AN INTRODUCTION TO GEOMETRIC STABILITY THEORY

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Abstract. In this paper, we will introduce some of the most basic concepts in geometric stability theory, and attempt to state a dichotomy theorem proved by Buechler. We will state and prove one of the requisite lemmas, though we will not provide a proof of the theorem itself.

Contents

0. Introduction

Stability theory is a field that grew out of an attempt to classify models of theories: Shelah introduced the notion of a stable theory (roughly, a theory is stable if it does not have too many types) in 1969, and this turned out to be incredibly useful in helping to distinguish between theories whose models could be classified, and those whose models couldn't. Geometric stability theory started, perhaps, when Zilber recognized in the 1970s that questions of classification were linked to the 'geometry' of independence. While we hope this is enough to paint some idea of the field for the reader, Pillay traces out the historical development of the field in considerable detail for those interested.

We will begin largely with a sequence of definitions, and leave most of the proofs for the final section. We start off with a few definitions from model theory — most of them revolving around the notion of a *type* — and then define what it means to be a (*pre-*)*geometry*. We will then construct a very natural pregeometry on a minimal set in an arbitrary (universal) model of a (complete) theory T.

We will assume some basic model theory — namely, we will assume knowledge of the definitions of a model, a formula and a complete theory. For the purposes of this paper, however, the details are not important. A formula is simply a 'grammatically correct' logical string which may or may not have free variables (for instance, $x = x$ where x is a variable is a formula); a sentence is just a formula with no unbound variables (for example, $\forall x(x = x)$ or $c = c$ for a variable x and a constant c). A theory T is a set of logical sentences with symbols from some language $\mathscr{L}$. A complete theory is a consistent set of sentences which is maximal with respect to inclusion. A model $\mathcal{M}$ of a theory T is a set equipped with some sort of structure that allows it to 'satisfy' sentences, with this satisfaction relation being required to behave as would be expected with the operators $\neg$, $\wedge$, $\vee$, $\forall$ and $\exists$ (the satisfaction relation is more accurately a part of the logic, but this distinction is irrelevant for the purposes of this paper).

For a more detailed introduction, we refer the reader to the first few pages of most basic model theory texts (for instance, Chang & Keisler).

Concerning notation, we will use $\overline{x}$ to mean an n-tuple for some fixed n (sometimes specified), and we will loosely use $\overline{x} \in X$ to mean $\overline{x} \in X^n$. For some subset A of the universe of the model, we will denote by $L(A)$ the language formed by adjoining constant symbols for each element in A to the language of the theory. When we are working with a generic $\mathcal{M}$, we will use M to mean the universe of the model; with a universal model $\mathfrak{C}$, we will never mention the universe - it will simply be assumed that all sets are subsets of the universal model of the theory. In most cases, there is no point in declaring $\mathfrak{C}$ to be the universal model, and we will often neglect to do so - it is simply assumed that all sets under consideration are subsets of some fixed universal model.

1. Types and Stability

One of the most basic definitions (and especially so in this section), is that of a type.

Definition 1.1. Let $\mathcal{M}$ be a model for a language L, and let $A \subset M$. An *n-type* over A in $\mathcal{M}$ is a consistent set $p(\overline{v})$ of formulas in $L(A)$ with at most n free variables $\overline{v} = (v_1, \ldots, v_n)$ so that, for any finite subset $p_0(\overline{v})$, there is $\overline{c} = (c_1, \ldots, c_n)$ for $c_1, \ldots, c_n \in M$ so that $\mathcal{M} \vDash \varphi(\overline{c})$ for all $\varphi(\overline{v}) \in p_0(\overline{v})$.

A *complete n-type* is an n-type which is maximal with respect to inclusion.

For a given model $\mathcal{M}$ and a subset $A \subset M$, we will denote by $S_n(A)$ the set of complete n-types over A in $\mathcal{M}$. We let $S(A) = \bigcup_{n<\omega} S_n(A)$.

A type is simply a set of formulas whose finite subsets carve out subsets of the model (we will soon see that this need not be true of the type itself). It is important to note here that the definition of a type does depend on the model in more than just a formal sense - the formulas are formed with symbols from the language $L(A)$ for some subset A of the model $\mathcal{M}$.

Finally, observe that there isn't necessarily an element of $\mathcal{M}$ that satisfies every formula in a type, though this will be true in some elementary extension of $\mathcal{M}$ by a compactness argument. To see this, and perhaps to better illustrate the definition, we provide the following example.

Example 1.2. Let $L = \{\leq, \cdot\}$ and let $\mathcal{M}$ be the rationals equipped with the usual ordering and multiplication. Let $A \subset M$ be the set of decimal approximations to $\sqrt{2}$, along with 2 (that is, $A = \{2, 1, 1.4, 1.41, \ldots\}$). Our theory T will be the axioms for a dense linear order, and we form a 1-type over A consisting of all formulas of the form $a < x \wedge x \cdot x \leq 2$ for every constant $a \in A$ except 2. Observe that any finite subset of this type is satisfiable, though the entire collection is not satisfiable in the rationals, but only in the reals. We could also have replaced $\leq$ with $<$ above, which would make the entire collection only satisfiable in the hyperreals!

The next definition associates a complete n-type to every n-tuple of elements from M.

Definition 1.3. Let $\overline{a}$ be an n-tuple from M and observe that there is a unique $p(\overline{v}) \in S(A)$ so that $\mathcal{M} \vDash p(\overline{a})$ (here, we say that $\overline{a}$ realizes the complete n-type $p(\overline{v})$ in $\mathcal{M}$). We call this $p(\overline{v})$ the *type of* $\overline{a}$ *over* A *in* $\mathcal{M}$, denoted by $tp_{\mathcal{M}}(\overline{a}/A)$.

To see uniqueness, note that this type is, written explicitly,

$$tp_{\mathcal{M}}(\overline{a}/A) = \{\varphi(\overline{v}) \mid \varphi(\overline{v}) \text{ is a formula over } A \text{ and } \mathcal{M} \vDash \varphi(\overline{a})\}$$

which, with completeness, gives uniqueness.

We are now in a position to introduce the concepts of saturation and stability. We begin with saturated models.

Definition 1.4. Let $\mathcal{M}$ be a model, and κ an infinite cardinal. We say that $\mathcal{M}$ is *κ-saturated* if $\mathcal{M}$ realizes every type in $S(A)$ for all $A \subset M$ with $|A| < \kappa$. We say that $\mathcal{M}$ is *saturated* if it is $|M|$-saturated.

For this to make sense, recall that models need not necessarily realize their types (we say they *omit* them if not): we only require this to be true for finite subsets.

On the other hand, we say a theory is stable when it does not have 'too many' types.

Definition 1.5. A complete theory T is said to be *κ-stable* for some cardinal κ if, for any set A with $|A| < \kappa$, we have $|S(A)| < \kappa$. A theory is *stable* if it is κ-stable for some infinite κ, and *superstable* if it is κ-stable for all sufficiently large κ.

It is a basic property of a saturated model $\mathcal{M}$ that any model $\mathcal{N}$ which is elementarily equivalent to $\mathcal{M}$ with $|N| \leq |M|$ is isomorphic to an elementary sub-model of of $\mathcal{M}$.

For a given complete theory T, we will - for the sake of convenience - fix a *universal domain* which we will often denote by $\mathfrak{C}$. The universal domain will be an arbitrarily large model (of some fixed cardinality larger than all cardinalities we are interested in) which contains all models of T that we are considering as elementary sub-models — assuming that there are saturated models of arbitrarily large cardinality, this makes sense given the above property. For a more detailed discussion on the existence of universal domains, we refer the reader to Buechler.

We now move on to notions of definability and rank.

2. Definability, Minimality and Rank

We begin with a relatively crude and natural notion of 'smallness' of formulas.

Definition 2.1. Let T be a theory. A formula $\varphi(\overline{v})$ is said to be *algebraic* in T if φ is consistent with T and for all $\mathcal{M} \vDash T$, the set $\{\overline{a} \in M \mid \mathcal{M} \vDash \varphi(\overline{a})\}$ is finite.

A type is algebraic if it implies an algebraic formula.

We will not be terribly interested in formulas which only carve out finite subsets of models. However, there is a more interesting notion: that of a *strongly minimal* formula.

Definition 2.2. Let $\mathcal{M}$ be a model and $\varphi(\overline{v})$ be a nonalgebraic formula. We say that $\varphi(\overline{v})$ over M is *strongly minimal* if for every elementary extension $\mathcal{N} \succ \mathcal{M}$ and every formula $\psi(\overline{w})$ over N, exactly one of $\{\overline{a} \in N \mid \mathcal{N} \vDash \varphi(\overline{a}) \wedge \psi(\overline{a})\}$ and $\{\overline{a} \in N \mid \mathcal{N} \vDash \varphi(\overline{a}) \wedge \neg\psi(\overline{a})\}$ is finite.

A complete theory T is *strongly minimal* if the formula $x = x$ is strongly minimal.

In the above definition, note that if one of the sets is finite, the other is necessarily infinite since we require $\varphi(\overline{v})$ to be nonalgebraic.

An extension of this concept leads to a notion of rank.

Definition 2.3. Let T be a complete theory. We define the *Morley rank* of a of a formula φ in n variables, $MR(\varphi)$, corresponding to a definable subset S of a model $\mathcal{M}$ inductively as follows:

- $MR(\varphi) = -1$ if φ is inconsistent.
- $MR(\varphi) = \alpha$ if $\{p \in S_n(\mathfrak{C}) : \varphi \in p \text{ and } \neg\psi \in p \text{ for all } \psi \text{ with } MR(\psi) < \alpha\}$ is nonempty and finite.

We define $MR(p)$ of an n-type p to be the infimum $\inf\{MR(\varphi) : p \text{ implies } \varphi\}$. $MR(S)$ and $MR(\varphi)$ will be used interchangeably and are equivalent.

Observe that being strongly minimal is equivalent to having Morley rank 1.

If p and q are complete n-types over A and B respectively, with $p \subset q$ and $A \subset B$, it is easy to see that $MR(q) \leq MR(p)$. Occasionally, we have equality, which leads to the following definition.

Definition 2.4. Let p be a complete type over A, and let q be a complete type over B such that $A \subset B$ and $p \subset q$. We say that q is a *non-forking* extension of p over B if $MR(p) = MR(q)$. If this is not the case, we say that q is a *forking* extension.

In particular, we will be interested in cases where a type not only has a nonforking extension over any $B \supset A$, but where this extension is also unique.

Definition 2.5. A complete type $p \in S(A)$ is said to be *stationary* if it has a unique nonforking extension over any $B \supset A$.

Finally, we will have need of a notion of definability.

Definition 2.6. Given a complete theory with with universal domain $\mathfrak{C}$, we say a subset $X \subset C$ is *definable* if $X = \{\overline{c} \in C \mid \mathfrak{C} \models \varphi(\overline{c})\}$ for some formula $\varphi(\overline{v})$. X is moreover said to be *strongly minimal* if $\varphi(\overline{v})$ is strongly minimal.

An analogous notion, with types rather than formulas, is that of $\bigwedge$-*definability.*

Definition 2.7. Let $\mathfrak{C}$ be the universal domain of a theory. A subset $X \subset C$ is called $\bigwedge$-*definable* over A if $X = \{\overline{c} \in C \mid \mathfrak{C} \models p(\overline{c})\}$ for some type $p(\overline{v})$ over A. X is $\bigwedge$-*definable* if it is $\bigwedge$-definable over A for some A.

3. Pregeometries

In this section, we will attempt to impose a natural geometrical structure on 'minimal' sets in the universal domain of an arbitrary complete theory. We begin with the definition of a pregeometry.

Definition 3.1. A *pregeometry* is a pair (S, cl) consisting of a set S and a (finitary) closure map $cl : \mathcal{P}(S) \to \mathcal{P}(S)$ satisfying:

(i) $X \subset cl(X)$ for any $X \in \mathcal{P}(S)$;
(ii) $cl(cl(X)) = cl(X)$ for any $X \in \mathcal{P}(S)$;
(iii) if $a \in cl(X \cup \{b\}) \setminus cl(X)$, then $b \in cl(X \cup \{a\})$, for any $a, b \in S$ and $X \in \mathcal{P}(S)$;
(iv) if $a \in cl(X)$, then $a \in cl(Y)$ for some finite $Y \subset X$, for any $X \in \mathcal{P}(S)$.

A pregeometry is said to be a *geometry* if, in addition, it has the property that the empty set $\emptyset$ and singleton sets $\{a\}$ for any $a \in S$ are closed (that is, they are equal to their own closure).

A set X is said to *trivial* if $cl(X) = \bigcup_{x \in X} cl(\{x\})$.

We note that 'geometry' here is used in the sense of combinatorial geometries (or matroids), rather than the traditional sense. A pregeometry can be defined as a *finitary* matroid.[1] It is often helpful to think of the closure operation as analogous to closure (or linear span) in a finite-dimensional vector space.

The third axiom allows us to give definitions for 'independence' and 'dimension'.

Definition 3.2. Let (S, cl) be a pregeometry, and let $A, B, C \subset S$. We say that A is *independent* over B if, for any $a \in A$, $a \notin cl((A \setminus \{a\}) \cup B)$.

A subset $A_0 \subset A$ is said to be a *basis* for A over B if $A \subset cl(A_0 \cup B)$ and A_0 is independent over B. The *dimension* $\dim(A/B)$ of A over B is defined to be the (unique) cardinality of a basis for A over B. We say that A is independent from B over C if for all finite subsets $A' \subset A$, we have $\dim(A'/(B \cup C)) = \dim(A'/C)$

Proving that the dimension of a set is well-defined is analogous to proving that the dimension of a vector space is well-defined.

With independence, we can give a definition of modularity (which, as will be clear from the definition, is a notion of geometric simplicity in some sense).

Definition 3.3. Let (S, cl) be a pregeometry. We say that (S, cl) is *modular* if, for any closed subsets $X, Y \subset S$, X is independent from Y over $X \cap Y$.

Given a subset $A \subset S$, we define the *localization* (S, cl_A) of (S, cl) at A by defining $cl_A(X) = cl(A \cup X)$. We say that a pregeometry (S, cl) is *locally modular* if, for some $a \in S$, the localization $(S, cl_{\{a\}})$ at $\{a\}$ is modular.

At this point, we will observe that with the notion of an 'algebraic' type, we can provide an analogous definition for n-tuples over a set by considering the associated n-type. Considering algebraic 1-types over a set gives us a relatively natural closure operation on subsets of models.

Definition 3.4. Let $\mathcal{M}$ be a model with $A \subset M$ and $\overline{a} \in M$. We say that $\overline{a}$ is *algebraic* over A if $tp_{\mathcal{M}}(\overline{a}/A)$ is algebraic. The *algebraic closure* of A in $\mathcal{M}$ is the set $\{a \in M \mid a \text{ is algebraic over } A\}$.

It is a relatively easy exercise to verify that this does, in fact, form a closure operator on the universe of a model.

4. A Dichotomy Theorem

We are now in a position to state the dichotomy theorem; while we cannot prove this, we will provide a proof for a preliminary lemma.

Theorem 4.1. *Let T be a complete superstable theory and D a minimal set in a universal model for T which is not locally modular. Then D is strongly minimal.*

We start a few definitions which will give us a notion of definability of types, which will become useful in the lemma.

[1] A matroid is simply a structure which abstracts the notion of independence in a vector space. We will not give a formal definition, but only mention it here in an attempt to make the fourth axiom in the definition seem a little less bizarre.

Definition 4.2. A formula $\varphi(x,y)$ is said to be stable if there do not exist a_i, b_i for $i \in \mathbb{N}$ such that $\models \varphi(a_i, b_j)$ if and only if $i \leq j$.

Definition 4.3. Let M be a model and $p(x)$ a complete type over M. A *δ-definition* of $p(x)$ is a formula $\psi(y)$ of L_M such that, for all $b \in M$, $\delta(x,b) \in p(x)$ if and only if $\models \psi(b)$.

It is a basic result that if $\delta(x,y)$ is stable and $p(x)$ is a complete type, then there is a δ-definition of $p(x)$. For a proof, we refer the reader to Pillay's book.

Finally, we will need to introduce two more notions of rank, after which we will be able to proceed.

Definition 4.4. Let T be a complete theory. We define the U-rank on complete types by the following recursion: $U(p) \geq \alpha$ if for all $\beta < \alpha$ there is a forking extension q of p such that $U(q) \geq \beta$. We say $U(p) = \alpha$ if $U(p) \geq \alpha$ but $U(p) \not\geq \alpha+1$. We say that $U(p) = \infty$ if $U(p) \geq \alpha$ for all α.

For consistent formulas $\varphi(x)$, we define the ∞-rank of φ, $R^\infty(\varphi)$, to be -1 if φ is inconsistent and $R^\infty(\varphi) = \alpha$ if

$$\{p \in S_n(\mathfrak{C}) \mid \varphi \in p \text{ and } \neg\psi \in p \text{ for all formulas } \psi \text{ with } R^\infty(\psi) < \alpha\}$$

is nonempty and has cardinality less than $|\mathfrak{C}|$.

The lemma reduces the scope of formulas we need to consider to only weakly minimal, nontrivial sets. It should be clear from the definition in the previous section that weak minimality implies local modularity.

At this point, we will blur the distinction between the properties of types and formulas and the properties of the sets they define for convenience.

Lemma 4.5. *Let T be a superstable theory and $p(x) \in S(A)$ a complete minimal stationary type (defining a set D). If $p(x)$ is non-trivial, then p is weakly minimal.*

Proof. Since p is nontrivial, there are realizations $a_1, \ldots, a_n, b$ of p such that $b \in acl(a_1, \ldots, a_n, A)$ but $b \notin acl(a_i, A)$ for all i. We minimize n so that $\{a_1, \ldots, a_n\}$ is independent over A (and observe that there are at least two elements in this set by the non-triviality of p).

Now, let $B = A \cup \{a_1, \ldots, a_{n-2}\}$. Since p is stationary, there is a unique non-forking extension p_1 of p over B. We will show that $R^\infty(p_1) = 1$, which will show that $R^\infty(p) = 1$. Observe that a_{n-1}, a_n, b all realize p_1 and are pairwise independent over B, but not independent together. Let $\varphi(x,y,z)$ be a formula over B such that $\models \varphi(a_{n-1}, a_n, b)$ and for all a', b', c', if $\models \varphi(a', b', c')$, then $a' \in acl(b', c', B)$ and $b' \in acl(a', c', B)$.

Let $\alpha = R^\infty(p_1)$ and let $\theta(x) \in p_1(x)$ be a formula with ∞-rank α (we can find such a formula since p_1 is complete). Let $\chi(x,y)$ be the formula $\exists z(\varphi(x,y,z) \wedge \theta(z))$, and let $\psi(y)$ be the $\chi(x,y)$-definition of the stationary type p_1. Since p_1 is over B, so is $\psi(y)$. Moreover, observe that $\models \psi(b)$ and $\models \chi(a,b)$. Hence, $\psi(y) \in p_1(y)$, since p_1 is complete.

We now claim that $R^\infty(\psi(y)) = 1$, which will immediately imply $R^\infty(p_1) = 1$. To see this, we will show that for any $a' \in \psi(\mathfrak{C})$, $U(a') \leq 1$. Pick any such a' - by the definition of ψ, there are b' and c' such that $b' \in D$, $\models \varphi(a', b', c') \wedge \theta(c')$, and b' is independent from a'. Since $b' \in acl(a', c')$, we see that $\alpha = R^\infty(b'/a') \leq R^\infty(c'/a') \leq R^\infty(c') \leq \alpha$. Thus, c' is independent from a' over B. Since $a' \in acl(b', c')$, $U(a'/c') \leq U(b'/c') \leq 1$, which completes the proof. ∎

Proving Theorem 4.1 requires a little more work, and a few more definitions to capture the necessary concepts, though it is hoped that the proof above gives some idea of how the geometry of a model (in the sense of an independence structure) could be connected with the model-theoretic structure.

5. Acknowledgements

I would like to thank my mentor, Matthew Wright, for his comments, and Professor May for organizing the REU.

The proof of the lemma is an adaptation of the proofs provided by Buechler and Pillay, and the definitions are largely taken from the two (full references to the books are provided below). The historical note at the beginning is largely a summary of Pillay's introduction to his book.

References

[1] Steven Buechler, *Essential Stability Theory.* Springer, New York, 1996.
[2] Anand Pillay, *Geometric Stability Theory.* Oxford University Press, New York, 1996.
[3] C. C. Chang & H. J. Keisler, *Model Theory.* Elsevier, New York, 1973.