SHIFTING AND MOVING: AN INTRODUCTION TO DYNAMICAL SYSTEMS

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Abstract. This paper is meant to provide an introduction to the ideas of dynamical systems. In order to do this, we must first provide an overview of measure theory and metric spaces. From there, we define the concepts of recurrence and ergodicity and analyze the classical example of Symbolic Spaces.

Contents

1. Introduction

This project was inspired by my experiences with classical dance and a subsequent desire to analyze the movement of bodies in space. I wanted to understand the movement of bodies from a mathematical perspective, leading to a project to analyze dynamical systems.

Traditionally, when analyzing dynamical systems, we do not consider time as a continuous dimension. Instead, time n represents the number of iterates of a function f. In other words, the notation $f^n(x)$ denotes f applied to the value x an n number of times.

Dynamical systems are a mathematically rich topics because they can be analyzed from both an analytical and a topological perspective. As such, the consequences of some of the theorems presented here have applications in both branches of mathematics. Because of the vast range of applications, each of which deserves analysis in its own right, we cannot devote the needed time and attention to each application of this theory. Instead, this paper serves as an overview of several major concepts and examples.

We begin with a general overview of dynamical systems by considering some basic definitions and a few classical examples. Then we introduce the fundamentals of measure theory. Section 3 focuses on the ideas of recurrence and ergodicity and presents the example of the rotation transformation. In Section 4, we illustrate concepts in dynamical systems through the example of Symbolic Spaces, shifts and the Cantor Set.

2. General Background in Dynamical Systems

As dynamical systems can be analyzed using several different mathematical perspectives, we must first establish some preliminaries. This section will define several key concepts that set the groundwork for analysis in later parts of the paper. Using those definitions, we will consider two classical examples of dynamical systems.

When we analyze discrete dynamical systems, we are concerned with the behavior of a sequence $\{x, f(x), f^2(x)\dots\}$ where $f(x)$ is any function on the real line. We want to know what happens to the sequence $f^n(x)$ as $n \to \infty$.

A function f is a **homeomorphism** if f is a bijection and both f and f^{-1} are continuous. We say that a set A is a **forward orbit of** x if $A = \{x, f(x), f^2(x), f^3(x)\dots\}$ for $n \in \mathbb{N}$. If the function f is a homeomorphism, then we can define the **complete orbit of x** to be the set

$$B = \{\dots f^{-n}(x), f^{-n+1}(x), \dots, x, f(x), f^2(x), f^3(x), \dots f^n(x)\dots\} \quad \text{for all} \quad n \in \mathbb{Z} \tag{2.1}$$

From here, we can also define the idea of a **fixed point** and a **periodic point**. It is important to define these concepts as they serve as the most basic example of invariant sets. They also help us consider if an iterate of a function $f(x)$ revisits a set A for $x \in A$. A point x is a fixed point if $f(x) = x$. The point x is a fixed point if $f(x) = x$. The point x is periodic if there exists some $n \in \mathbb{Z}$ such that $f^n(x) = x$. The smallest value of n where the above holds is called the **prime period** of x and is usually denoted by p.

Now that we have these definitions, it is important to consider a few applications. We look at two classical examples: Quadratic Maps and Rotations of a Circle.

Example 2.1. One of the most important classes of functions to consider is the quadratic map. We take such maps to be of the form

$$f(x) = \mu x(1-x) \quad \text{where} \quad \mu \in \mathbb{R} \tag{2.2}$$

We know that

(1) The function takes its zeros at 0 and 1.
(2) The function has a fixed point at p at $p = 1 - 1/\mu$. Furthermore, it follows that if $\mu > 1$, then $p \in (0, 1)$.

For values of $x < 0$ or $x > 1$, we can generalize what happens as we iterate $f(x)$ an infinite number of times.

Proposition 2.2. *If $\mu > 1$ and $x < 0$ or $x > 1$, then $f^n(x) \to -\infty$ as $n \to \infty$*

Proof. We can divide this proof into two cases. Case 1 is where $x < 0$ and Case 2 is where $x > 1$.

Suppose $x < 0$, meaning both $\mu > 1$ and $(1 - x) > 1$, then we know that $f(x) = \mu x(1 - x) < 0$ and that $|f(x)| > |x|$. This holds for all $x < 0$. Because $f(x) < 0$, then by above $f(f(x)) < 0$. Since this is true for all iterates of $f(x)$, this implies that $\{f^n(x)\}$ is decreasing for all n.

From here, we proceed with a proof by contradiction. Suppose that $f^n(x)$ does not converge to $-\infty$. This means that there exists a such that $f^n(x) \to a \quad \text{as} \quad n \to \infty$.

If this is true, then $f^{n+1}(x) \to f(a)$. In other words, given $\epsilon > 0$, there exists N such that for all $n > N$, $|f^{n+1}(x) - f(a)| < \epsilon$. Since this sequence converges, the sequence is Cauchy, which means that $|f^{n+1}(x) - f^n(x)| < \epsilon$.Then, as $n \to \infty$, we know that given $\epsilon > 0, |f(a) - a| < \epsilon$.

However, the sequence $\{f^n(x)\}$ is always decreasing, meaning that $f(a) < a$ for all a. This implies that $|f(a) - a| > 0$ which means that as $n \to \infty$, $|f^{n+1}(x) - f^n(x)| > 0$. Therefore, $f^n(x)$ cannot converge for a value of a, meaning that $f^n(x) \to -\infty$ as $n \to \infty$.

Now take Case 2 and suppose that $x > 1$. Because $x > 1$ implies that $(1 - x) < 0$, $f(x) = \mu x(1-x) < 0$. Then, since $f(x) < 0$, $f(f(x)) < f(x) < 0$. It follows from here that we can use the same proof as above. This implies that $f^n(x) \to -\infty$ as $n \to \infty$. ■

For $x \in [0, 1]$, the analysis becomes much more difficult because the function's behavior depends on the value of μ. Furthermore, for $\mu \in [1, 3]$, as $n \to \infty$, $f^n(x) \to p$ where $p = 1 - 1/\mu$. The proof of this fact follow from the analysis of the graph of $f(x)$. For $\mu \geqslant 4$, analysis of the function becomes more technical. We invite the reader to see [1] for more in-depth analysis.

Example 2.3. Another important class of functions rotate the unit circle. These functions are represented by

$$T(\theta) = \theta + 2\imath\lambda \mod 2\pi \quad \text{for some real} \quad \lambda \tag{2.3}$$

Consider the iterates of the function $T^n(\theta)$. We know that $T(T(\theta)) = \theta + 2\pi\lambda + 2\pi\lambda \mod 2\pi = \theta + 4\pi\lambda \mod 2\pi$. We extrapolate that $T^n(\theta) = \theta + 2n\pi\lambda \mod 2\pi$. As the rotation function is invertible, $T^{-n}(\theta) = \theta - 2n\pi\lambda \mod 2\pi$. We want to consider the behavior of $T(\theta)$ when λ is rational and irrational.

If λ is rational, then $\lambda = p/q$ for some integers p and q. To find fixed and periodic points, we are looking for values of λ such that $2\pi\lambda \cong 0 \mod 2\pi$.

From this, we know that $T(\theta)$ has a fixed point for all integer values of λ. By the property described above, we also know that $T^q(\theta) = \theta + 2q(p/q)\pi = \theta + 2q\pi = \theta$, meaning that q is the prime period of $T(\theta)$. Then, we see that all rational rotations give rise to periodic orbits of period q. When λ is irrational, the behavior is more interesting.

Proposition 2.4. *Suppose λ is irrational. Then, the orbit of $T(\theta)$ is dense on the unit circle S.*

Proof. To prove that a set is dense, we must prove that within every open interval I on the unit circle S, there exists a point $T^n(\theta)$ for some n so that $T^n(\theta) \in I$.

First we claim that each $T^n(\theta)$ is distinct. To prove this, we look at two iterates, $T^n(\theta)$ and $T^m(\theta)$. Suppose that each $T^n(\theta)$ is not distinct. Then, for $n \neq m, T^n(\theta) = T^m(\theta)$. This implies that $\theta + 2n\pi\lambda = \theta + 2m\pi\lambda$. However, this would mean that $\lambda(n - m)$ is an integer. As λ is irrational, this is only possible if $n = m$. This is a contradiction, meaning that every $\{T^n(\theta)\}$ must be distinct.

We also know that as $\{T^n(\theta)\}$ is bounded by our definition of $T(\theta)$. By the Bolzano-Weierstrass Theorem, we know that there exists a subsequence of $\{T^n(\theta)\}$ which is Cauchy. This means that given $\epsilon > 0$, there exists integers n and m such that $|T^n(\theta) - T^m(\theta)| < \epsilon$.

Define $k = n - m$. Then the above implies that $|T^k(\theta) - \theta| < \epsilon$. This means that instead of analyzing the set $\{T^n(\theta)\}$, we can look at iterates of $T^k(\theta)$. In other words, we can consider $\left\{T^{kn}(\theta)\right\}$ for some fixed k.

We know that $|T^k(\theta) - \theta| = |T^{nk}(\theta) - T^{k(n-1)}(\theta)| < \epsilon$. This means that the rotations of $T^k(\theta)$ are isometric, meaning that the length of the arc formed does not change no matter how many times we rotate the interval. However, we also know that every $T^{nk}(\theta)$ is distinct. Therefore, we are essentially partitioning the entire unit circle S into intervals of length smaller than ϵ.

As we consider the orbit of $\left\{T^{kn}(\theta)\right\}$, we fill S with an infinite number arcs of length smaller than ϵ. As ϵ is arbitrary, this implies that any given open interval on the unit circle, we can find a value $T^{nk}(\theta)$ in that interval I for some integer value of n. This implies that $T(\theta)$ is dense.

■

3. Metric Spaces and Measure Theory

In this section, we will introduce the fundamentals of topology and measure theory needed to define dynamical concepts more precisely. Because it is meant to serve as an overview, most proofs have been omitted in this section.

Definition 3.1. To analyze sets, we presume a metric space (X, d). We say that a metic space (X, d) is **compact** if every sequence $\{x_n\}$ in X has a subsequence converging to a point in X. X is said to be **totally bounded** if for all $\epsilon > 0$, there exist finitely many points $x_1, \dots x_n$ such that for all $x \in X$, there exists $1 \leqslant i \leqslant n$ such that $x \in B(x_i, \epsilon)$. $B(x_i, \epsilon)$ denotes a ball of radius ϵ centered at x_i.

We have already used the idea of a Cauchy sequence but it helps to define it as well. A sequence $\{x_n\}$ is **Cauchy** if for all $\epsilon > 0$, there exists N such that $d[x_n, x_m] < \epsilon$ for all $n, m > N$. A metric space X is **complete** if every Cauchy sequence in X converges to a point in X. It follows that if X is complete, then so is every closed subset of X.

A set G is called **open** if for every $x \in G$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset G$. We know from this definition that both the empty set and the real line are open sets. A set F is **closed** when, given a sequence of points $\{x_n\}$ in the set F, if $\{x_n\}$ converges to a point x, then $x \in F$. If F is closed, all limit points of F are contained in F. It also follows that a set F is closed if and only if its complement, F^c is open.

We can now begin to cover the basics of measure theory. Measure theory is concerned with rigorously defining the measure of a set from an analytic perspective. For now, we are concerned with developing this theory on $\mathbb{R}$. To build up to the formal definition of a measure, we first discuss some natural properties that the measure would satisfy. The simplest sets for us to measure are intervals. If we have an open interval $I = (a - b)$, the measure of this interval, denoted $\mu(I)$ should be the length of the interval, or $b - a$.

For a generic set A, it should be natural to approximate the measure using a sequence of open intervals I_i such that each interval is disjoint and the set $A \subseteq \bigcup_{i=1}^{\infty} I_i$. Of course, this only approximates the measure of A and does not give us a precise value for the measure of A. In order to calculate most accurate approximation of the measure of A, denoted $\lambda(A)$, we want to take the smallest possible collection of such intervals. We denote a generic measure function with μ and the more specific Lebesgue Measure function with λ. From this, we solidify the idea of outer measure.

Definition 3.2. Given a set A, we say that the outer measure $\lambda^*(A) = \inf \{\sum_{i=1}^{\infty} \mu(I_i)\}$ where I_i are disjoint and $A \subseteq \bigcup_{i=1}^{\infty} I_i$.

At this point, some might wonder why we do not simply define the outer measure to be the measure of a set. While the outer measure is the normal way to define a measure, it does not satisfy a certain property that we expect measures to have. This is the idea of countable additivity, which expresses the following. Intuitively, if we have a collection of disjoint sets, the measure of the collection should be the sum of the measures of every set in the collection.

Definition 3.3. Suppose we have a collection of disjoint sets $\{A_n\}_{n=1}^{\infty}$. We say that μ is countably additive if

$$\mu(\bigcup_{n=1}^{\infty}(A_n) = \sum_{n=1}^{\infty} \mu(A_n) \tag{3.1}$$

The outer measure λ^* does not satisfy this property for all sets. If we can characterize the sets which satisfy this property, then λ^* would be a fine way of defining this measure. We call these sets **Lebesgue Measurable Sets**.

Definition 3.4. A set A in $\mathbb{R}$ is said to be Lebesgue Measurable if for any $\epsilon > 0$, there exists an open set $G = G_\epsilon$ such that

$$A \subset G \quad \text{and} \quad \lambda^*(G \setminus A) < \epsilon \tag{3.2}$$

Equipped with this definition, we can say that these sets satisfy countable additivity (See [2] for proof). The Lebesgue Measure is only one example of a measure function μ. All properties of the Lebesgue Measure apply to general measures μ.

Definition 3.5. The measure function μ maps from a set A to $[0, \infty)$ and satisfies the following

(1) $\mu(\emptyset) = 0$
(2) μ satisfies the property of countable additivity

For different μ, we need different ways of expressing the class of sets on which μ can be defined. This class is called a σ-algebra.

Definition 3.6. We say that S is a σ-algebra if S is a sub collection of subsets of X such that

(1) S is non-empty.
(2) S is closed under complements, meaning if $A \in S$, then $A^c \in S$.
(3) If $\{A_n\} \in S$, then $\bigcup_{n=1}^{\infty} A_n \in S$.
(4) By above, it follows that $\bigcap_{n=1}^{\infty} A_n \in S$.

It turns out that the sets which are Lebesgue Measurable form a σ-algebra.

We denote a measure space by (X, S, μ). We say that X is a finite measure space if $\mu(X) < \infty$.

Example 3.7. The Cantor Middle Third Set To construct the Cantor Middle Third Set, denoted by K, we begin with the interval $[0, 1)$. From this interval, we remove the middle third $[1/3, 2/3)$ and are left with the intervals $[0, 1/3) \cup [2/3, 1)$. From these two intervals, we again remove the middle third and are left with the intervals $[0, 1/9) \cup [2/9, 1/3) \cup [2/3, 7/9) \cup [8/9, 1)$. We continue this process until we are left with a set of discrete points. However, the process is infinite.

Proposition 3.8. $\mu(K) = 0$

Proof. Consider the complement of K, called K^c. We can consider K^c to be the union of disjoint intervals and use the property of countable additivity to compute $\mu(K^c)$. We know that $\mu(K^c) = 1/3 + 2/9 + 4/27 + \ldots$ This means that $\mu(K^c) = \sum_{k=1}^{\infty} \frac{1}{3}\frac{2}{3}^k$. This is a geometric series, meaning that $\mu(K^c) = \frac{1}{3}\frac{1}{(1-2/3)} = 1$. However, since we are concerned with the space $X = [0, 1)$ and $\mu(X) = 1$, this implies that $\mu(K) = 0$. ■

While the Cantor Middle Third Set has a measure of 0, it is possible to define Cantor sets with a positive measure. Called Fat Cantor Sets, these sets are constructed in the same way as the Cantor Middle Third Set. However, instead of removing the middle third from $[0, 1)$ and successively from each remaining interval, they are constructed by removing the middle $1/4$, middle $1/5$ or other such interval. It is also possible to construct a set by removing two intervals of a given length. The Cantor Set is important to our discussion of Symbolic spaces in section 4.

4. Recurrence and Ergodicity

Now that we have defined the fundamentals of measure theory, we consider how those ideas are applied to Dynamical Systems. To do this, we look at a transformation T which maps from a finite measure space X to itself. T is said to be a **measurable transformation** if for all sets A contained in the σ-algebra, S of X, $T^{-1}(A)$ is also contained in S. T is an **invertible measurable transformation** if both T and T^{-1} are measurable.

From these two definitions, we say that T is **measure preserving** if T is a measurable transformation and $\mu(T^{-1}(A)) = \mu(A)$. We use A and $T^{-1}(A)$ rather than $T(A)$ because $T^{-1}(A)$ preserves the topological properties of A. (For example, if A is open, $T^{-1}(A)$ is also open). In the case where T is invertible and continuous, T and T^{-1} are equitable.

The idea of a measure preserving transformation is essential to our definition of the concept of recurrence. The idea of recurrence is concerned with whether or not a point $x \in A$ revisits the set A after n iterations of the function $T(x)$.

We say that a measure preserving transformation T is **conservative** if for any set A of positive measure, there exists an integer n such that $\mu(A \cap T^{-n}(A)) > 0$.

Definition 4.1. Suppose we have a measure preserving transformation T on a finite measure space X. T is **recurrent** if, for every measurable set A such that $\mu(A) > 0$, there exists $N \subset A$ such that $\mu(N) = 0$ for all $x \in A \setminus N$, there is an integer $n = n(x) > 0$ with $T^n(x) \in A$.

One of the most important elements of recurrence is the idea of **Poincare Recurrence**, explained by Henri Poincare in 1890. Before we can solidify Poincare Recurrence, we first state a lemma. For the proof of this Lemma, see [2].

Lemma 4.2. *Let X be a finite measure space and let $T : X \to X$ be a measure preserving transformation. Then, T is recurrent if and only if T is conservative.*

Theorem 4.3. ***Poincare Recurrence*** *Let X be a finite measure space. If $T : X \to X$ is measure preserving, then T is a recurrent transformation.*

Before we begin the proof, we first explain why such a result is to be expected. Suppose we have the worst possible scenario where A and $T^{-n}(A)$ are disjoint for all n. Since T is measure preserving, $\mu(T^{-n}(A)) = \mu(A)$. Now suppose we iterate $T^{-n}(A)$ an infinite number of times. Because each iterate has the same measure and all are disjoint, if we consider an infinite number of iterates, by countable additivity, we need an infinite measure space to accommodate every iterate. However, we are dealing with a finite measure space, meaning after some N, $T^{-n}(A)$ will have to intersect some other iterate $T^{-j}(A)$ where $j \leqslant N$.

Now that we have presented the idea of the proof, we proceed by contradiction to formalize it.

Proof. By 4.2, it suffices to show that for any set of positive measure A, there exists an integer $n > 0$ such that $\mu(A \cap T^{-n}(A)) > 0$.

Suppose that $\mu(A \cap T^{-n}(A)) = 0$ for all $n > 0$. Then, for any integers $l, k > 0$ where $l \neq k$, we can assume without loss of generality that $l > k$ and write $l = n + k$ for some $n > 0$.

Since μ is translation invariant

$$\mu(T^{-l}(A) \cap T^{-k}(A)) = \mu(T^{-n-k}(A) \cap T^{-k}(A)) = \mu(T^{-k}[T^{-n}(A) \cap A])$$

However,

$$\mu(T^{-k}[T^{-n}(A) \cap A]) = \mu(T^{-n}(A) \cap A) = 0$$

By hypothesis, this implies that

$$\mu(T^{-l}(A) \cap T^{-k}(A)) = 0$$

By countable additivity, the elements of the set $\{T^{-n}(A)\}_{n \leqslant 0}$ are pairwise disjoint, which tells us that

$$\mu(\bigcup_{n=0}^{\infty}(T^{-n}(A)) = \sum_{n=0}^{\infty}\mu(T^{-n}(A))$$

Because T is a measure preserving transformation,

$$\sum_{n=0}^{\infty}\mu(T^{-n}(A)) = \sum_{n=0}^{\infty}\mu(A) = \infty$$

This is a contradiction as, by hypothesis, $\mu(A) < \infty$. This implies that there exists $n > 0$ such that $\mu(A \cap T^{-n}(A)) > 0$ for all $n > 0$. ■

In addition to the idea of recurrence, another element of dynamical systems that is important to consider is the concept of ergodicity. However, before we define ergodicity, we must first define the concept of a **T-invariant measurable set**. We say that a measurable set A is T-invariant if $T^{-1}(A) = A$.

Definition 4.4. We say that a measure-preserving transformation T is ergodic if, for any T-invariant set A, either $\mu(A) = 0$ or $\mu(A^c) = 0$.

However, this definition of Ergodicity is equivalent to several others which are easier to work with when trying to prove that a transformation is ergodic.

Proposition 4.5. *Suppose A and B are sets of positive measure. If there exists an integer $n > 0$ such that $\mu(T^{-n}(A) \cap B) > 0$, then T is recurrent and ergodic.*

Proof. Let A be a strictly invariant set of positive measure. Then we know that $T^{-n}(A) = A$ for all $n > 0$. Let $B = A^c$. For the purpose of contradiction, suppose that B has a positive measure.

If B has a positive measure, then there exists an integer $n > 0$ such that $\mu(T^{-n}(A) \cap B) > 0$. This means that $\mu(A \cap A^c) > 0$. However, this is a contradiction as $A \cap A^c = \emptyset$.

Therefore, $\mu(A^c) = \mu(B) = 0$ which shows that T is ergodic.

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From here, it makes sense to consider an example of an ergodic transformation. In section 1, we considered the Rotation Transformation in terms of the unit circle. However, here we consider the transformation in terms of the interval $[0, 1)$.

Example 4.6. We define the Rotation Transformation $R_\alpha(x) = x + \alpha \mod 1$ This transformation is invertible, meaning we can define the full orbit of the set $\{R_\alpha^n(x)\}$ for all integers n. It follows from our definition of $R_\alpha(x)$ that it is an invertible, measure preserving transformation.

Theorem 4.7. *Irrational rotations are ergodic.*

Proof. First define A_1 and B_1 as two sets of positive measure on the finite measure space X. Also define I and J, two dyadic intervals such that

$$\lambda(A_1 \cap I) > 3/4\lambda(I) \quad \text{and} \quad \lambda(B_1 \cap J) > 3/4\lambda(J)$$

Without loss of generality, we can assume $\lambda(I) = \lambda(J)$.

Define $A = A_1 \cap I$ and $B = B_1 \cap J$. Suppose that $I = [a, b)$ and $J = [c, d)$. Suppose that $a \leqslant c$, meaning that J is to the right of I. As R_α is dense on $[0, 1)$, there exists $n > 0$ such that

$$d - \frac{d-c}{4} < R^n(b) < d$$

Therefore, $\lambda(R^n(I) \cap J) > 3/4\lambda(J)$.

Sine $A \subset I$ and $B \subset J$, this implies that

$$\lambda(R^n(A) \cap B) \geqslant \lambda(R^n(I) \cap J) - \lambda(I \setminus A) - \lambda(J \setminus B) > 3/4\lambda(J) - 1/4\lambda(I) - 1/4\lambda(J) > 0$$

By 4.5, this implies that R_α is ergodic. ■

5. An Example: Symbolic Dynamics

A classical example of dynamical systems is that of a symbolic space.

Definition 5.1. We define a Symbolic Space Σ_N as the set of all bi-infinite sequences comprised of the integers in $\{0, \ldots N-1\}$.

Every element x in the space Σ_N is of the form $x = \ldots x_{-n-1}X_{-n} \ldots x_0x_1x_2 \ldots x_n \ldots$ for all integer values of n. Each x_i is an element of $\{0, \ldots N-1\}$.

It is also possible to define the set Σ_N^+ so that elements are of the form $x = x_0x_1x_2 \ldots$. For this space ,we are only concerned with values of n that are positive, or one-sided infinite sequences.

We can also think of an element x in Σ_N as a representation of the positive real numbers in base N. For example, the binary representation of a real number $x \geqslant 0$ is an element in the space Σ_2.

Theorem 5.2. *Symbolic spaces are metric spaces.*

Proof. We begin by defining a metric $d[x, y]$ on the space Σ_N^+. We say that for $x, y \in \Sigma_N^+$, $I[x, y] = \inf \{i \geqslant 0 : x_i \neq y_i\}$. In other words, $I[x, y]$ is the first positive integer i where x and y are different.

From this, we define the metric $d[x, y] = 2^{-I[x,y]}$ for all $x \neq y$ and $d[x, y] = 0$ for all $x = y$. To show that $(\Sigma_N^+, d[x, y])$ is a metric space, we must show the following

(1) For all $x, y \in \Sigma_N, d[x, y] = 0$ if and only if $x = y$.
   This follows from our definition of $d[x, y]$ as, if $x \neq y, d[x, y] \neq 0$.
(2) $d[x, y]$ is symmetric.
   It is easy to see that $I[x, y] = I[y, x]$ because of how we define the function $I[x, y]$. Therefore, $d[x, y] = d[y, x]$.
(3) $d[x, y]$ satisfies the triangle inequality
   We want to show that $d[x, y] \leqslant d[x, z] + d[z, y]$ for some $x, y, z \in \Sigma_N^+$.
   We know that $I[x, y]$ is the first positive integer i where x and y are different.

Suppose $I[x,z] = i$ and $I[z,y] = j$. Suppose that $j \geqslant i$. Then, we know that $x_n = y_n$ for all $n \leqslant i$. This means that $I[z,y] \geqslant i$. Now suppose the opposite. Suppose that $j \leqslant i$. Then $I[x,z] \geqslant j$, which means that $x_n = y_n$ for all $n \leqslant j$.

From this, we know that $I[x,y] \geqslant \min\{I[x,z], I[z,y]\}$. From our definition of $d[x,y]$ it follows that $d[x,y] \leqslant \min\{d[x,z], d[z,y]\}$. We know then that $d[x,y]$ must satisfy the triangle inequality.

■

Example 5.3. One application of symbolic spaces is with its connection to the Cantor set. As stated above, we can express every positive real number as an element of the symbolic space Σ_N. It is possible to define the elements of the Cantor Set in a similar way.

Recall the construction of the Cantor Middle Third Set. To define the elements of the Cantor Middle Third Set, we look at elements of Σ_3. By the nature of its construction, the Cantor Middle Third Set

$$K = \{\sigma \in \Sigma_3 : \sigma_i = 0 \quad \text{or} \quad 2 \forall i\}$$

This is true because we always remove the middle third from every interval when constructing the set. The same holds for any Fat Cantor Set. For example, suppose we have a Cantor Set in which we remove the second and fourth 1/5th from $[0,1)$. Then, symbolically, the points in this set would be represented by the following expression:

$$K = \{\sigma \in \Sigma_5 : \sigma_i \in \{0,2,4\} | \forall i\}$$

This is an isomorphism between a Cantor Set and a Symbolic Space, which means we can use the symbolic representation to analyze the dynamics of the Cantor Set.

Example 5.4. The most common example of a symbolic space is is the space Σ_2. All elements σ of this space can be represented solely by 0's and 1's and elements of the space are the binary representations of all real numbers.

By above, we know that this space must be a metric space. However, we also define a metric $d^*[s,t]$ that is limited to Σ_N^+ to show the same property.

Suppose that for $s,t \in \Sigma_N^+$, we define $d^*[s,t] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^t}$. We note that $d^*[s,t]$ is dominated by the series $\sum_{i=0}^{\infty} 1/2^i$, which always converges.

Theorem 5.5. *Σ_2^+ is a metric space under $d^*[s,t]$.*

Proof. In order to prove that $(\Sigma_2^+, d^*[s,t])$ is a metric space, we must prove the following:

(1) For all $s,t \in \Sigma_2^+, d^*[s,t] = 0$ if and only if $s = t$.

By our definition of $d^*[s,t]$, we know that if $s \neq t$, then $d^*[s,t] \neq 0$ for any $s,t \in \Sigma_2^+$. Furthermore, if $d^*[s,t] = 0$, then this implies that for all $n \geqslant 0, s_i = t_i$, implying that $s = t$.

(2) $d^*[s,t]$ is symmetric.

Because $|s_i - t_i| = |t_i - s_i|, d^*[s,t] = d^*[t,s]$.

(3) $d^*[s,t]$ satisfies the triangle inequality.

We want to show that if we have $s, t, u \in \Sigma_2^+$, then $d^*[s,t] \leqslant d^*[s,u] + d^*[u,t]$. We know that $|s_i - t_i| \leqslant |s_i - u_i| + |u_i - t_i|$. Therefore, it follows that $d^*[s,t] \leqslant d^*[s,u] + d^*[u,t]$.

■

Example 5.6. On the space Σ_2^+, we can define a shift transformation σ which maps from Σ_2^+ to Σ_2^+ and is continuous.

Suppose we have an element $s \in \Sigma_2^+$ with $s = s_0 s_1 s_2 \ldots$. We define $\sigma(s) = s_1 s_2 s_3 \ldots$. This function, σ, is called the one-sided shift function and simply forgets the first entry of s. σ is a two-to-one function and therefore is not invertible.

In Σ_2, however, we can define the two sided shift function τ, which is defined similarly to σ. However, because we are dealing with an element $s = \ldots s_{-1} s_0 s_1 \ldots$, we not longer forget the first entry. Instead, we simply shift to the right. In other words,

$$\tau(\ldots s_{-1} s_0 s_1 \ldots) = \ldots s_0 s_1 s_2 \ldots$$

The function is invertible and we define τ^{-1} as a shift to the left or

$$\tau^{-1}(\ldots s_{-1} s_0 s_1 \ldots) = \ldots s_{-2} s_{-1} s_0 \ldots$$

Theorem 5.7. *The two sided shift function τ is a homeomorphism*

Proof. (1) τ is invertible

We know that the shift function is onto as every element of Σ_2 can be represented as $\tau(s)$ for some $s \in \Sigma_2$. Therefore, τ is onto.

We also know that τ is one-to-one. By our definition of τ, if we have $s, t \in \Sigma_2$ and $s \neq t$, then $\tau(s) \neq \tau(t)$. More importantly, if $s \neq t$, then $\tau^{-1}(s) \neq \tau^{-1}(t)$. This is true specifically because we are dealing with a two sided shift function. If we were concerned with σ rather than τ, this would not be true.

Therefore τ is invertible.

(2) τ and τ^{-1} are continuous on the metric $d[s,t] = 2^{-I[s,t[}$ as defined above

We claim that if we have two elements s and t in the space Σ_2, if $s_i = t_i$ for $i = 0, 1, \ldots n$, then $d[s,t] \leqslant 1/2^n$. If $s_i = t_i$ for $i \leqslant n$, then $d[s,t] \leqslant 2^{-n}$ as we define $I[s,t] = \inf\{i \in \mathbb{N} : s_i \neq t_i\}$.

For $\epsilon > 0$ and $s = s_0 s_1 \ldots$, take an n such that $1/2^n < \epsilon$. Now suppose that we have a $\delta = 1/2^{n+1}$. We can take another element $t = t_0 t_1 \ldots$ such that t satisfies $d[s,t] < \delta$. By above, we can say that we have $s_i = t_i$ for $i \leqslant n+1$, meaning that the i'th entries of $\tau(s)$ and $\tau(t)$ are equal for all $i \leqslant n$. We know that $d[\tau(s), \tau(t)] \leqslant 1/2^n < \epsilon$. Therefore τ is continuous.

As τ^{-1} is merely a shift to the left instead of to the right, by the same argument used above, we can say that τ^{-1} is continuous.

■

6. Acknowledgements

I would like to sincerely thank my mentor, Jessica Lin, for helping me through this project. I could not have done it without her help, patience and guidance. I would also like to thank the REU Program for funding this project and giving me a chance to explore an area of Mathematics that I had not previously considered. I have enjoyed working on this project immensely and I sincerely thank the program for giving me this opportunity.

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