EULER'S FORMULA AND THE FIVE COLOR THEOREM

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Abstract. In this paper, we will define the necessary concepts to formulate map coloring problems. Then, we will prove Euler's formula and apply it to prove the Five Color theorem.

Contents

1. Introduction

Many have heard of the famous Four Color Theorem, which states that any map drawn on a plane can be colored with 4 colors under the rule that neighboring countries must have different colors. The Five Color Theorem which we will discuss in this paper is implied by the Four Color Theorem, but it is worth mentioning because it is considerably easier to prove. To elaborate, while the Four Color Theorem has only been proved by the brute force of computers, the Five Color Theorem can be proved within a few logical steps using Euler's formula and combinatorics. Therefore, we will focus on Euler's formula and introduce the Five Color Theorem as one of its many applications.

2. Preliminaries for Map Coloring

Map coloring problems can be easily understood by anyone. All we have to remember is to color the neighboring countries with different colors. However, approaching the problem mathematically is not so easy because theorems related to map coloring require rigorous definitions of its concepts.

First of all, we need to define what a map is and what it means to color a map. Concepts such as countries, borderlines also need to be defined in order to convert the problem into an abstract format. After defining all the relevant concepts, we can solve the problem using combinatorics and Euler's formula.

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Definition 2.1. A subset C of the plane $\mathbb{R}^2$ is

(a) an *arc* if there exists an injective continuous mapping $c : [0,1] \to \mathbb{R}^2$ such that

$$C = \text{Image } c := \{c(t) : \text{t} \in [0,1]\}$$

(b) a *simple closed curve*, or a *closed Jordan curve*, if there exists a continuous mapping $c : [0,1] \to \mathbb{R}^2$ for which

$$C = \text{Image } c$$

and such that $c(0)$=$c(1)$ and $c|_{[0,1)}$ is injective

(c) a *simple curve* if it is either an arc or a closed Jordan curve.

Definition 2.2. A *map* is a finite set L of arcs in the plane $\mathbb{R}^2$ having the property that the intersection of any two distinct arcs in L is either empty or is a common end point of these arcs.

For a given map L, we denote the arcs belonging to L as *edges* of L.

Definition 2.3. Let L be a map. A point in $\mathbb{R}^2$ is a *vertex* of L if it is the end point of an edge of L.

Hence, a vertex of a map is what we commonly refer to as "multinational corner" or "border-stone." A vertex is called a *final vertex* if it is the vertex of only one edge.

If L is a map, then, its corresponding *set of vertices* will be denoted by E_L. The pair $G = (E_L, L)$ then forms a finite *plane graph.* Since one can immediately obtain a map from a graph and vice versa, we will use the terms "map" and "graph" almost synonymously.

If L is a map, then we call all points that belong to an edge of L *neutral points* of L. The set of all neutral points, that is, the *neutrality set* of L, will be denoted by N_L, or, in the case where L is a fixed map, simply N. A map K is a *circuit* if its neutrality set N_K is a closed Jordan curve.

Using the definitions aforementioned, we can now define a "country."

Definition 2.4. Let L be a map. A *country* of L is a path component of the complement of the neutrality set of L, that is, of $\mathbb{R}^2 \backslash N_L$.

From this definition, a few properties of maps emerge.

Theorem 2.5. *Let L be a map.*

(a) Every point of the plane that is not a neutral point belongs to exactly one country.

(b) A country is an open subset of the plane.

(c) There exists exactly one unbounded country.

The theorem will not be proved here for the sake of conciseness. However, the theorem itself is intuitive and the proofs are rather straightforward. Interested readers can find proofs in ref [1].

Definition 2.6. A map is said to be *connected* if every pair of vertices can be joined by an arc consisting of concatenated edges.

This is precisely the case if the neutrality set is path connected.

Definition 2.7. Let L be a map and C a country of L
(a) An edge that lies entirely in the boundary of C is called a *borderline* of C.
(b) The set of all borderlines of C, denoted by G_C, is called the *border* of C.
(c) A set of edges of L is called a *national border* if it is the border of a country.

Definition 2.8. Let L be a map. An edge is a *common borderline* of two countries of L if it belongs to the borders of both countries.

Definition 2.9. Two countries of a map that have a common borderline are said to be *neighboring countries*, or simply *neighbors*.

If L is a map, we denote by M_L the set of all countries of L. Map coloring is a mapping from the set of countries of L to the set of colors. Yet, the colors do not necessarily need to be "red" or "blue". What matters is whether the colors are same or different. Hence, we will designate n colors by the numbers 1,2,3,...,n.

Definition 2.10. Let L be a map and n $\in \mathbb{N}$. An $n-coloring$ of L is a mapping $\phi : M_L \to \{1,...,n\}$. An n-coloring is *admissible* if neighboring countries always have distinct function values (i.e., "colors").

Lemma 2.11. *Let $\phi : M_L \to \{1,...,n\}$ be an admissible n-coloring of a map L and π :$\{1,...,n\} \to \{1,...,n\}$ a permutation. Then the composition $\pi \circ \phi$ is also an admissible n-coloring.*

We consider two colorings of a map to be *equivalent* if they differ only by a permutation of colors.

Theorem 2.12. (Five Color Theorem) *For every map there exists an admissible 5-coloring.*

Now we are in a position to tackle the Five Color problem. But before moving on, let us end this section with one of the fundamental results in topology since we will be needing it later on in the proof.

Theorem 2.13. (Jordan Curve Theorem) *Let K be a closed Jordan curve. Then $\mathbb{R}^2 \backslash K$ is the disjoint union of two open sets I(K) ("interior domain" of K) and A(K) ("exterior domain" of K) in such a way that:*
1. I(K) is bounded, but A(K) is unbounded.
2. I(K) and A(K) are path connected.
3. Every arc that joins a point of I(K) to a point of A(K) has at least one point in common with K.
4. Every neighborhood of a point of K has a nonempty intersection with I(K) and with A(K).

3. Euler's Formula

Theorem 3.1. (Euler's Formula) *Suppose that G is a connected planar graph that has V vertices, E edges, and F faces. Then, the following holds:*

$$V - E + F = 2. \tag{3.2}$$

The proof is by induction on vertices.

Proof. If G has only one vertex, then each edge is a Jordan curve that does not intersect with the other. For E edges, there are $(E+1)$ faces. Hence, $1-E+(E+1)=2$.

If G has more than one vertex, choose an edge e connecting two different vertices of G and contract it. This decreases both the vertex and edge by one, and the result then holds by induction. □

We can also see that Euler's formula is valid for convex polyhedra. In fact, every convex polyhedron corresponds to a connected planar graph with the same number of edges and vertices. This can be demonstrated by puncturing any face of the polyhedron and pulling the hole open until the polyhedron is stretched out flat. This process can be carried out mathematically via stereographical projection.

Moreover, Steinitz's theorem says that every convex polyhedron forms a 3-connected planar graph, and every 3-connected planar graph can be represented as the graph of a convex polyhedron. Hence, Euler's formula may also be referred to as Euler's polyhedral formula.

Theorem 3.3. (Euler's Polyhedral Formula) *Let P be a polyhedron which satisfies:*

(a) Any two vertices of P can be connected by a chain of edges.

(b) Any loop on P which is made up of straight line segments (not necessarily edges) separates P into two pieces.

Then, the following holds for P:

$$V - E + F = 2.$$

4. The Five Color Theorem

The basic approach to the Five Color theorem is quite simple. It involves the investigation of "minimal counterexamples"- sometimes referred to as "minimal criminals." Suppose a map cannot be colored with five colors. Of the counterexamples, then, there must be one such map having the fewest number f of countries. We call a map with f countries that cannot be colored with five colors a "miminal criminal." The essence of the proof lies in showing that there cannot exist such a minimal criminal.

Definition 4.1. Let L be a map. An edge of L is

(a) a *circuit edge* if it is an edge of a circuit in L.

(b) a *final edge* if it is incident with a final vertex of L.

(c) a *bridge* if it is neither a circuit edge or a final edge.

Definition 4.2. A map is said to be *regular* if it fulfills the follwing conditions:

1. It is not empty.
2. It is connected.
3. It contains no bridges and no final edges.
4. Any two distinct countries have at most one common borderlines.

To prove the Five Color theorem we will assume that the maps are regular. This is because regular maps are easier to picture and have some nice properties. In addition, a theorem guarantees that dealing with regular maps is sufficient.

Theorem 4.3. *If there exist miminal criminals at all, then there must be regular maps among them.*

Proof outline. Let L be a minimal criminal. Since L must have at least 6 countries, it is not empty.

Also, we can assume that L is without bridges or final edges because removing bridges or final edges does not affect the coloring. In admissible colorings of a map, only the edges that are common borderlines play a role. However, bridges or final edges are not common borderlines since they cannot belong to two distinct countries. Hence, we can assume that L is without bridges or final edges. In this instance, however, L is also connected since minimal criminals without bridges or final edges are connected.

Circuit edges that abut at vertices of degree 2 can always be merged, so we can assume that L contains only vertices of degree ≥ 3. In this case, however, two distinct countries have at most one common borderline. If two distinct countries C_0 and C_1 meet at more than two common borderlines, then there will be some countries that are surrounded by the union of C_1 and C_2. Denote the set of such "inner" countries as $I(C_{0,1})$. We can merge two adjacent countries in $I(C_{0,1})$ and form a new map L'. L' is 5-colorable since it has less countries than L. We now color L as follows. The countries not in $I(C_{0,1})$ will be colored with the colors provided by the 5-coloring of L'. The countries in $I(C_{0,1})$ will be colored by taking into consideration the colors of C_0 and C_1. This shows that L has an admissible 5-coloring, which is a contradiction. Therefore, if minimal criminals exist at all, there are regular maps among them. For a complete proof, see ref[1].

One can establish basic estimates as to the number of vertices, edges, and countries of a map. More important, however, are a few of the relationships among these numbers. In that regard, we consider a map L with the countries $C_1, C_2, ..., C_{f_L}$. In addition, let n_s denote the number of borderlines of C_s for s $\in\{1,...,f_L\}$.

Proposition 4.4. *Let L be a regular map. Then the following hold:*

$$\sum_{s=1}^{f_L} n_s = 2 \cdot e_L \tag{4.5}$$

$$3 \cdot v_L \leq 2 \cdot e_L \tag{4.6}$$

Proof. (4.4) Every borderline is an edge and is shared by two countries. Therefore, if we sum the number of borderlines, it will be exactly twice the number of edges since each edge will be counted twice.

(4.5) Let d_r denote the degree of the r-th vertex. Every vertex of a regular map has at least degree 3. Hence,

$$\begin{aligned} 3 \cdot v_L &= 3 \cdot \sum^{v_L} 1 \\ &= \sum^{v_L} 3 \\ &\leq \sum^{v_L} d_r \\ &= 2 \cdot e_L \end{aligned}$$

$\square$

Theorem 4.7. *The following holds for a regular map:*

$$\sum_{s=1}^{f_L}(6-n_s) \geq 12 \tag{4.8}$$

Proof. We can use the results obtained in Proposition 4.3 and Euler's formula to prove these inequalities.

$$\begin{aligned}\sum_{s=1}^{f_L}(6-n_s) &= 6 \cdot v_L - 2 \cdot e_L \\ &= 6 \cdot v_L - 6 \cdot e_L + 4 \cdot e_L \\ &\geq 6 \cdot v_L - 6 \cdot e_L + 6 \cdot f_L \\ &= 6 \cdot (v_L - e_L + f_L) \\ &= 12\end{aligned}$$

□

Now we are in a position to prove the Five Color theorem.

Theorem 4.9. (Five Color Theorem) *For every regular map there exists an admissible 5-coloring.*

Proof. Suppose not. Then, there exists a minimal criminal. This minimal criminal will be denoted by L.

Claim 1. Every country in map L has at least 5 neighbors.

Proof. Suppose not. Then, there exists a country C_0 with at most 4 neighbors. Denote one of the neighbors as C_1 and erase the common borderline with C_0. Denote this newly formed country as C'. Then, the new map L' is 5-colorable since it has less countries than the minimal criminal.

Now draw the borderline again. Since C_0 has at most 4 neighbors, at most 4 colors are needed to color C' and its neighbors. If we assign C_1 the color of C', we still have one color remaining for C_0. Therefore, the map L is 5-colorable, which is a contradiction since we assumed that L was a minimal criminal. □

Claim 2. There exists a country in map L with exactly 5 neighborss

Proof. By Theorem 4.6,

$$\sum_{s=1}^{f_L}(6-n_s) \geq 12 \tag{4.10}$$

The sum must be positive. Hence, there exists a country with at most 5 neighbors. However, Claim 1 states that every country has at least 5 neighbors. Therefore, there exists a country with exactly 5 neighbors. □

By Claim 2, we can find a country C_0 in L with exactly 5 neighbors. Pick one neighboring country and label it C_1. Then, proceed counter-clockwise and label the next one C_2 and so on. From this figure, we can find 2 countries that are not neighboring. This follows from the Jordan Curve theorem. Suppose C_1 and C_3 are neighbors. Draw a Jordan curve K that lies entirely in the union of these two countries and C_0, and passes through these three countries. Then, by the

Jordan Curve theorem, any path connecting C_2 and C_4 cannot be drawn without intersecting curve K. Therefore, C_2 and C_4 are not neighbors.

Now take C_0, C_2, C_4 and merge these countries. Denote the newly formed country as C'. Then the map is now 5-colorable because it has less countries than the minimal criminal. Color the maps and put the borderlines back. If we take the color of C' to be the colors of C_2 and C_4, we have one color remaining for C_0. Therefore, L is also 5-colorable, which is a contradiction. Therefore, any regular map is 5-colorable.

□

5. Another Application: Platonic Solids

Euler's formula can also be used derive a fact about platonic solids.

Definition 5.1. A *platonic solid* is a convex polyhedron all of whose vertices have the same degree and all of whose faces are congruent to the same regular polygon.

Theorem 5.2. *There are only five platonic solids.*

Proof. For platonic solids,

$$pF = 2E = qV \tag{5.3}$$

where p stands for the number of edges of each face and q for the number of edges meeting at each vertex. We can apply (5.3) to Euler's formula,

$$V - E + F = 2$$

and obtain

$$2E/q - E + 2E/p = 2$$

Simple algebraic manipulation then gives

$$1/q + 1/p = 1/2 + 1/E$$

Since E is strictly positive, we have

$$1/q + 1/p > 1/2$$

Also, p and q must both be at least 3. Therefore, there are only five possibilities for (p, q):

$$(3, 3), (4, 3), (3, 4), (3, 5), (5, 3).$$

□

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