_0) and (E', e_0') be objects of $\overline{\mathrm{Cov}}(B, b_0)$. If there is a covering $p\colon (E, e_0) \to (E', e_0')$, then by our discussion above, p is a based map of coverings of the base space (B, b_0), and so is unique by the Lifting Criterion. Furthermore, if there is a covering $q\colon (E', e_0') \to (E, e_0)$, then the two based spaces are isomorphic and hence must be the same object by construction. □

The set of subgroups of a group G can be ordered under inclusion to form a poset, which we shall denote $\mathcal{P}(G)$. If we let $\Lambda\colon \mathcal{P}(\pi_1(B, b_0)) \to \overline{\mathrm{Cov}}(B, b_0)$ be the set bijection taking a subgroup of $\pi_1(B, b_0)$ to the based space realizing it, we can ask whether Λ is a functor.

Theorem 6.4. *The map $\Lambda\colon \mathcal{P}(\pi_1(B, b_0)) \to \overline{\mathrm{Cov}}(B, b_0)$ is an isomorphism of categories. In fact, it is a lattice isomorphism.*

Proof. We have already shown that it is a bijection on objects. To prove Λ is a functor, we just need to show it is order-preserving in both directions, i.e. that it is bijective on morphisms. Now, suppose we have a morphism $H \to K$ in $\mathcal{P}(\pi_1(B, b_0))$, so that H is a subgroup of K. Write $(E_H, e_0) = \Lambda(H)$, $(E_K, e_0') = \Lambda(K)$, and let $p\colon (E_H, e_0) \to (B, b_0)$, and $p'\colon (E_K, e_0') \to (B, b_0)$ denote the respective covering maps. Then by the Lifting Criterion, there is lift of p to a covering map $q\colon (E_H, e_0) \to (E_K, e_0')$, and so a morphism $\Lambda(H) \to \Lambda(K)$. The converse is proved by reversing the above argument.

Since $\mathcal{P}(\pi_1(B, b_0))$ is a lattice, we automatically get that $\overline{\mathrm{Cov}}(B, b_0)$ is a lattice, and that Λ is a lattice isomorphism. □

Remark 6.5. Many texts and online sources on covering spaces call attention to the analogy between the classification of covering spaces and the following fact from Galois Theory: If K/F is a Galois extension, then there is a lattice isomorphism between the poset of subfields of K containing F and the poset of subgroups of $\mathrm{Gal}(K/F)$ given by the inverse functors $\mathrm{Aut}(K/-)$ and $\mathrm{Fix}(-)$. This is not entirely helpful, because the analogy breaks down when we take a look at the bigger picture.

If K/F is an arbitrary field extension, $\mathrm{Aut}(K/-)$ and $\mathrm{Fix}(-)$ are no longer isomorphisms, and it is possible to have $\mathrm{Fix}(\mathrm{Gal}(K/L)) \neq L$. On the other hand, they are still order-reversing maps of posets, and the correspondence they give satisfies the conditions for what is known in order theory as a *Galois connection*. There is no such generalization for $\overline{\mathrm{Cov}}(B, b_0)$.

Moreover, there is still another analogy between covering space theory and this fact from Galois theory that arises when we consider the automorphism groups of covers. In this case, we actually have a full analogue of the Fundamental Theorem of Galois Theory. We pursue this observation in a later section.

7. Interlude - Homotopy Relationships with the Base Space

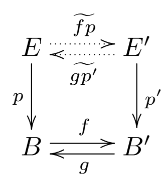
Let us take a brief detour from our main exposition to examine how we can use covering spaces as a tool for studying the homotopy properties of spaces.

Proposition 7.1. *A covering map $p\colon (E, e_0) \to (B, b_0)$ induces isomorphisms $p_*\colon \pi_n(E, e_0) \to \pi_n(B, b_0)$ for $n \geq 2$.*

Proof. Since S^n is simply connected for $n \geq 2$, any map $f\colon (S^n, s_0) \to (B, b_0)$ satisfies the Lifting Criterion trivially, and so has a lift $\tilde{f} : (S^n, s_0) \to (E, e_0)$. This proves surjectivity. Injectivity follows from the fact that a homotopy in the base space lifts to a homotopy in the covering space. □

Proposition 7.2. *Let E_0 and E_0' be universal covers for B and B' respectively. If B and B' are homotopically equivalent, then so are E_0 and E_0'.*

Proof. Let $f\colon B \to B'$ and $g\colon B' \to B$ be homotopy inverses, and let $p\colon E_0 \to B$ and $p'\colon E_0' \to B'$ denote the covering maps. By the Lifting Criterion, we have lifts $\widetilde{fp}\colon E_0 \to E_0'$ and $\widetilde{gp'}\colon E_0' \to E_0$, giving the following commutative diagram:



We claim that the two lifts are homotopy inverses. To see this, observe that $p'\widetilde{fp} = fp$, so we have $\widetilde{gp'}\widetilde{fp} = \widetilde{gfp}$. Now let H be the homotopy between gf and id_B. Then $H \circ (p \times id_B)\colon E_0 \times I \to B$ is a map from a simply connected space into B. The Lifting Criterion applies, and we obtain a lift that one can check to be a homotopy between $\widetilde{gfp}$ and $\tilde{p}$. But $\tilde{p}$ is an automorphism of covers that fixes a point, so by the uniqueness portion of Theorem 4.5, it must be the identity. Arguing similarly for $\widetilde{fp}\widetilde{gp'}$ completes the proof of our claim. □

The converse is not true as seen from the fact that $\mathbb{R}^2$ is a covering space for both the torus and the Klein bottle, but these two base spaces are not homotopically equivalent.

Proposition 7.3. *Let X be path-connected and locally path-connected, and fix a base point x_0. If $\pi_1(X, x_0)$ is finite, then every map $X \to S^1$ is nullhomotopic.*

Proof. Let $f : (X, x_0) \to (S^1, s_0)$ be a map. Then $f_*(\pi_1(X, x_0))$ is isomorphic to a quotient of $\pi(X, x_0)$, and is hence finite. In particular, it is torsion. On the other hand, $f_*(\pi_1(X, x_0)) \leq \pi(S_1, s_0) \cong \mathbb{Z}$, and as a subgroup of a free group, must itself be free. So the only possibility is that $f_*(\pi_1(X, x_0)) = 0$, so by the Lifting Criterion, there is a lift $\tilde{f} : (X, x_0) \to (\mathbb{R}, e_0)$, which is nullhomotopic. This nullhomotopy projects down to give a nullhomotopy for f. □

Corollary 7.4. *Every map $f : S^n \to S^1$ is nullhomotopic.*

Observe that the proof of the proposition still works when we replace S^1 by any other space with a free (or free abelian) fundamental group and a contractible universal cover, and if we allow $\pi_1(X, x_0)$ to be infinite so long as it remains torsion. We thus also have:

Corollary 7.5. *Let X be path-connected and locally path-connected, and let W be homotopically equivalent to a wedge of circles or to a torus of any dimension. If every element of $\pi_1(X, x_0)$ has finite order, then every map $X \to W$ is nullhomotopic.*

8. Covering of Groupoids

Fundamental groups serve as natural algebraic models for connected based topological spaces. They are not, however, entirely convenient when the space we consider is disconnected, in which case the fundamental group depends on the choice of path component, or when the space is not based, so that we have to make an arbitrary choice of a base point. We hence introduce something far more natural.

Definition 8.1. Let X be a space. Then the *fundamental groupoid* of X, denoted $\Pi(X)$, is the category with objects the points of X, and whose morphisms $x \to y$ are the homotopy classes of paths between x and y.

A map $f: X \to Y$ takes homotopy classes of paths to homotopy classes of paths. In other words, it induces a map of groupoids $f_*: \Pi(X) \to \Pi(Y)$. Letting Π act on morphisms by $f \mapsto f_*$ then defines a functor $\Pi(-): \mathbf{Top} \to \mathbf{Gpd}$ from the category of topological spaces to the category of groupoids.

Now, recall that a *skeleton* $\mathrm{Sk}(\mathcal{C})$ of a category $\mathcal{C}$ is defined as a full subcategory with one object from each isomorphism class of objects of $\mathcal{C}$. It is an important notion because the inclusion functor $J: \mathrm{Sk}(\mathcal{C}) \to \mathcal{C}$ gives an equivalence of categories.

Given a point $x_0 \in X$, the fundamental group based at that point, $\pi_1(X, x_0)$, comprises all homotopy classes of paths starting and ending at x_0, and so embeds as a full subcategory of $\Pi(X)$. As such, a skeleton for $\Pi(X)$ comprises the disjoint union of a collection of fundamental groups, one for each path-component of X. In particular, we have

Proposition 8.2. *Let X be a path-connected space. Then for each point $x_0 \in X$, $\pi_1(X, x_0)$ is a skeleton for $\Pi(X)$, and the inclusion functor $\pi_1(X, x_0) \hookrightarrow \Pi(X)$ is an equivalence of categories.*

It turns out that much of the theory of covering spaces has an analogue for groupoids. In fact, both theories have a beautiful confluence, which we shall present in the next section. In this section, we first develop the fundamentals of a theory for covering groupoids, loosely following the approach of [6].

We say that a groupoid is *connected* if any two objects have at least one morphism between them. It is easy to see that a space X is path-connected if and only if $\Pi(X)$ is connected. With the same rationale as with spaces, we restrict our attention to connected groupoids.

Let us first develop an analogue of a fundamental neighborhood for groupoids.

Definition 8.3. Let $\mathcal{C}$ be a small groupoid, and x an object of $\mathcal{C}$. The *star* of x, denoted $\mathrm{St}_{\mathcal{C}}(x)$ is the set of morphisms of $\mathcal{C}$ with domain x. We also use the suggestive notation $\pi(\mathcal{C}, x)$ to denote $\mathcal{C}(x, x) \subset \mathrm{St}_{\mathcal{C}}(x)$, the group of automorphisms of the object x.

Remark 8.4. If X is a space and x_0 a point of X, we have $\pi_1(X, x_0) = \pi(\Pi(X), x_0)$.

Definition 8.5. Let $\mathcal{E}$ and $\mathcal{B}$ be small groupoids. A *covering of groupoids* $p\colon \mathcal{E} \to \mathcal{B}$ is a functor that is surjective on objects and restricts to a bijection $p\colon \mathrm{St}(e) \to \mathrm{St}(p(e))$ for each object e of $\mathcal{E}$.

We immediately get

Proposition 8.6. *Let* $p : (\mathcal{E}, e_0) \to (\mathcal{B}, b_0)$ *be a covering of groupoids, then the restriction* $p : \pi(\mathcal{E}, e_0) \to \pi(\mathcal{B}, b_0)$ *is injective.*

Observe that if we let F_b denote the set of objects of $\mathcal{E}$ such that $p(e) = b$, then $p^{-1}(\mathrm{St}(b))$ is the disjoint union over $e \in F_b$ of $\mathrm{St}(e)$. This is the analogue of the partition of the preimage of a fundamental neighborhood into slices. Recall that with covering spaces, the first step was to use this property of covering maps to obtain unique lifting of paths and homotopies. The case for groupoids is more spartan: the analogue of a homotopy class of paths between two points b and b' is simply a morphism between them, and given a choice of source $e \in F_b$, this morphism lifts uniquely by definition. We can hence immediately prove

Theorem 8.7 (Lifting Criterion)**.** *Let* $p\colon (\mathcal{E}, e_0) \to (\mathcal{B}, b_0)$ *be a covering of groupoids, and* $f\colon (\mathcal{X}, x_0) \to (\mathcal{B}, b_0)$ *any map with* $\mathcal{X}$ *connected. Then a lift* $\tilde{f}\colon (\mathcal{X}, x_0) \to (\mathcal{E}, e_0)$ *exists if and only if* $f(\pi(\mathcal{X}, x_0)) \subset p(\pi(\mathcal{E}, e_0))$*. Furthermore, if such a lift exists, it is unique.*

Proof. If $\tilde{f}$ exists, then by definition, we have $f = p\tilde{f}$, so the image of f is contained in that of p. To prove the converse, let x be an object of $\mathcal{X}$, and let $\alpha\colon x_0 \to x$ be a morphism in $\mathcal{X}$. Let $\tilde{\alpha}$ be the unique morphism with source e_0 projecting down to $f(\alpha)$. We define $\tilde{f}$ by setting $\tilde{f}(x)$ to be the target of $\tilde{\alpha}$, and check that it is a well-defined functor.

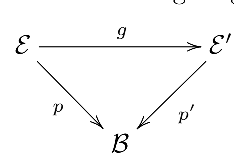
Suppose we have another morphism $\alpha'\colon x_0 \to x$. Then there is a unique lift of $f(\alpha^{-1}\alpha')$ to a morphism β in $\mathcal{E}$ with source e_0. But by assumption, β is an element of $\pi(\mathcal{E}, e_0)$ and thus also has target e_0. We then have the following chain of equalities:

$$p(\tilde{\alpha}\beta) = p(\tilde{\alpha})p(\beta) = f(\alpha)f(\alpha^{-1}\alpha') = f(\alpha')$$

which implies that by definition, $\tilde{\alpha}' = \tilde{\alpha}\beta$. As such, $\tilde{\alpha}$ has the same target as $\tilde{\alpha}'$, so $\tilde{f}$ is well-defined on objects and morphisms. Finally, note that $\tilde{f}$ is functorial because every morphism in $\mathcal{X}$ can be written in terms of elements of $\mathrm{St}_{\mathcal{X}}(x_0)$, and on these, we can write $\tilde{f}$ as a composition of functors, namely $\tilde{f} = (p|_{\mathrm{St}_{\mathcal{B}}(b_0)})^{-1} f$. □

As with covering spaces, this theorem allows us to prove the analogues of our results from Sections 4 and 5: we can classify covers by associating them with subgroups of $\pi(\mathcal{B}, b_0)$. Since the proofs are similar to those for covering spaces, we leave these to the reader.

Definition 8.8. Let $p\colon \mathcal{E} \to \mathcal{B}$ and $p'\colon \mathcal{E}' \to \mathcal{B}$ be coverings of groupoids, and let $g\colon \mathcal{E} \to \mathcal{E}'$ be a functor such that the following diagram commutes:



We then say that g is a *map of coverings over* B. If g is in addition a groupoid isomorphism, we say that it is an *isomorphism* of coverings.

Theorem 8.9. *Let* $p\colon (\mathcal{E}, e_0) \to (\mathcal{B}, b_0)$ *and* $p\colon (\mathcal{E}, e_0) \to (\mathcal{B}, b_0)$ *be covering maps. There is an isomorphism of coverings* $h : (\mathcal{E}, e_0) \to (\mathcal{E}', e_0)$ *if and only if the groups* $H = p(\pi(\mathcal{E}, e_0))$ *and* $H' = p(\pi(\mathcal{E}', e_0'))$ *of* $\pi(\mathcal{B}, b_0)$ *are equal. If* h *exists, it is unique.*

Lemma 8.10. *Let* $p : \mathcal{E} \to \mathcal{B}$ *be a covering of groupoids. Let* e_0 *and* e_1 *be points of* $p^{-1}(b_0)$*, and let* $H_i = p(\pi(\mathcal{E}, e_i))$ *for* $i = 0, 1$*. Then* H_0 *and* H_1 *are conjugate. Conversely, given* e_0*, and given a subgroup* H *of* $\pi(\mathcal{B}, b_0)$ *conjugate to* H_0*, there exists a point* e_1 *of* $p^{-1}(b_0)$ *such that* $H_1 = H$*.*

Theorem 8.11. *Let* $p : \mathcal{E} \to \mathcal{B}$ *and* $p' : \mathcal{E}' \to \mathcal{B}$ *be covering maps, and let* $p(e_0) = p'(e_0) = b_0$*. There is an isomorphism of coverings* $h : E \to E'$ *if and only if the subgroups* $H = p(\pi(\mathcal{E}, e_0))$ *and* $H' = p(\pi(\mathcal{E}', e_0'))$ *of* $\pi(\mathcal{B}, b_0)$ *are conjugate.*

Proposition 8.12. *Let* $\mathcal{B}$ *be a small groupoid. Then for every subgroup* $H \leq \pi(\mathcal{B}, b_0)$*, there is a covering* $p\colon \mathcal{E}_H \to \mathcal{B}$ *such that* $p(\pi(\mathcal{E}_H, e_0)) = H$ *for a suitably chosen base object* $e_0 \in \mathcal{E}_H$*.*

Theorem 8.13 (Classification of connected coverings of groupoids)**.** *Let* $\mathcal{B}$ *be a groupoid. Fix a base object* b_0*. Then*

(1) There is a bijection between the set of isomorphism classes of based covering groupoids $p\colon (\mathcal{E}, e_0) \to (\mathcal{B}, b_0)$ *and the set of subgroups of* $\pi(\mathcal{B}, b_0)$ *obtained by associating the covering space* $p\colon \mathcal{E} \to \mathcal{B}$ *with the subgroup* $p(\pi(\mathcal{E}, e_0))$*.*
(2) This association also induces a bijection between the set of isomorphism classes of covering spaces $p\colon \mathcal{E} \to \mathcal{B}$ *and conjugacy classes of subgroups of* $\pi(\mathcal{B}, b_0)$*.*

Remark 8.14. It is possible to first classify covering groupoids and then use some of the results to provide immediate analogues for the theory of covering spaces. Indeed, this is the approach of [6]. We have chosen to present the concepts in the inverse order, however, because it is instructive to see the specific instances where considering groupoids rather than spaces simplifies the theory.

9. Group Actions and Groupoid Actions

Having classified covering spaces and covering groupoids up to isomorphism type, we would next like to classify the maps between these objects. To do so, we need to make a brief detour by considering the theory of group actions and how we can generalize this to obtain the notion of a groupoid action. Let us first recall some basic definitions.

Definition 9.1. Let G be a group, and S a set.

(1) A (left) *action* of a group G on a set S is a function $\cdot\colon G \times S \to S$ such that $e \cdot s = s$ and $(g'g) \cdot s = g' \cdot (gs)$ for all $s \in S$. The axiom of associativity allows us to omit writing $\cdot$ without causing ambiguity.
(2) The *stabilizer* or *isotropy group* G_s of a point s is the subgroup of G with elements $\{g \mid gs = s\}$.
(3) An action is *free* or *semiregular* if all stabilizers are trivial.
(4) An action is *transitive* if for every pair of set elements s and $s' \in S$, there is a group element $g \in G$ such that $gs = s'$.
(5) An action is *regular* if it is both free and transitive.

Remark 9.2. We call a set S with a G-action a *G-set*, and we often refer to it by simply 'S'. This is not entirely rigorous, for we can put two different G-actions on the same set, so one should specify the associated map $G \times S \to S$. Nonetheless, we seldom do this in practice.

Definition 9.3. Let G be a group, and let S and S' be G-sets. A *G-map* $\phi\colon S \to S'$ is a set function that respects the G-set structure, i.e. such that $\phi(gs) = g\phi(s)$ for all $s \in S$ and $g \in G$. If ϕ is also a bijection, we call it an *isomorphism* of G-sets.

Example 9.4. Let H be a subgroup of G. Left multiplication by elements of G makes the set of cosets G/H into a transitive G-set. If we fix a coset gH, its stabilizer is simply gHg^{-1}.

It turns out that this example, in a sense, accounts for all possible transitive G-sets. More precisely, we have

Proposition 9.5. *Let G act transitively on a set S. Pick any element $s \in S$, and let G_s denote its stabilizer. Then S is isomorphic to G/G_s as a G-set.*

Proof. Define $\phi\colon S \to G/G_s$ by sending gs to gG_s for each $g \in G$. We then have $\phi(gs) = gG_s = g\phi(s)$. The map is clearly surjective. It is injective as well because if $gG_s = hG_s$, then $gh^{-1} \in G_s$, so $gs = hs$. □

A G-set can be interpreted as a functor from G seen as a category with one object to the category of sets, so studying group actions of G is equivalent to studying the functor category $\mathbf{Set}^G$. If we restrict our attention to the subcategory consisting of transitive actions, then Proposition 9.5 tells us precisely that the following full subcategory contains a skeleton:

Definition 9.6. Let G be a group. The *orbit category* of G, denoted $\mathcal{O}(G)$ is defined to be the category with orbits the G-sets G/H and morphisms the G-maps between them. These G-sets are called *canonical orbits*.

Having dealt with the objects, we now study the morphisms in more detail. It is easy to see that a G-map of transitive G-sets is fully determined by where it sends a single element. We then have

Proposition 9.7. *Let $\phi\colon G/H \to G/K$ be a G-map, and let $\gamma \in G$ be such that $\phi(H) = \gamma^{-1}K$. Then $\gamma H \gamma^{-1}$ is a subgroup of K. Conversely, for every $\gamma \in G$ such that $\gamma H \gamma^{-1}$ is a subgroup of K, the map $\phi_\gamma\colon gH \mapsto g\gamma^{-1}K$ defines a G-map. Finally $\phi_\gamma = \phi_{\gamma'}$ if and only if both γ and γ' belong to the same coset of K.*

Proof. For all $h \in H$, we have $\gamma^{-1}K = \phi(H) = \phi(hH) = h\phi(H) = h\gamma^{-1}K$, which implies that $\gamma h \gamma^{-1} \in K$, so the first statement follows. The second statement and third statements are obvious. □

As is the case with any other mathematical object, the automorphisms of a G-set form a group. The following proposition characterizes its structure.

Proposition 9.8. *The automorphism group of the G-set G/H is isomorphic to $N_G(H)/H$, the quotient of the normalizer of H in G by H.*

Proof. We define a map $\Phi\colon N_G(H)/H \to \mathrm{Aut}_G(G/H)$ by $\gamma H \mapsto \phi_\gamma$, where ϕ_γ is the G-map defined in Proposition 9.7. This is independent of our choice of coset representative. Indeed, if we pick another representative γ', then $\gamma'\gamma^{-1} \in H$, and we have

$$\phi_{\gamma'}(H) = \gamma'^{-1}H = \gamma'^{-1}(\gamma'\gamma^{-1})H = \gamma^{-1}H = \phi_\gamma(H).$$

Proposition 9.7 tells us that Φ is a well-defined bijection. Furthermore, given two elements αH and βH of $N_G(H)/H$, we have

$$\phi_{\beta\alpha}(H) = (\beta\alpha)^{-1}H = \alpha^{-1}\beta^{-1}H = \alpha^{-1}\phi_\beta(H) = \phi_\beta(\alpha^{-1}H) = \phi_\beta(\phi_\alpha(H))$$

which proves that Φ is a homomorphism. □

Remark 9.9. Such groups $N_G(H)/H$ are sometimes denoted $W(H)$ and called *Weyl* groups.

It is possible to generalize the notion of group actions to groupoids. Since a group action is a functor from a group G to the category **Set**, it is natural to define a groupoid action as a functor Ψ from a groupoid $\mathcal{G}$ to **Set**. Studying groupoid actions is then tantamount to studying the functor category $\mathbf{Set}^{\mathcal{G}}$.

For each object $x \in \mathcal{G}$, restricting Ψ to $\pi(\mathcal{G}, x)$ gives a group action on $\Psi(x)$. Recall that if objects x and x' are in the same connected component of $\mathcal{G}$, then $\pi(\mathcal{G}, x)$ and $\pi(\mathcal{G}, x')$ are isomorphic. We can then ask whether we can make the respective actions of both groups comparable by generalizing the notion of G-map. Let us formalize this notion as follows:

Definition 9.10. Form the category $\mathcal{A}$ with objects all triples $(G, S, \cdot)$, where G is a group, S a set, and $\cdot\colon G \times S \to S$ is a group action. Given two objects $(G, S, \cdot)$ and $(G', S', \cdot')$, we define a morphism $(\alpha, \phi)\colon (G, S, \cdot) \to (G', S', \cdot')$ whenever $\alpha\colon G \to G'$ is a homomorphism and $\phi\colon S \to S'$ is a G-map satisfying $\phi(g \cdot s) = \alpha(g) \cdot' \phi(s)$ for all $g \in G$ and $s \in S$. We call a morphism in this category a *map of group actions*, and an isomorphism, an *isomorphism of group actions*.

One can check that this is a well-defined category. Furthermore, a G-map is then the special case of a map of group actions in which the source and target objects have the same first coordinate G.

Proposition 9.11. *Let $\Psi\colon \mathcal{G} \to$ **Set** be a groupoid action, and let x and x' be objects in $\mathcal{G}$. Then the induced actions of $\pi(\mathcal{G}, x)$ on $\Psi(x)$ and $\pi(\mathcal{G}, x')$ on $\Psi(x')$ are isomorphic group actions.*

Proof. First, write $G = \pi(\mathcal{G}, x)$ and $G' = \pi(\mathcal{G}, x')$. Let $\gamma\colon x \to x'$ be a morphism in $\mathcal{G}$, and let $c_\gamma\colon G \to G'$ denote the conjugation map defined by $g \mapsto \gamma g \gamma^{-1}$ for all $g \in G$. We claim that $(c_\gamma, \Psi(\gamma))\colon (G, \Psi(x), \cdot) \to (G', \Psi(x'), \cdot')$ is an isomorphism of group actions. But given $g \in G$ and $s \in \Psi(x)$, we have

$$\Psi(\gamma)(g \cdot s) = \Psi(\gamma g)(s) = \Psi(\gamma g \gamma^{-1})\Psi(\gamma)(s) = \gamma g \gamma^{-1} \cdot' \Psi(\gamma)(s) = c_\gamma(g) \cdot' \Psi(\gamma)(s)$$

which proves the claim. □

Example 9.12. Let $p\colon \mathcal{E} \to \mathcal{B}$ be a covering of groupoids. We can view $\mathcal{E}$ as the collection of fibers F_b as b runs over objects of $\mathcal{B}$. Then $\mathcal{B}$ acts on $\mathcal{E}$ via the functor T defined on objects by $T\colon b \mapsto F_b$ and on morphisms as follows: Given $\gamma\colon b \to b'$, consider any $e \in F_b$. Let $\tilde{\gamma}$ be the unique lift of γ with source e, and let e' be the target of $\tilde{\gamma}$. We then set $\gamma \cdot e = e'$. One easily checks that this gives a well-defined functor. We call T a *fiber translation functor*.

Since $\mathcal{B}$ is connected, Proposition 9.11 tells us that every action of $\pi(\mathcal{B}, b)$ on its corresponding fiber F_b is isomorphic. In particular, there is a bijection between every pair of fibers, so they must all have the same cardinality. We now characterize this action.

Fix $e \in F_b$, and let G denote $\pi(\mathcal{B}, b)$. Then for $\gamma \in G$, we have $\gamma e = e$ if and only if the unique lift of γ with source e is an automorphism, and this happens if and only if $\gamma \in p(\pi(\mathcal{E}, e))$, so $p(\pi(\mathcal{E}, e))$ is precisely the stabilizer of e in G. This action is transitive since $\mathcal{E}$ is connected, so by Proposition 9.5, F_b is isomorphic to $G/p(\pi(\mathcal{E}, e))$ as a G-set.

10. Some Equivalences of Categories

In this section, we finally weave together all the different threads we have been developing so far. The main goal is to contextualize covering spaces in relation to other mathematical objects. Now recall from Section 6 that given a base space B with a choice of basepoint b_0, we have that the functor $\Lambda\colon \mathcal{P}(\pi_1(B, b_0)) \to \overline{\mathrm{Cov}}(B, b_0)$ is an isomorphism of categories. We can place this result in a bigger picture as follows:

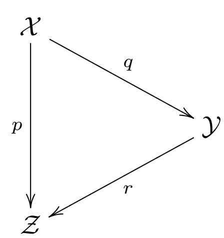
We observed that $\mathrm{Cov}(B, b_0)$ constitutes a skeleton for $\mathrm{Cov}(B, b_0)$, which in turn embeds as a subcategory of $\mathbf{Top}_*$. Now the fundamental group can be seen as a functor from the category of based topological spaces to the category of groups, i.e. we have $\pi_1(-)\colon \mathbf{Top}_* \to \mathbf{Grp}$. One can then check that $\Lambda^{-1}(-)$ is just the restriction of $\pi_1(-)$ to $\overline{\mathrm{Cov}}(B, b_0)$. This also implies that $\pi_1(-)$ gives an equivalence of categories between $\mathrm{Cov}(B, b_0)$ and $\mathcal{P}(\pi_1(B, b_0))$.

If we forget about the base point b_0, we can construct another category $\mathrm{Cov}(B)$ with objects all covers of B and morphisms all maps of coverings. We then embed $\mathrm{Cov}(B)$ as a subcategory of $\mathbf{Top}$ and it is natural to ask: What is the image of the fundamental groupoid functor $\Pi\colon \mathbf{Top} \to \mathbf{Gpd}$ when restricted to $\mathrm{Cov}(B)$? To answer this, we first have to define a groupoid analogue of the category $\mathrm{Cov}(B)$.

Definition 10.1. Let $\mathcal{B}$ be a small groupoid. We define $\mathrm{Cov}(\mathcal{B})$ to be the category with objects the covering groupoids of $\mathcal{B}$, and morphisms the maps of coverings.

The reason why $\mathrm{Cov}(B, b_0)$ is well-defined is because we can use Proposition 6.1 to prove that covering maps satisfy the 2-out-of-3 property when the base space has a universal cover. When we pass to groupoids, we get this almost automatically. Specifically, we have

Proposition 10.2. *Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be small groupoids. Let p, q and r be functors such that the following the diagram commutes:*



If any two of these maps are covering maps, then the third is as well.

Proof. This follows simply from the fact that bijections on sets satisfy the 2-out-of-3 property, and that covering maps are defined by bijections on stars. ∎

We would like to compare the structures of $\mathrm{Cov}(B)$ and $\mathrm{Cov}(\mathcal{B})$, but first we need to understand these better. The classification theorems tell us what the objects are for both these categories, so it remains to classify the morphisms. In a way, we have already done so: In Theorems 4.5 and 8.8, we gave conditions for the existence of based isomorphisms for spaces and groupoids. Arguing similarly using the Lifting Criterion actually allow us to characterize all maps of coverings.

Lemma 10.3. *Let $p : (E, e_0) \to (B, b_0)$ and $p' : (E', e_0) \to (B, b_0)$ be coverings of spaces. There is a map of coverings $h : (E, e_0) \to (E', e_0)$ if and only if $p_*(\pi_1(E, e_0))$ is a subgroup of $p'_*(\pi_1(E', e'_0))$ of $\pi_1(B, b_0)$. If h exists, it is unique.*

Lemma 10.4. *Let $p\colon (\mathcal{E}, e_0) \to (\mathcal{B}, b_0)$ and $p'\colon (\mathcal{E}, e_0) \to (\mathcal{B}, b_0)$ be coverings of groupoids. There is a map of coverings $h : (\mathcal{E}, e_0) \to (\mathcal{E}', e_0)$ if and only if $p(\pi(\mathcal{E}, e_0))$ is a subgroup of $p'(\pi(\mathcal{E}', e'_0))$. If h exists, it is unique.*

We can now properly explain the significance of the following observation:

Proposition 10.5. *If $p\colon (E, e_0) \to (B, b_0)$ is a covering of spaces, then the induced functor $\Pi(p)\colon (\Pi(E), e_0) \to (\Pi(B), b_0)$ is a covering of groupoids. Furthermore, if we pick a base object $b_0 \in \Pi(B)$ and $e_0 \in \Pi(p)^{-1}(b_0)$, then the restriction of $\Pi(p)$ to $\pi(\Pi(E), e_0) = \pi_1(E, e_0) \to \pi_1(B, b_0) = \pi(\Pi(B), b_0)$ is simply p_*.*

Proof. This follows from the path and homotopy lifting lemmas. □

The first statement in this proposition tells us that when we restrict Π to $\mathrm{Cov}(B)$, the image of the functor lies in the subcategory $\mathrm{Cov}(\Pi(B))$. Furthermore, we have

Lemma 10.6. *The fundamental groupoid functor Π induces a bijection between sets of morphisms*

$$\Phi\colon \mathrm{Cov}_B(E, E') \to \mathrm{Cov}_{\Pi(B)}(\Pi(E), \Pi(E')).$$

Proof. Pick a base point $b_0 \in B$, and pick preimages $e_0 \in p^{-1}(b_0)$ and $e'_0 \in p'^{-1}(b_0)$. If there is a map of coverings $\eta\colon (\Pi(E), e_0) \to (\Pi(E'), e'_0)$, then Lemma 10.4 tells us that $\Pi(p)(\pi(\Pi(E), e_0)) \subset \Pi(p')(\pi(\Pi(E'), e'_0))$. But changing notation gives $p_*(\pi_1(E, e_0)) \subset p'_*(\pi_1(E', e'_0))$, so by Lemma 10.3, there is a map of coverings $h\colon (E, e_0) \to (E', e'_0)$, with $\Pi(h) = \eta$. This proves the surjectivity of Φ. Injectivity follows from the uniqueness conditions of the lemmas. □

We can now finally answer the question we asked in the introduction to this section.

Theorem 10.7. *The fundamental groupoid functor Π induces an equivalence of categories $\mathrm{Cov}(B) \to \mathrm{Cov}(\Pi(B))$.*

Proof. The classification theorems tell us that Π takes the objects comprising a skeleton of $\mathrm{Cov}(B)$ (i.e. a space realizing each subgroup of $\pi_1(B, b_0)$) to the objects comprising a skeleton of $\mathrm{Cov}(B')$ (i.e. a groupoid realizing each subgroup of $\pi(\Pi(B), b_0) = \pi_1(B, b_0)$.) Lemma 10.6 implies that Π takes a full subcategory to a full subcategory. This proves the equivalence. □

The story is still only half-finished, however, because we can also establish a connection between group actions and either covering spaces or covering groupoids. (Either would do because of the equivalence we just proved.) We choose the latter.

Lemma 10.8. *Let $p\colon \mathcal{E} \to \mathcal{B}$ and $p'\colon \mathcal{E}' \to \mathcal{B}$ be coverings, choose a base object $b_0 \in \mathcal{B}$, and let $G = \pi(\mathcal{B}, b_0)$. If $g\colon \mathcal{E} \to \mathcal{E}'$ is a map of coverings, then g restricts to a map $F_{b_0} \to F'_{b_0}$ of G-sets, and restriction to fibers specifies a bijection between $\mathrm{Cov}_{\mathcal{B}}(\mathcal{E}, \mathcal{E}')$ and the set of G-maps $F_{b_0} \to F'_{b_0}$.*

Proof. Let γ be an element of $\pi(\mathcal{B}, b_0)$, and let $\tilde{\gamma}$ be the unique lift of γ with source e_0, and let its target be e_0'. Then by definition, we have $\gamma e_0 = e_0'$. This implies that $g(\gamma e_0) = g e_0'$. On the other hand, $p(g(\tilde{\gamma})) = p(\tilde{\gamma}) = \gamma$, so $g(\tilde{\gamma})$ is the unique lift of γ with source $g(e_0)$. But $g(\tilde{\gamma})$ has target $g(e_0')$, so by definition, $\gamma g(e_0) = g(e_0')$. This proves that restricting g to F_b gives a G-map. By Lemma 10.4, this gives an injection on $\mathrm{Cov}_{\mathcal{B}}(\mathcal{E}, \mathcal{E}')$. To show surjectivity, let α be a G-map. Now choose $e_0 \in F_{b_0}$, and let $e_0' = \alpha(e_0)$. Given $\gamma \in p(\pi(\mathcal{E}, e_0))$, we then have $\gamma\alpha(e_0) = \alpha(\gamma e_0) = \alpha(e_0)$, so γ is in the stabilizer of $\alpha(e_0)$, which is $p(\pi(\mathcal{E}', e_0'))$. This means that $\gamma \in p(\pi(\mathcal{E}, e_0)) \subset p(\pi(\mathcal{E}', e_0'))$, so there is a map of coverings $h\colon (\mathcal{E}, e_0) \to (\mathcal{E}', e_0)$ that restricts to a G-map α' taking e_0 to e_0'. But transitive G-maps are fully determined by where they send a single element, so $\alpha = \alpha'$. □

Theorem 10.9. *Let $\mathcal{B}$ be a groupoid. Pick a base object b_0, and let $G = \pi(\mathcal{B}, b_0)$. There is a functor $\Omega\colon \mathrm{Cov}(\mathcal{B}) \to \mathcal{O}(G)$ that is an equivalence of categories.*

Proof. We define Ω on objects by sending each covering groupoid $\mathcal{E}$ to the G-set $G/\pi(\mathcal{E}, e_0)$, where the base object e_0 is arbitrarily chosen from F_{b_0}. We next define Ω on morphisms by restricting each map of coverings to fibers of b_0. The previous lemma then tells us that this functor is well-defined, and that it induces bijections $\mathrm{Cov}_{\mathcal{B}}(\mathcal{E}, \mathcal{E}') \to \mathcal{O}(G)(\Omega(E), \Omega(E'))$. By Proposition 9.7, isomorphism classes of objects in $\mathcal{O}(G)$ correspond to conjugacy classes of subgroups. Choosing a representative from each class and the full subcategory they span then gives a skeleton for $\mathcal{O}(G)$. As such, Ω maps $\mathrm{Cov}(\mathcal{B})$ onto a skeleton of $\mathcal{O}(G)$, giving an equivalence of categories. □

In summary, we have proved the following diagram of equivalences.

$$\mathrm{Cov}(\mathcal{B}) \xrightarrow{\Pi} \mathrm{Cov}(\Pi(B)) \xrightarrow{\Omega} \mathcal{O}(\pi(\Pi(B), b_0))$$

11. Orbit Spaces and a "Galois Theory" of Covering Spaces

There is one last concept to cover before we can claim to have given a comprehensive account of covering spaces, and that is the notion of a *deck transformation*. We have reserved it for the very last because we want to consider it in the context of a "Galois Theory" of Covering Spaces leading up to a covering space analogue of the Fundamental Theorem of Galois Theory, and this requires the full panoply of tools we have developed so far.

Suppose we have a covering $p\colon E \to B$. Then a deck transformation of E is just another name for a covering space isomorphism of E. We can consider the set of isomorphisms, $\mathrm{Aut}_B(E) \subset \mathrm{Cov}_B(E, E)$, and study the structure of the group they generate.

Remark 11.1. Because we can conjugate a subgroup into a proper subgroup of itself, it is possible for an endomorphism to be non-invertible, and hence not an automorphism.

The following definition plays an important role.

Definition 11.2. A covering $p\colon E \to B$ is called *regular* if for each $b_0 \in B$ and each pair of elements e_0 and e_0' in F_{b_0}, there is a covering map $h\colon (E, e_0) \to (E, e_0')$.

Proposition 11.3. *Let $p\colon (E, e_0) \to (B, b_0)$ be a covering, and let G and H denote $\pi_1(B, b_0)$ and $p_*(\pi_1(E, e_0))$ respectively. Then*

(1) $\mathrm{Aut}_B(E)$ is isomorphic to the quotient $N_G(H)/H$.
(2) E is a regular covering if and only if H is a normal subgroup of G.

Proof. The composite functor $\Omega\Pi$ is bijective on morphisms sets, and takes invertible morphisms to invertible morphisms by functoriality. It therefore restricts to a group isomorphism between $\mathrm{Aut}_B(E)$ and the group of G-set automorphisms $\mathrm{Aut}(G/H)$. By Proposition 9.8, $\mathrm{Aut}(G/H)$ is isomorphic to $N_G(H)/H$, so (1) follows. Now by definition, E is regular if and only if $\mathrm{Aut}_B(E)$ is transitive. Then Proposition 9.7 tells us that $\mathrm{Aut}(G/H)$ is transitive if and only if H is a normal subgroup of G, proving (2). □

Some authors, such as [3], use the label *normal* instead because of the correspondence between regular covers and normal extensions in Galois theory. Like normal extensions, regular covers are those that have sufficient symmetry, and one can best get an intuition for them by considering covers for $S^1 \vee S^1$. In this case, the isomorphism classes of covers are in bijection with 4-regular graphs, and the regular ones are precisely those that are vertex-transitive. A more detailed account can be found in [3].

Now, there is a way to generate covers that are automatically regular. It involves the notion of a group acting on a space.

Definition 11.4. Let G be a group that acts on a space X. The *orbit space* of the action, denoted X/G, is the quotient space of X generated by the equivalence relation $x \sim y$ if they belong to the same orbit.

Definition 11.5. Let G be a group that acts on a space X. We say that the action if *properly discontinuous* if every point $x \in X$ has a neighborhood U such that all the images $g(U)$ for $g \in G$ are disjoint.

One can check that this condition is equivalent to requiring $g(U)$ to be disjoint form U for all $g \in G$. It is also easy to see that a fixed-point free action of a finite group on a Hausdorff space is always properly discontinuous. We then have the following proposition:

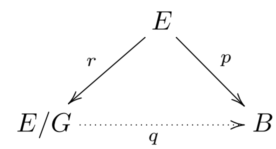
Proposition 11.6. *Let G be a group that acts on a space X. The quotient map $p\colon X \to X/G$ is a covering map if and only if the action is properly discontinuous. In this case, the covering is regular and G is its automorphism group.*

Proof. Suppose p is a covering map. Given a point $x \in X$, pick a fundamental neighborhood U in X/G containing $p(x)$, and a partition of its preimage into slices. Let V_α be the slice containing x. Then for all $g \in G$, we must have $gx \notin V_\alpha$, otherwise p would not restrict to a bijection on V_α. Hence the action is G is properly discontinuous, which proves the only if part of the statement.

Conversely, if the action is properly discontinuous, for any point $y \in X/G$ we can pick a lift x which has a neighborhood V such that the neighborhoods $g(V)$ for $g \in G$ are all disjoint. The quotient map then induces a homeomorphism of each of these neighborhoods with $p(V)$, and so is a covering map.

Now, each $g \in G$ is clearly an automorphism of $p\colon X \to X/G$. Furthermore, if h is any automorphism, then picking $x \in X$ arbitrarily, we have $h(x) = g(x)$ for some $g \in G$. Then by the classification of maps between covers, we must have $h = g$, which proves that $G = \mathrm{Aut}_{X/G}(X)$. The fact that the covering is regular now follows by definition. □

Corollary 11.7. *Let $p\colon E \to B$ be a covering map with automorphism group G, and let $r\colon E \to E/G$ be the usual quotient map. Then there is a map of coverings $q\colon E/G \to B$ such that the following diagram commutes:*



If E is in addition a regular cover, then q is a homeomorphism.

Proof. This follows immediately from the classification theorem □

Remark 11.8. This observation gives us a method of computing the fundamental group of a space B: If we can identify a universal cover E_0, then its automorphism group $\mathrm{Aut}_B(E_0)$ is precisely $\pi_1(B, b_0)$.

We now set out to prove the "Fundamental Theorem" for covering space theory, and we first state its Galois Theory equivalent for comparison.

Theorem 11.9 (Fundamental Theorem of Galois Theory)**.** *Let K/F be a Galois extension and set $G = \mathrm{Gal}(K/F)$. Let $\mathcal{F}$ be the lattice of subfields of K containing F, and let $\mathcal{G}$ be the lattice of subgroups of G. We then have lattice isomorphisms given by*

$$\mathcal{F}^{op} \underset{\mathrm{Fix}(-)}{\overset{\mathrm{Aut}(K/-)}{\rightleftarrows}} \mathcal{G}.$$

Under this correspondence,

1. *If $H = \mathrm{Aut}(K/E)$, we have $[K : E] = |H|$ and $[E : F] = |G : H|$.*
2. *K/E is always Galois, with Galois group $\mathrm{Gal}(K/E) = H$.*
3. *E is Galois over F if and only if H is a normal subgroup in G. If this is the case then we have the isomorphism $\mathrm{Gal}(E/F) \cong G/H$. More generally, even if H is not necessarily normal in G, the isomorphisms of E (into a fixed algebraic closure of F containing K) which fix F are in one to one correspondence with the cosets $\{\sigma H\}$ of H in G.*

Several obstacles need to be navigated before being able to translate this into the language of covering space theory. The first is the need to introduce compatible notation.

We notice that the automorphism group of a field changes depending on what we take to be the fixed field. The same thing is true for covering spaces. We have so far considered the automorphism group of a family of covering space with respect to a single fixed base space B, but it is possible to swap B for any intermediate covering space. We hence write $\mathrm{Aut}(E/B)$ instead of $\mathrm{Aut}_B(E)$ to indicate that B is also a variable. In accordance with Galois theory, we shall call regular covers

Galois covers, and write $\mathrm{Gal}(E/B)$ for $\mathrm{Aut}(E/B)$ whenever E is a Galois cover for B. We call this the *Galois group* of E over B.

Next, we need to provide an analogue for the index of a subfield. This is given by the following definition.

Definition 11.10. Let $p\colon E \to B$ be a covering map. The number of *sheets* of the covering is the cardinality of F_b, the fiber of any point $b \in B$.

Example 9.12 tells us that this number is well-defined, and we shall denote it using $[E : B]$.

Finally, we would like to have a lattice of covering spaces analogous to the subfield lattice $\mathcal{F}$. But here we run into the following problem: a lattice requires in particular that no two distinct objects are isomorphic, so how do we make an arbitrary choice of covering space for each isomorphism class? The solution is suggested by one of the implicit assumptions made in the Fundamental Theorem of Galois Theory, namely, that we work in the context of a fixed algebraic closure of the base field F.

Now, an algebraic closure of a field F can be defined as an algebraic extension of F which has no nontrivial algebraic extensions. Similarly, a universal cover for a base space B is one that has no nontrivial covers. We immediately see that the two concepts are analogous. We then see that, just as the isomorphism problem is solved in Galois theory by only considering subfields of the algebraic closure, we can solve ours by only considering orbit spaces, i.e. quotients, of our universal cover.

This makes us ready to prove:

Theorem 11.11 ("Fundamental Theorem" of Covering Space Theory)**.** *Let $p\colon E \to B$ be a Galois covering, and let $G = \mathrm{Gal}(E/B)$. Let $\mathcal{C}$ be the lattice with objects the spaces E/H for subgroups H of G and morphisms quotient maps (one can check that this gives a well-defined lattice.) Let $\mathcal{G}$ be the lattice of subgroups of G. We then have lattice isomorphisms given by*

$$\mathcal{C}^{op} \underset{E/(-)}{\overset{\mathrm{Aut}(E/-)}{\rightleftarrows}} \mathcal{G}.$$

Under this correspondence,

1. *If $H = \mathrm{Aut}(E/E')$, we have $[E : E'] = |H|$ and $[E' : B] = |G : H|$.*
2. *E/E' is always Galois, with Galois group $\mathrm{Gal}(E/E') = H$.*
3. *E' is a Galois cover for B if and only if H is a normal subgroup in G. If this is the case then we have the isomorphism $\mathrm{Gal}(E'/B) \cong G/H$. More generally, even if H is not necessarily normal in G, the isomorphisms of E' with covers of B (that are quotients of a fixed universal cover of B for which E is a quotient) are in one to one correspondence with the cosets $\{\sigma H\}$ of H in G.*

Proof. $\mathrm{Aut}(E/-)$ and $E/(-)$ are inverse bijections on objects by Proposition 11.6. Now suppose H and K are subgroups of G such that $H \geq K$. Let $p\colon E \to E/H$ and $r\colon E \to E/K$ be the respective projection maps. Then by Corollary 11.7, there is a quotient map q that makes the following diagram commute:

$$\begin{array}{ccc} & E & \\ {}^{r}\swarrow & & \searrow^{p} \\ E/K & \xrightarrow[q]{} & E/H \end{array}$$

Hence, the function $E/(-)$ is order-preserving. It is obvious that $\mathrm{Aut}(E/-)$ is also order-preserving. This proves that both of these are functors and more specifically, lattice isomorphisms.

Note that (2) just paraphrases Proposition 11.6. To prove (1), we observe that E' is just E/H, so given a point $y \in E'$, we pick a point x belonging to its fiber F_y. Then F_y is simply $\{hx \mid h \in H\}$ and so has cardinality $|H|$, which gives the first half of the statement. Now given $y \in B$, let F_y and F'_y denote its fibers in E and E/H respectively. By the above discussion, the elements of G are in bijection with the elements of F_y, and the quotient map $p\colon E \to E/H$ identifies those points in F_y corresponding to elements of the same coset. This implies that the points of F'_y are in bijection with the cosets of H in G, and so $[E' : B] = |G : H|$.

Now, pick a base point $b \in B$, e an element of the fiber of b in E, and let G_B and G_E denote the fundamental groups $\pi_1(B, b)$ and $\pi_1(E, e)$ respectively. Proposition 11.3 then tells us that G_E is a normal subgroup of G_B, and that the automorphism group G is isomorphic to the quotient group G_B/G_E. Picking an element e' of the fiber of b in E', let $G_{E'}$ denote the fundamental group $\pi_1(E', e')$. Then the automorphism group H is isomorphic to the quotient group $G_{E'}/G_E$. We then have that E' is a Galois cover of B if and only if $G_{E'}$ is a normal subgroup of G_B, which by the correspondence theorem happens if and only if $H = G_{E'}/G_E$ is a normal subgroup of $G = G_B/G_E$.

Finally, we have that a fiber of B in E' is isomorphic to $G_B/G_{E'}$ as a G_B-set. An analogue of Lemma 10.8 tells us that covering isomorphisms are in bijection with G_B-set isomorphisms of $G_B/G_{E'}$, which by Proposition 9.7 are in bijection with the cosets of $G_{E'}$ in G_B. Invoking the correspondence theorem once again tells us that this set is in bijection with the cosets of H in G, thereby completing the proof of (3). □

Remark 11.12. The parallels between Theorems 11.9 and 11.11 are striking and suggest that something deeper is at work. It turns out that we can abstract some of the common properties of both into something called a *Galois category*. This is, however, outside the scope of this paper, and the interested reader is invited to consult [4].

12. Postlude - Some Philosophical Remarks

We end this paper by asking: What does it mean in general to *classify* a given class of mathematical objects?

Where there are only finitely many objects, classification is synonymous with enumeration, and for instance, we classify platonic solids simply by stating the five that exist. When we have infinitely many objects, the situation can be a lot more complex, and sometimes the best we can hope for is to divide the objects into a finite number of families, as is the case with finite simple groups.

Often, however, the class of objects comes along with sets of morphisms that make the class of objects into a category $\mathcal{C}$. In this new situation, classification takes on a different texture: Since we are interested in studying the structure of objects up to isomorphism type, it becomes the same thing as finding and understanding a skeleton in $\mathcal{C}$. But, by an easy thought exercise, two categories are equivalent if and only if they have isomorphic skeletons. Therefore, classification then becomes entangled with finding equivalences of $\mathcal{C}$ with other categories with which we are more familiar.

This is precisely what has happened to us in our journey with covering spaces. The first part of the classification theorem (5.9) is equivalent to the statement that $\text{Cov}(B, b_0)$ and $\mathcal{P}(\pi_1(B, b_0))$ are equivalent categories, and the second, to the statement that $\text{Cov}(B)$ and $\mathcal{O}(\pi_1(B, b_0))$ are likewise equivalent. We could have proceeded in either direction, using Theorem 5.9 to prove these, or conversely, using these to prove Theorem 5.9.

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References

[1] P. Aluffi. *Algebra: Chapter 0*. Graduate Studies in Mathematics. American Mathematical Society, 2009.
[2] D.S. Dummit and R.M. Foote. *Abstract Algebra*. Wiley, 3rd edition, 2004.
[3] A. Hatcher. *Algebraic Topology*. Cambridge University Press, 2001.
[4] M. Lynn. *Galois Categories*. UChicago REU preprint, 2009.
[5] S. Mac Lane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics. Springer, 2nd edition, 1998.
[6] J. P. May. *A Concise Course in Algebraic Topology*. University of Chicago Press, 1999.
[7] J. Munkres. *Topology*. Prentice Hall, 2nd edition, 2000.