AN INTRODUCTION TO THE FUSION SYSTEMS OF BLOCKS

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Abstract. This paper will present an introduction to modular representations of finite groups via the techniques of block theory, first introduced by Brauer. After developing an understanding of block idempotents, Brauer pairs, and defect groups, we will present applications to the theory of finite groups and their fusion systems.

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1. Introduction

A representation of a finite group over a field k can be thought of as a module over the group algebra kG. If k is a field of characteristic zero, the results obtained are stronger than those obtained in positive characteristic. In particular, we have the following theorem by Maschke.

Theorem 1.1 (Maschke)**.** *Let k be a field, and G a group, such that the order of G does not divide the characteristic of the field k. Then, if M is a kG-module, M is completely reducible.*

Thus, the irreducible representations of a finite group G are the key to understanding arbitrary finite-dimensional representations of G in characteristic zero. In positive characteristic, irreducible representations are not as fundamental, as we have representations that cannot be expressed as the direct sum of irreducible ones. Many representations contain subrepresentations whose complements are not representations themselves, as seen in the following example.

Example 1.2. Let k be the finite field $\mathbb{F}_2$, and let $G = \mathbb{Z}/2\mathbb{Z} = \{g|g^2 = e\}$ (written multiplicatively). Then, kG is a 2-dimensional vector space over k, generated by e and g. The trivial representation, corresponding to the subspace generated by

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$e+g$, is an irreducible subspace. However, any other 1 dimensional subspace is not a subrepresentation, as it is not invariant under the action of $g \in \mathbb{Z}/2\mathbb{Z}$.

Instead, we desire to understand representations via their indecomposable subrepresentations. It turns out that while the positive characteristic representations do not necessarily decompose into irreducible representations, they do decompose into the sum of indecomposable subrepresentations, via the following lemma.

Lemma 1.3. *Let k be a field, G a finite group, and M a finitely generated kG-module. Then there is a decomposition of M into indecomposable two-sided ideals. That is,*

$$M = \bigoplus_{i=1}^{r} B_i$$

where B_i is an indecomposable two-sided ideal of M.

For convenience, we will call an indecomposable two-sided ideal summand of kG a *block* of kG.

There is a correspondence between the blocks of kG and certain central idempotents of the group algebra that will prove crucial to our theory. Recall that a non-zero element e in kG is an *idempotent* if $e^2 = e$. We say that two idempotents, e and e' are *orthogonal* if $ee' = e'e = 0$. Finally, an idempotent is called *central* if lies in the center of kG, and an idempotent is called *primitive* if it cannot be written as the sum of two orthogonal idempotents.

A central idempotent will be called primitive if it is primitive in the center of kG, rather than in kG itself. A central idempotent e therefore may be written as $e = e_1 + e_2$, for two idempotents e_1 and e_2 if they do not lie in the center of kG. The following propositions establish the correspondence between primitive central idempotents and blocks of kG.

Proposition 1.4. *Let R be a Noetherian ring, and M a finitely-generated R-algebra.*

(i) There is a one-to-one correspondence between decompositions of M into two-sided ideals and decompositions of 1 as the sum of pairwise orthogonal, central idempotents. In particular, a decomposition $1 = e_1 + e_2 + \cdots + e_r$ is in correspondence with $M = Me_1 \oplus Me_2 \oplus \cdots \oplus Me_r$, and a decomposition $M = B_1 + \cdots + B_r$ is in correspondence with the decomposition of 1 as an element of M in this ideal decomposition.

(ii) If B is a block of kG, with $M = B \oplus B'$, and $1 = e + e'$ as the corresponding decomposition, then e is a primitive central idempotent. Conversely, if e is a primitive central idempotent, then Me is a block of M.

Proposition 1.5. *Let R be a Noetherian ring, and M a finitely generated R-algebra, and let $M = Me_1 \oplus \cdots \oplus Me_r$ be a decomposition of M into blocks. Then if $B = Me$ is a block of M, then $B = Me_i$, for some i.*

Given a decomposition as in the previous theorem, we call the e_i the *block idempotents* of M. Now that we know more of the theory, we can provide a more interesting example.

Example 1.6. Let k be a field of characteristic 2, and let $G = S_3$. We want to find the decomposition of kG into indecomposable representations by computing block idempotents. Recall that the center of the group algebra $Z(kG)$ is generated by class sums of conjugacy classes of G. Since the block idempotents are central, this is where we will begin searching. The conjugacy classes of S_3 correspond exactly with the cycle types of the permutations. Thus, $Z(kG)$ is generated by 1, $(1\ 2) + (1\ 3) + (2\ 3)$, and $(1\ 2\ 3) + (1\ 3\ 2)$. A computation yields that $b_0 = 1 + (1\ 2\ 3) + (1\ 3\ 2)$ and $b_1 = (1\ 2\ 3) + (1\ 3\ 2)$ are both block idempotents.

Note that in general, the group algebra kG will not contain all indecomposable representations as subrepresentations. Indeed, it is often the case that there are infinitely many indecomposable representations.

Finally, we will state the following lemma.

Lemma 1.7 (Rosenberg's Lemma)**.** *Let R be a Noetherian ring, and M a finitely-generated R-module, and let e be a primitive idempotent of M. If e lies in a sum of two-sided ideal of M, then e lies in at least one of them.*

For the remainder of the paper, we will let G be a finite group, p be a prime dividing the order of G, and k an algebraically closed field of characteristic p.

2. The Brauer Morphism and Relative Trace

We now define the Brauer morphism and relative trace map. Despite the fact that the definitions of these maps are fairly straightforward, they will be incredibly useful in the future.

Definition 2.1. Let P be a p-subgroup of G. The *Brauer morphism*, $\mathrm{Br}_P : kG \to kC_G(P)$ is the surjective k-linear map given by the following:

$$\sum_{g \in G} \alpha_g g \mapsto \sum_{g \in C_G(P)} \alpha_g g$$

The Brauer morphism is the projection of kG onto the $kC_G(P)$. This is not a k-algebra homomorphism in general. For example, consider kS_3 and let $P = \langle (1\ 2) \rangle$. The 3-cycles are both mapped to zero, but since they are inverses in S_3, their product is not mapped to zero. However, if we restrict our domain, then Br_P is an algebra homomorphism. Let H be a subgroup of the group G acting on G by conjugation. This action can be extended linearly to an action on kG. The Brauer map is an algebra homomorphism if we restrict the domain to the P-stable elements of kG, that is, the elements of kG fixed under the action of P.

Notation 2.2. The H-stable elements of an arbitrary set X will be denoted by X^H. In the case that $X = kG$, we write $(kG)^H$.

Proposition 2.3. *Let P be a p-subgroup of G. The map Br_P is multiplicative when restricted to $(kG)^P$, the P-stable elements of kG.*

Corollary 2.4. *Let P be a p-subgroup of G.*

(i) If e is a central idempotent of kG, then $\mathrm{Br}_P(e)$ is either zero or a central idempotent of $kC_G(P)$.
(ii) If b is a block idempotent of kG, such that $\mathrm{Br}_P(b) \neq 0$, then $\mathrm{Br}_P(b) = b_1 + \cdots + b_r$ is a sum of block idempotents of $kC_G(P)$.
(iii) If e is a block idempotent of $kC_G(P)$, then $\mathrm{Br}_P(b)e \neq 0$ if and only if $e = b_i$ for some i. In this case, $\mathrm{Br}_P(b)b_i = b_i$.

The next step is to define the relative trace map.

Definition 2.5. Let G be a finite group acting on an abelian group X, and let H be a subgroup of G. The *relative trace*, denoted $\mathrm{Tr}_H^G : X^H \to X^G$, is the map

$$\mathrm{Tr}_H^G(x) = \sum_{g \in T} x^g$$

where T is a right transversal to H in G.

Note that the relative trace is well-defined, since, using the notation above, if $Hg = Hh$, and $x \in X$, then $x^g = x^{hg^{-1}g} = x^h$.

Notation 2.6. Given a group M with a G-action, then we can consider the group algebra kM as an abelian group under addition with a G-action by extending linearly. We write $(kM)_H^G$ to denote the image of $(kM)^H$ under the trace map Tr_H^G.

We will require several properties of the relative trace, but we omit the proofs as they are tangential to the goal of this paper.

Theorem 2.7. *Let e be a block idempotent of kG. There exists a p-subgroup D such that, for any subgroup H of G, we have that $e \in (kG)_H^G$ if and only if H contains a conjugate of D. The subgroup D is unique, up to G-conjugacy.*

Lemma 2.8. *Let H be a subgroup of G, and let P be a p-subgroup of H. Then,*

$$\mathrm{Ker}(\mathrm{Br}_P) \cap (kG)^H = \sum_{Q \in \mathcal{Q}} (kG)_Q^H$$

where $\mathcal{Q}$ is the set of all subgroups of H not containing a subgroup H-conjugate to P.

3. Brauer Pairs

We now define the concept of a Brauer pair, which will be a powerful tool in our study of fusion systems of blocks.

Definition 3.1. Let G be a finite group and let p be a prime dividing the order of the group G. A *Brauer pair* consists of a p-subgroup Q of G, and a block idempotent e of $kC_G(Q)$. We write (Q, e) to denote such a Brauer pair.

Since $C_G(Q^g) = C_G(Q)^g$, this gives a bijection between the block idempotents of $kC_G(Q)$ and $kC_G(Q^g)$. Thus, G acts on the Brauer pairs by conjugation, sending Q to Q^g and e to e^g.

Lemma 3.2. *Let G be a finite group, and let $Q \trianglelefteq R$ be p-subgroups of G. If e is a block idempotent of $kC_G(R)$, then there is a unique R-stable block idempotent f of $kC_G(Q)$ such that*

$$\mathrm{Br}_R(f)e = e.$$

If f' is any other R-stable block idempotent of $kC_G(Q)$, then $\mathrm{Br}_R(f')e = 0$.

Proof. We first prove the second part of the lemma. Suppose f is a block idempotent of $kC_G(Q)$ such that $\mathrm{Br}_R(f)e = e$, and let $f' \neq f$ be a an R-stable block idempotent. Then, the result follows from the following calculation:

$$\mathrm{Br}_R(f')e = \mathrm{Br}_R(f')(\mathrm{Br}_R(f)e) = \mathrm{Br}_R(f'f)e = 0.$$

To show existence, first note that if $g \in G$, and $r \in R$, then $\mathrm{Br}_R(g^r) = \mathrm{Br}_R(g)$, since $g \in C_G(R)$ if and only if $g^r \in C_G(R)$. As a result, $\mathrm{Br}_R(f^r) = \mathrm{Br}_R(f)$. So, the group R acts on the block idempotents of $kC_G(Q)$ via conjugation, and any idempotent that is not R-stable belongs to an orbit whose length is a multiple of p. Since k has characteristic p, this gives us that

$$1 = \mathrm{Br}_R(1) = \sum_{b \in \mathcal{B}(Q)} \mathrm{Br}_R(b) = \sum_{b \in \mathcal{B}(Q)^R} \mathrm{Br}_R(b)$$

where $\mathcal{B}(Q)$ is the set of block idempotents of $kC_G(Q)$, and $\mathcal{B}(Q)^R$ is the R-stable subset of $\mathcal{B}(Q)$. By multiplying on the right by some $e \in \mathcal{B}(R)$, we notice

$$e = 1 \cdot e = \sum_{b \in \mathcal{B}(Q)^R} \mathrm{Br}_R(b)e.$$

In particular, $\mathrm{Br}_R(b)e \neq 0$ for some b. Letting f be such a block idempotent, we apply Corollary 2.4 to see that $\mathrm{Br}_R(f)e = e$. □

We can define a partial ordering on the set of Brauer pairs of a group G. We first define the non-transitive relation $\trianglelefteq$ as follows. We say $(Q, f) \trianglelefteq (R, e)$ if $Q \trianglelefteq R$, the block idempotent f is R-stable, and $\mathrm{Br}_R(f)e = e$. We take the partial order $\leq$ on the Brauer pairs to be the transitive closure of the relation $\trianglelefteq$. The lemma we just proved shows that given $Q \trianglelefteq R$, and a block idempotent e of $kC_G(R)$, there is a unique Brauer pair (Q, f) such that $(Q, f) \leq (R, e)$. It turns out we have a much stronger result.

Theorem 3.3. *Let $Q \leq R$ be p-subgroups of G. If (R, e) is a Brauer pair, then there is a unique Brauer pair (Q, f) such that $(Q, f) \leq (R, e)$.*

In order to prove this theorem, we require the following lemma.

Lemma 3.4. *Let R be a p-subgroup of G. Let P and Q be normal subgroups of R with $P \leq Q$. Suppose that e is a block idempotent of $kC_G(R)$, and let f_1 and f_2 be the unique R-stable block idempotents of $kC_G(P)$ and $kC_G(Q)$ respectively, such that $\mathrm{Br}_R(f_1)e = e$, and $\mathrm{Br}_R(f_2)e = e$. If f is the unique Q-stable block idempotent of $kC_G(P)$ with $\mathrm{Br}_Q(f)f_2 = f_2$, then $f = f_1$.*

As a result, if $P \trianglelefteq R$, and e is a block idempotent of $kC_G(R)$, then there is a unique Brauer pair (P, f) such that $(P, f) \leq (R, e)$.

Proof. First, we show that f is indeed R-stable. Let $x \in R$, and note that f^x is a block idempotent of $kC_G(P)$ and is Q-stable, since $Q \trianglelefteq R$. Thus, $\mathrm{Br}_Q(f^x) = \mathrm{Br}_Q(f)^x$. But, f_2 is R-stable, and so

$$\mathrm{Br}_Q(f^x)f_2 = \mathrm{Br}_Q(f)^x f_2^x = f_2^x = f_2.$$

Thus, $f^x = f$, and f is R-stable by Lemma 3.2.

Now, we want to check that $\mathrm{Br}_R(f)e = e$, via the following computation:

$$\begin{aligned} \mathrm{Br}_R(f)e = \mathrm{Br}_R(f)\,\mathrm{Br}_R(f_2)e &= \mathrm{Br}_R(\mathrm{Br}_Q(f))e \\ &= \mathrm{Br}_R(\mathrm{Br}_Q(f)f_2)e = \mathrm{Br}_R(f_2)e = e. \end{aligned}$$

Thus, $f = f_1$. This shows that if $(P, f) \trianglelefteq (Q, f_2) \trianglelefteq (R, e)$, and $(P, f_1) \trianglelefteq (R, e)$, then $(P, f) = (P, f_1)$, and that if $(P, f_1) \trianglelefteq (Q, f_2) \trianglelefteq (R, e)$ and $P \trianglelefteq R$, then $(P, f_1) \trianglelefteq (R, e)$.

Now suppose we have another Brauer pair $(P, f') \leq (R, e)$, with $P \trianglelefteq R$, via the chain

$$(P, f') \trianglelefteq (Q_1, f_1') \trianglelefteq \cdots \trianglelefteq (Q_r, f_r') \trianglelefteq (R, e).$$

We can apply the previous process inductively on $(P, f') \trianglelefteq (Q_1, f_1') \trianglelefteq (Q_2, f_2')$, and onwards to see that $(P, f') \trianglelefteq (R, e)$. □

Proof of Theorem. We induct on the index $[R : Q]$. The base case, when $[R : Q] = p$, is proven in the lemma, since in this case, $Q \trianglelefteq R$. Furthermore, given a subnormal series of subgroups of R, that is a chain of subgroups $Q = P_1 \trianglelefteq P_2 \trianglelefteq \cdots \trianglelefteq P_r = R$, we have a unique chain of inclusions of Brauer pairs.

We now consider the case of some arbitrary subgroup Q of R. We can exhibit a subnormal series by taking the iterated normalizers. To see that this does create a subnormal series, first notice that by the class equation, any p-group has non-trivial center. Now suppose for contradiction that $N_R(S) = S$ for some proper subgroup S of R. Then $S_1 = S/Z(S)$ is a subgroup of $R_1 = R/Z(S)$ with normalizer $N_R(S)/Z(S) = S/Z(S)$. Continuing inductively for $S_i = S_{i-1}/Z(S_{i-1})$ and $R_i = R_{i-1}/Z(S_{i-1})$, we get that $|S_i| < |R_i|$, for all i. Then, S_i must be the trivial group for some i. For this i, R_i must also be the trivial group, since otherwise $N_{R_i}(S_i) \neq S_i$, which is a contradiction.

Suppose we have a subnormal series

$$Q = R_0 \trianglelefteq \cdots \trianglelefteq R_s \trianglelefteq R.$$

We assume that the R_i are distinct. Furthermore, since $R_1 \leq N_R(Q)$, we apply induction to see that there is a unique Brauer pair $(R_1, f') \leq (R, e)$, and a unique Brauer pair $(N_R(Q), f'') \leq (R, e)$. Thus, by the lemma,

$$(Q, f) \trianglelefteq (N_R(Q), f'') \trianglelefteq (R, e).$$

Since both (R_1, f') and $(N_R(Q), f'')$ are less than (R, e), we apply the inductive hypothesis to get that

$$(R_1, f') \leq (N_R(Q), f'').$$

Therefore, $(Q, f) \trianglelefteq (N_R(Q), f'')$ and $(R_1, f') \leq (N_R(Q), f'')$, so $(Q, f) \trianglelefteq (R_1, f')$. □

The Brauer pairs contained in (P, e) therefore form a poset identical to the poset of subgroups of P. Note that although if given a Brauer pair (P, e) and $Q \leq P$, there is a unique Brauer pair $(Q, e_Q) \leq (P, e)$, this does not hold in the opposite direction. Furthermore, the pair (Q, e_Q) may be contained in another Brauer pair (P, f) with $f \neq e$.

4. Defect groups

The machinery of Brauer pairs puts us one step closer to developing the notion of the fusion system of a block. We continue on this path by introducing the following definition.

Definition 4.1. Let G be a finite group and let b be a block idempotent of kG. A *b-Brauer pair* is a Brauer pair $(R,\ e)$ such that $\mathrm{Br}_R(b)e = e$. A *maximal* b-Brauer pair is a b-Brauer pair (D,e) such that $|D|$ is maximal. We call the subgroup D a defect group of the block b.

Note that we can also characterize b-Brauer pairs in the following two ways. First, a Brauer pair (R,e) is a b-Brauer pair if e is a term of $\mathrm{Br}_R(b)$, when expressed as a sum of block idempotents of $kC_G(R)$. Second, a Brauer pair $(R,\ e)$ is a b-Brauer pair if $(1,\ b) \leq (R,\ e)$.

Example 4.2. Let $p = 2$. Let $V = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, the elementary abelian group of order 9, with generators u and v. Let $Q = \langle x, y | x^4 = y^4 = 1, xyx^{-1} = y^3\rangle$ be the quaternion group. We let Q act on V on the left as follows:

$$x \cdot u = u^2 \qquad y \cdot u = u$$
$$x \cdot v = v \qquad y \cdot v = v^2$$

Let $G = V \rtimes Q$. Given $g \in G$, denote the class sum of g, that is, the sum of all elements of the conjugacy class of g, by $\hat{g}$. The group algebra kG has 4 blocks:

$$b_0 = 1 + \hat{u} + \hat{v} + \hat{uv} \qquad b_1 = \hat{v} + \hat{uv}$$
$$b_2 = \hat{u} + \hat{uv} \qquad b_3 = \hat{uv}$$

The subgroups of Q and their centralizers are as follows:

$R \leq Q$

$$\begin{tikzcd}
& 1 \arrow[d, dash] & \\
& Z(Q) = \langle x^2\rangle = \langle y^2\rangle \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\langle x\rangle \arrow[dr, dash] & \langle y\rangle \arrow[d, dash] & \langle xy\rangle \arrow[dl, dash] \\
& Q &
\end{tikzcd}$$

$C_G(Q)$

$$\begin{tikzcd}
& G \arrow[d, dash] & \\
& G \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\langle v\rangle \times \langle x\rangle \arrow[dr, dash] & \langle u\rangle \times \langle y\rangle \arrow[d, dash] & \langle xy\rangle \arrow[dl, dash] \\
& Z(Q) &
\end{tikzcd}$$

Under the Brauer morphism, the blocks map to the following idempotents

$\mathrm{Br}_R(b_0)$

$$\begin{tikzcd}
& b_0 \arrow[d, dash] & \\
& b_0 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
(1+v+v^2) \arrow[dr, dash] & (1+u+u^2) \arrow[d, dash] & 1 \arrow[dl, dash] \\
& 1 &
\end{tikzcd}$$

$\mathrm{Br}_R(b_1)$

$$\begin{tikzcd}
& b_1 \arrow[d, dash] & \\
& b_1 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
(v+v^2) \arrow[dr, dash] & 0 \arrow[d, dash] & 0 \arrow[dl, dash] \\
& 0 &
\end{tikzcd}$$

$\mathrm{Br}_R(b_3)$

$$\begin{tikzcd}
& b_3 \arrow[d, dash] & \\
& b_3 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
0 \arrow[dr, dash] & 0 \arrow[d, dash] & 0 \arrow[dl, dash] \\
& 1 &
\end{tikzcd}$$

The case for b_2 is omitted since it is analogous to b_1. This describes the b-Brauer pairs contained in Q.

We now analyze the conjugacy action of G on its b-Brauer pairs. We begin with a few useful lemmas. The proof of the first is omitted [3].

Lemma 4.3. *Let b be a block idempotent of kG, and D be a defect group of b. Then, $b \in (kG)^G_D$, i.e. there is some $a \in (kG)^D$ such that $b = \mathrm{Tr}^G_D(a)$.*

Lemma 4.4. *Let G be a finite group, let b be a block idempotent of kG, and let Q be a p-subgroup of G.*

(i) There exists a b-Brauer pair (Q, e) if and only if $\mathrm{Br}_Q(b) \neq 0$.
(ii) The subgroup Q is a defect group if and only if $\mathrm{Br}_Q(b) \neq 0$, and for all p-subgroups R properly containing a conjugate of Q, we have $\mathrm{Br}_R(b) = 0$.

Proof. If $\mathrm{Br}_Q(b) = 0$, then $\mathrm{Br}_Q(b)e = 0$, and (Q, e) is not a b-Brauer pair, for any block idempotent e of $kC_G(Q)$. Conversely, if $\mathrm{Br}_Q(b) \neq 0$, then we can write $\mathrm{Br}_Q(b) = b_1 + b_2 + \cdots + b_r$, where the b_i are block idempotents of $kC_G(Q)$, and $\mathrm{Br}_Q(b)e = e$ when $e = b_i$ for some i.

Suppose there is some $g \in G$ and subgroup R of G such that Q^g is properly contained in R, with $\mathrm{Br}_R(b) \neq 0$. Then, (Q, f) is a b-Brauer pair if and only if (Q^g, f^g) is a b-Brauer pair. Furthermore, $(Q^g, f^g) \leq (R, e)$. Then (R, e) is a b-Brauer pair with strictly greater order than Q, so Q cannot be a defect group.

Now suppose that Q is not a defect group of b, and (Q, f) is a b-Brauer pair. Then, there is some defect group R and some $g \in G$ such that $Q^g \leq R$. By Lemma 4.3, $b \in (kG)^G_R$, that is $b = \mathrm{Tr}^G_R(a)$ for some $a \in (kG)^R$. Let Cl_R denote the set of conjugacy classes of G containing an element x such that R contains a Sylow p-subgroup of $C_G(x)$. The set of class sums of Cl_R, that is $\{\hat{X} | X \in \mathrm{Cl}_R\}$, is a k-basis for $(kG)^G_R$. Thus, we can write

$$b = \sum_{X \in \mathrm{Cl}_R} \alpha_X \hat{X}, \qquad \alpha_X \in k$$

Because $\mathrm{Br}_Q(b) \neq 0$, there is some $X \in \mathrm{Cl}_R$ such that $\mathrm{Br}_Q(\hat{X}) \neq 0$, and $X \cap C_G(Q)$ is not empty, and there is some $x \in X$ that commutes with Q. In particular, $Q \leq C_G(x)$, and thus Q is contained in some Sylow p-subgroup of $C_G(x)$, and hence conjugate to a subgroup of R. □

Theorem 4.5. *Let b be a block idempotent of kG. Suppose D is a minimal p-subgroup such that $b \in (kG)^G_D$. Then, D is a defect group of b. Furthermore, G acts transitively by conjugation on the set of defect groups of b.*

Proof. Let D denote a minimal p-subgroup of G such that $b \in (kG)^G_D$. We claim that given a p-subgroup Q, $\mathrm{Br}_Q(b) \neq 0$ if and only if Q is contained in D. First, suppose that $\mathrm{Br}_Q(b) = 0$. By Lemma 2.8, $b \in \sum_{R \in \mathcal{R}} (kG)^G_R$, where $\mathcal{R}$ is the collection of all p-subgroups, R, not containing a conjugate of Q. By Rosenberg's Lemma (1.7), b lies in $(kG)^G_R$, for some $R \in \mathcal{R}$. Thus, Q is not contained in D.

Now, suppose that $\mathrm{Br}_Q(b) \neq 0$. Then, again by Lemma 2.8, b does not lie in any $(kG)^G_R$, where R does not contain any conjugate of Q. Thus, if $b \in (kG)^G_S$ for some p-subgroup S, then S does contain a conjugate of Q. Since $b \in (kG)^G_D$, we see that D contains a conjugate of Q as claimed.

Next, Theorem 2.7 tells us that minimal p-subgroups P of G such that $b \in (kG)^G_P$ must be conjugate. Thus, G acts transitively on the defect groups of b. □

It turns out that this conjugation action is compatible with the the Brauer pairs. That is, the group G acts transitively not only on the defect groups of a block idempotent b, but on the maximal b-Brauer pairs. We will first show this for the case of normal defect groups, but omit the proof of the first claim.

Proposition 4.6. *Let P be a normal p-subgroup of G, and let b be a block idempotent of kG. Then defect groups of b contain P, and $b \in (kC_G(P))^G$. Furthermore, write $b = b_1 + b_2 + \cdots + b_r$, where b_i is a block idempotent of $kC_G(P)$, and let D_i be a defect group of b_i, and H_i be the stabilizer in G of b_i. Then, for each i, we have that $b = \mathrm{Tr}^G_{H_i}(b_i)$. Moreover, if b has defect group P, then the Brauer pairs (P, b_i) are the maximal b-Brauer pairs, and G acts transitively on the set of maximal b-Brauer pairs.*

Proof. Let T_j be a transversal to H_j in G. If $t \in T_j$, then $b^t = b$ and $\left(\sum b_i\right)^t = \sum b_i$. Moreover, b_i^t is a block of $kC_G(P)$, since $C_G(P)^t = C_G(P^t) = C_G(P)$ and hence conjugation by t induces an automorphism of $kC_G(P)$. We know H_j stabilizes b_j.

Additionally, G acts transitively on the block idempotents, since otherwise, there is a j such that for all i and $g \in G$, $b_i^g \neq b_j$, and $b_j = \sum b_i^g - \sum_{i \neq j} b_i$, which is impossible since b_j is primitive. Therefore, we can arrange each $T_i = \{t_1, t_2, \dots t_r\}$, where $b_j^{t_i} = b_i$. Then, $b = \mathrm{Tr}_{H_j}^G(b_j)$.

Now suppose b has defect group P. Since $b \in kC_G(P)$, $\mathrm{Br}_P(b) = b$. In particular, given the decomposition $b = \sum b_i$ into block idempotents, we have that the (P, b_i) are maximal b-Brauer pairs. Since G acts transitively on the b_i, it acts transitively on the pairs. □

The following is a major result due to Brauer. The proof is omitted, but can be found in David Craven's *Theory of Fusion Systems*[1], or most standard introductory texts in modular representation theory.

Theorem 4.7 (Brauer's First Main Theorem). *Let P be a p-subgroup of G. The map Br_P induces a bijection between the blocks of kG with defect group P and the blocks of $kN_G(P)$ with defect group P.*

Corollary 4.8. *If b is a block idempotent of kG, then G acts transitively on the set of maximal b-Brauer pairs.*

Proof. We've already shown that G acts transitively on the defect groups of b in Theorem 4.5. Fix such a defect group D. By Brauer's First Main Theorem, we know Br_D induces a bijection between the block idempotents of kG with defect group D and the block idempotents of $kN_G(D)$ with defect group D. Furthermore, D is normal in $N_G(D)$, so we can apply Proposition 4.6 to see that $N_G(D)$ acts transitively on the maximal $\mathrm{Br}_D(b)$-Brauer pairs. However, since Br_D is a projection, the maximal $\mathrm{Br}_D(b)$-Brauer pairs are exactly the maximal b-Brauer pairs, proving the corollary. □

The relationship between representations and the defect group is not completely understood and is the source of many conjectures in modern group representation theory, including Brauer's $k(B)$ Conjecture and Broue's Abelian Defect Conjecture. The final theorem of this section will give a sample of how the defect group can yield information about the representation. We provide a proof of one direction of the theorem. The converse is omitted, but can be found in Radha Kessar's *Introduction to Block Theory* [3].

Theorem 4.9. *Let b be a block idempotent of kG and let P be a defect group of b. Then, P is trivial if and only if kGb is isomorphic to a matrix algebra over k.*

Proof. Suppose that P is trivial. Then $b = \mathrm{Tr}_1^G(a)$ for some $a \in kG$ by Theorem 4.5. Let M and N be kG-modules that are also kGb-modules, and let $\varphi : M \to N$ be a surjective map of kG-modules. Furthermore, let $\psi : N \to M$ be a k-linear map (not necessarily a module homomorphism) that is a splitting of φ. For each $y \in N$, define $\hat{\psi}(y) = \sum_{x \in G} x\psi(ax^{-1}y)$. Then $\hat{\psi}$ is a kG-linear splitting of φ by the following calculation:

$$\begin{aligned}\varphi(\hat{\psi}(y)) &= \varphi\left(\sum_{x\in G} x\psi(ax^{-1}y)\right)\\ &= \sum_{x\in G} x\left(\varphi(\psi(ax^{-1}y))\right)\\ &= \sum_{x\in G} xax^{-1}y\\ &= by = y.\end{aligned}$$

Thus, every kGb-module is projective, and hence, kGb is semisimple (completely decomposable). Since kGb is also indecomposable, kGb must be a simple algebra. By Wedderburn's Theorem on classifying semisimple rings, kGb must therefore be isomorphic to a matrix algebra over k. □

5. Fusion Systems of Groups

We now have enough machinery to be able to define fusion systems of blocks, but first we will explore fusion systems on finite groups.

Definition 5.1. A *fusion system* $\mathcal{F}$ of a finite p-group P is a category whose objects are all subgroups of P, and given two subgroups Q and R, the set of morphisms $\mathrm{Hom}_{\mathcal{F}}(Q,R)$ is a subset of all injective homomorphisms from Q to R satisfying the following properties.

(i) For each $g \in P$ with $Q^g \leq R$, the associated conjugation map $c_g : Q \to R$ is in $\mathrm{Hom}_{\mathcal{F}}(Q,R)$
(ii) For each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$, the associated isomorphism obtained by restricting the codomain to the image of φ lies in $\mathrm{Hom}_{\mathcal{F}}(Q,\varphi(Q))$
(iii) If $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ is an isomorphism, then its inverse $\varphi^{-1} : R \to Q$ lies in $\mathrm{Hom}_{\mathcal{F}}(R,Q)$.

The main examples of fusion systems are those of the Sylow p-subgroups of a group G. The fusion system of G on a Sylow p-subgroup P, denoted $\mathcal{F}_P(G)$, is the category with the subgroups of P as its objects. Given two subgroups R and Q of P, the set $\mathrm{Hom}_{\mathcal{F}_P(G)}(Q,R)$ consists of maps induced by conjugation by elements g of G such that $Q^g \leq R$. Note that these maps do not have to be surjective. The fusion system of a group G thus encapsulates the information of how p-subgroups are embedded in the group G.

Example 5.2. Recall the group $G = V \rtimes Q$, as in Example 4.2. We let $p = 2$, and consider the fusion system of the Sylow 2-subgroup Q. Since the subgroups 1 and $Z(Q)$ are central, $\mathrm{Hom}(1,R)$ and $\mathrm{Hom}(Z(Q),R)$ are trivial, for any subgroup R of Q. The subgroup $\langle x\rangle$ has one non-trivial automorphism, which sends x to x^{-1}, and is induced by conjugation by y, y^3, xy, or $(xy)^3$. This extends to the unique non-trivial map from $\langle x\rangle$ to Q. The situation for the other subgroups of order 4 is analogous. Finally, there are 3 non-trivial automorphisms of Q in the category, induced by conjugation by x, y, and xy respectively. All other sets of morphisms are empty. Since the full automorphism group $\mathrm{Aut}(Q)$ has order 4 and $|V| = 4$, no non-trivial morphisms in the fusion system are induced by elements in V.

In general, however, fusion systems do not always come from groups, however, such as the following example.

Example 5.3. Let D_8 be the dihedral group of order 8. We define the *universal fusion system* on D_8, $\mathcal{U}(D_8)$, such that $\mathrm{Hom}_{\mathcal{U}(D_8)}(Q, R)$ contains all injective homomorphisms from Q to R. In particular, $\mathrm{Aut}_{\mathcal{U}(D_8)}(D_8) = \mathrm{Aut}(D_8)$, which has order 8. However, given a group G with Sylow 2-subgroup P isomorphic to D_8, $\mathrm{Aut}_G(P) = N_G(P)/C_G(P)$ is a 2-group. Since P is a Sylow 2-subgroup of $N_G(P)$ and $|Z(P)| = 2$, $|\mathrm{Aut}_G(P)| \leq |P|/2 = 4$. Thus, $\mathcal{U}(D_8)$ is not the fusion system of a group.

Fusion systems also allow us to restate some classical results concerning the extension of local properties in groups to global properties. The following theorem by Frobenius is such an example [1].

Theorem 5.4 (Frobenius)**.** *The following are equivalent:*

(i) The group G possesses a normal p-complement, that is $G = H \rtimes P$ for some subgroup H with order coprime to p, and some Sylow p-subgroup P of G;
(ii) If P is a Sylow p-subgroup of G, then $\mathcal{F}_P(G) = \mathcal{F}_P(P)$;
(iii) Every normalizer of a p-subgroup possesses a normal p-complement;
(iv) For every p-subgroup Q of G, $\mathrm{Aut}_G(Q)$ is a p-group.

6. Fusion Systems of Blocks

Having built up the theory of Brauer pairs and defect blocks, we are now ready to provide the definition of the fusion system of a block. If we recall the poset on Brauer pairs, the poset of pairs contained in a pair (P, e) is the same as the poset of subgroups of P. Furthermore, the group G induces a conjugation structure on the poset. This motivates the following definition.

Definition 6.1. Let b be a block idempotent of kG, and let (D, e_D) be a maximal b-Brauer pair. The *fusion system* of the block kGb, denoted $\mathcal{F} = \mathcal{F}_{(D,e_D)}(G, b)$, is the category whose objects are the subgroups of D, and whose morphisms are given by the following. Let Q and R be subgroups of D, and e_Q and e_R be the unique block idempotents such that $(Q, e_Q) \leq (D, e_D)$ and $(R, e_R) \leq (D, e_D)$. The set $\mathrm{Hom}_{\mathcal{F}}(Q, R)$ is the set of conjugation maps, denoted $c_x : Q \to R$, for all elements x of G such that $(Q, e_Q)^x \leq (R, e_R)$.

A priori, this definition seems to depend on a choice of maximal b-Brauer pair, but it turns out that this isn't the case.

Proposition 6.2. *Let b be a block idempotent of kG, and let (D, e_D) and (E, e_E) be maximal b-Brauer pairs. Then $\mathcal{F}_{(D,e_D)}(G, b)$ is a fusion system on D, and $\mathcal{F}_{(D,e_D)}(G, b)$ and $\mathcal{F}_{(E,e_E)}(G, b)$ are isomorphic.*

Proof. Let $\mathcal{F} = \mathcal{F}_{(D,e_D)}(G, b)$, and let $g \in D$. If $(Q, e_Q) \leq (D, e_D)$, then $(Q^g, e_Q^g) \leq (D^g, e_D^g) = (D, e_D)$. By the uniqueness of the inclusion of Brauer pairs, we have that $(Q^g, e_Q^g) = (Q^g, e_{Q^g})$. Thus, if $Q^g \leq R$, then $\mathrm{Hom}_{\mathcal{F}}(Q, R)$ contains the map c_g, satisfying the first axiom. For the remaining axioms, let φ be a morphism in $\mathcal{F}$. The associated isomorphism and its inverse are both maps in $\mathcal{F}$.

Recall by Corollary 4.8, that G acts transitively on the set of maximal b-Brauer pairs by conjugation. This induces an isomorphism on the fusion systems of $\mathcal{F}_{(D,e_D)}(G, b)$ and $\mathcal{F}_{(E,e_E)}(G, b)$. □

Just as in the case of groups, we can now rephrase certain theorems in terms of fusion systems of blocks. For example, Corollary 4.8 can be restated as the following.

Definition 6.3. Let b be a block idempotent of kG, let D be a defect group of b, and let $\mathcal{F}$ be a fusion system of D. The normalizer fusion system, $N_{\mathcal{F}}(D)$, is a fusion system with the subgroups of $N_G(D)$ as objects, and the following morphisms: given $P, R \leq N_G(D)$, the morphisms from P to R are the ones given by $\mathcal{F}$ such that there is a map $\bar{\varphi} : PD \to RD$ that extends φ and when restricted to D, $\bar{\varphi}|_D \in \mathrm{Aut}_{\mathcal{F}}(D)$.

Corollary 6.4 (Restatement of Corollary 4.8)**.** *Let b be a block idempotent of kG, and let D be a defect group of b. Then any maximal b-Brauer pair (D, e) is a maximal $\mathrm{Br}_D(b)$-Brauer pair, and letting $\mathcal{F} = \mathcal{F}_{(D,e)}(G, b)$, we have that $\mathcal{F}_{(D,e)}(N_G(D), \mathrm{Br}_D(b)) = N_{\mathcal{F}}(D)$, where $N_{\mathcal{F}}(D)$ is the normalizer fusion system.*

A natural question that arises is whether there exist fusion systems of blocks that do not occur as fusion systems of groups. In the case of blocks, it is believed that all fusion systems of blocks arise as the fusion system of a Sylow p-subgroup, but this has not been proven.

We do know that all fusion systems of blocks satisfy certain necessary conditions, but we will not discuss these in this paper. We will prove two results that show certain fusion systems must arise as fusion systems of finite groups. The first theorem is Brauer's Third Main Theorem. We begin with the following definition.

Definition 6.5. The *principal block* of kG is the block B_0 of kG containing the trivial kG-module.

Proposition 6.6. *Let B_0 denote the principal block of kG, with block idempotent b_0. The defect groups of b_0 are the Sylow p-subgroups of G.*

Proof. Let D be a defect group of b_0, and write $b_0 = \sum\limits_g \beta_g g$. Note that all $g \in G$ must act trivially on the trivial module. Likewise, since 1 acts trivially, and 1 is the sum of the block idempotents of kG, b_0 must act trivially. Thus, $\sum\limits_g \beta_g = 1$. Since D is a defect group, we can apply Lemma 4.3 to write $b_0 = \mathrm{Tr}_D^G(a)$ for some $a = \sum\limits_g \alpha_g g \in (kG)^D$. We note that, writing $a^t = \sum\limits_g \alpha_g g^t$, the sum of the coefficients is invariant under conjugation. Thus, letting T be a transversal to D in G,

$$1 = \sum_g \beta_g = \sum_{t \in T} \left(\sum_g \alpha_g \right) = |G : D| \sum_g \alpha_g.$$

In particular, p does not divide $|G : D|$, so D is a Sylow p-subgroup. □

Corollary 6.7. *The principal block idempotent b_0 is the only block idempotent that does not lie in the augmentation ideal of kG (i.e. whose coefficients, with respect to the basis of group elements, sum to zero). In particular, $\mathrm{Br}_P(b_0)$ is non-zero for any p-subgroup P.*

Theorem 6.8 (Brauer's Third Main Theorem)**.** *Let P be a Sylow p-subgroup of G, and let b and e denote the principal block idempotents of kG and $kC_G(P)$ respectively. The fusion systems $\mathcal{F} = \mathcal{F}_{(P,e)}(G, b)$ and $\mathcal{F}_P(G)$ are isomorphic.*

Proof. Let e_P be the principal block idempotent of $kC_G(P)$. Recall that since each subgroup Q of P corresponds to a unique Brauer pair, we can think of the objects of $\mathcal{F}$ as the Brauer pairs. We claim that in fact, the Brauer pairs are exactly the ones of the form (Q, e_Q) where e_Q is the principal block idempotent of $kC_G(Q)$. By Corollary 6.7, b lies outside the augmentation ideal of kG. Thus, $\mathrm{Br}_Q(b)$ lies outside the augmentation ideal of $kC_G(P)$. Since $\mathrm{Br}_Q(b)$ is a sum of block idempotents of $kC_G(P)$, and all non-principal block idempotents of $kC_G(P)$ lie inside the augmentation ideal, the principal block idempotent of $kC_G(P)$, e_Q, must appear in $\mathrm{Br}_Q(b)$ when written as a sum of block idempotents of $kC_G(P)$, and $\mathrm{Br}_Q(b)e_Q = e_Q$. Thus, (Q, e_Q) is a b-Brauer pair. Furthermore, if $Q \leq R$ are subgroups of P, then $(Q, e_Q) \leq (R, e_R)$, because the inclusion of Brauer pairs is unique.

Let Q and R be subgroups of P and let $g \in G$, such that $Q^g = R$. We want to show that $e_Q^g = e_R$. Conjugation by g induces an isomorphism from $kC_G(Q)$ to $kC_G(R)$. Furthermore, the image of e_Q is a block idempotent of $kC_G(R)$, which does not lie within the augmentation ideal, so must be e_R. Thus, $c_g : Q \to R$ is an element of $\mathrm{Hom}_{\mathcal{F}}(Q, R)$. □

Brauer's Third Main Theorem shows that all fusion systems of principal blocks must arise from groups. The following theorem, due to Lluis Puig, gives us another case in which this holds. In particular, the fusion systems of blocks of the symmetric group arise as the fusion system of a Sylow p-subgroup of some smaller symmetric group.

Theorem 6.9. *Let G be the symmetric group S_n, and let b be a block idempotent of kG, and P a defect group of b. Let $\mathcal{F}$ be the fusion system of the block b. There exists a non-negative integer w such that $pw \leq n$,P is a Sylow p-subgroup of S_{pw}, and $\mathcal{F} = \mathcal{F}_Q(S_{pw})$.*

The proof of this theorem will require the following proposition. A proof of this statement is provided in Peter Webb's *Finite Group Representations for the Pure Mathematician*, but will be omitted from this paper for brevity[4].

Proposition 6.10. *Let k be a field of characteristic p, and let G be a finite p-group. The only simple kG-module is the trivial module.*

Lemma 6.11. *Let X be a finite set, and let $S(X)$ be the symmetric group on X. Let Q be a p-subgroup of $S(X)$ which acts fixed point freely on X. Then the principal block is the unique block of $kC_{S(X)}(Q)$.*

Proof. Let $H = C_{S(X)}(Q)$ and let e be a block of kH. If Y is an orbit of Q on X, then we can consider $S(Y)$, and include Q into the product of the $S(Y)$ over all orbits, $\prod_Y S(Y)$.

Let $\tilde{Q} = \prod_Y Q_Y$, where for a given orbit U, Q_U is the image of Q under the projection of $\prod_Y S(Y)$ onto $S(U)$.

Consider $Z(\tilde{Q})$, the center of $\tilde{Q}$, which is a normal subgroup of H. By Proposition 4.6, $e \in kC_H(Z(\tilde{Q}))$. We note that $kC_H(Z(\tilde{Q})) = kC_{S(X)}(QZ(\tilde{Q}))$. As a result, $Z(\tilde{Q}) \leq Q$.

We have that
$$C_{S(X)}(Q) = \prod_Y C_{S(Y)}(Q_Y),$$
where the Y range over the Q-orbits of X. Since group algebra of the direct product of groups is the tensor product of their group algebras, $kC_{S(X)}(Q)$ is isomorphic to the tensor product of the $kC_{S(Y)}(Q_Y)$, and under this isomorphism, the block idempotent e is the tensor product of the block idempotents of the $C_{S(Y)}(Q_Y)$. Thus, we can reduce to the case where Q is transitive on X.

Thus, $|X| = p^m$, for some m, since otherwise, there is no Sylow p-subgroup of $S(X)$ that acts transitively on X. Let $\sigma \in H$, with $\sigma \neq 1$, and let $q \in Q$. Then, $\sigma^q = \sigma$. We notice that $x \cdot \sigma^i = x \cdot (\sigma^i)^q = x \cdot (q^{-1}\sigma q)$ for $x \in X$ if and only if σ^i fixes $x \cdot q^{-1}$. If the orbit of x has length i, then the orbit of xq^{-1} has length at most i. By transitivity of Q, $x \cdot q^{-1}$ can be any element of X, and all orbits of σ have the same length. In particular, each orbit of σ has length a power of p, and H is a p-group, and has exactly one block idempotent. □

Proof of Theorem 6.9. Consider $G = S(Y)$, where Y has cardinality n, with P a Sylow p-subgroup. Let b be a block idempotent of kG, $Q \leq P$, and let Y^Q denote the set of elements of Y fixed by Q. Then we have that
$$C_G(Q) = S(Y^Q) \times C_{S(Y-Y^Q)}(Q).$$

Since the group algebra of the direct product of two groups is the tensor product of their group algebras,
$$kC_G(Q) \cong kS(Y^Q) \otimes kC_{S(Y-Y^Q)}(Q).$$

Thus, using our previous notation to let e_Q be the principal block idempotent, e_Q is of the form $e'_Q \otimes e''_Q$, where e'_Q is a block of $kS(Y^Q)$ and e''_Q is a block of $kC_{S(Y-Y^Q)}(Q)$. By the lemma, e''_Q is the principal block of $kC_{S(Y-Y^Q)}(Q)$, hence 1, so $e_Q = e'_Q \otimes 1$. We notice that we have the same direct product decomposition for the normalizer as we did for the centralizer.
$$N_G(Q) = S(Y^Q) \times N_{S(Y-Y^Q)}(Q).$$

Since e'_Q is $S(Y^Q)$-stable, we have that e_Q is $N_G(Q)$-stable, and hence, $N_G(Q, e_Q) = N_G(Q)$. Thus, by the fact that $\mathrm{Aut}_G(Q) \cong N_G(Q)/C_G(Q)$, for any group G and subgroup Q,
$$\mathrm{Aut}_{\mathcal{F}}(Q) = \mathrm{Aut}_G(Q) = \mathrm{Aut}_{S(Y-Y^Q)}(Q).$$
Next, we show that P is a Sylow p-subgroup of $S(Y - Y^P)$. If this is the case, then given any $Q \leq P$, we have that $Y - Y^Q$ is contained in $Y - Y^P$, and $S(Y - Y^Q) \leq S(Y - Y^P)$, i.e. $\mathrm{Aut}_{\mathcal{F}}(Q) = \mathrm{Aut}_{S(Y-Y^P)}(Q)$, for all $Q \leq P$. This proves that $\mathcal{F} = \mathcal{F}_{S(Y-Y^P)}$, and since P acts fixed point freely on $Y - Y^P$, the action is non-trivial, and $S(Y - Y^P) \cong S_{pw}$, for some $w \geq 0$, and $\mathcal{F} = \mathcal{F}_{S_{pw}}(P)$, proving the theorem.

To show that P is a Sylow p-subgroup of $S(Y - Y^P)$, we let R be a p-subgroup of $S(Y - Y^P)$ containing P, and show it must equal P. Since P is fixed point free on $Y - Y^P$, so is R. Thus, the group algebras $kC_{S(Y-Y^P)}(P)$ and $kC_{S(Y-Y^P)}(R)$ both have 1 as their unique block idempotent. Thus, we have an inclusion of Brauer pairs, $(P, 1) \leq (R, 1)$. Now consider $kC_G(R)$. Since $C_G(R) = S(Y^P) \times C_{S(Y-Y^P)}(R)$, we can take $f = e \otimes 1$ as a block idempotent of $kC_G(R)$. This gives us the inclusion of Brauer pairs $(P, e_P) \leq (R, f)$, with $e_P = e \otimes 1$. Since P is a defect group of b, this implies that $P = R$. Thus, P is a Sylow p-subgroup of $S(Y - Y^P)$. $\square$

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References

[1] David A. Craven. *The Theory of Fusion Systems.* Cambridge Studies in Advanced Mathematics. 2011.

[2] Radha Kessar. Fusion and Representation Theory. *Fusion Systems in Algebra and Topology.* The London Mathematical Society. 2011.

[3] Radha Kessar. Introduction to Block Theory. *Group Representation Theory.* Meinolf Geck, Donna Testerman, Jacques Thevenaz, Eds., EPFL Press, pp.47-77. 2011.

[4] Peter Webb. Finite Group Representations for the Pure Mathematician. http://www.math.umn.edu/ webb/RepBook/RepnControl.pdf Accessed July 2012.