CLASSIFYING THE REPRESENTATIONS OF $\mathfrak{sl}_2(\mathbb{C})$.

MIRANDA SEITZ-MCLEESE

Abstract. This paper will present an introduction to the representation theory of semisimple Lie algebras. We will prove several fundamental theorems in the subject, including Engel's and Lie's theorems. Throughout the paper $\mathfrak{sl}_2(\mathbb{C})$ will serve as a key example, and the paper finishes with a classification of the finite dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$.

Contents

1. Introduction

The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is an important canonical example of a simple Lie algebra. The irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ form the foundation of the representation theory of finite dimensional Lie algebras. In light of this fact, this paper aims to provide an introduction to Lie algebras and their representations, using the representation theory of $\mathfrak{sl}_2(\mathbb{C})$ as a guiding example.

In this paper I will assume that the reader has a strong working knowledge of linear algebra; familiarity with eigenvectors, eigenspaces, dual bases, and quotient spaces is critical for understanding the content of this paper. Additionally, readers should be familiar with the basic ideas in the study of matrix Lie groups.

Finally, readers should have basic familiarity with module theory as it relates to representations, particularly in the case of reducibility.

Our study will begin in Section 2 with basic definitions and some discussion of the relationship between Lie groups and Lie algebras. A detailed discussion of this relationship, however, is outside the focus of this paper. The author instead refers interested readers to chapters eight and nine of [2]. In this section we will also introduce the epsilon algebra, a useful method for calculating Lie algebras. Later in this section we will also define solvable and nilpotent algebras and prove some important theorems about their representations. In Section 3 we will discuss representations of semisimple Lie algebras and prove some of their important properties. Finally, in Section 4 we will classify the irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$.

2. Lie Groups and Lie Algebras

2.1. **Basic definitions and the Epsilon Algebra.** We begin with the definition of a matrix Lie Group:

Definition 2.1.1. A *matrix Lie group* over a field $\mathbb{F}$ is a subgroup G of $GL_n(\mathbb{F})$, such that group multiplication and inversion are smooth maps.

In this paper, we will be focusing primarily on the Lie algebra associated with the Lie group $SL_2(\mathbb{C})$, the group of matrices with determinant 1. Generally, the Lie algebra that corresponds to a given Lie group is the tangent space of the manifold at the identity element of the group, together with a bracket operation that is the derivative of conjugation in the group. If the original Lie group is a matrix Lie group, the bracket corresponds to the commutator $AB-BA$. Lie algebras contain much information about their associated Lie groups. In this paper, however, we will focus on the algebraic structure of the Lie algebras themselves. For more information on this topic, see chapter nine of [2].

The epsilon algebra is a useful construction when calculating tangent spaces. Define ϵ to be a formal element such that $\epsilon^2 = 0$. The idea here is that ϵ is a first order infinitesimal. Therefore the tangent space at the identity is simply all matrices A such that $I + A\epsilon$ is still in G. For a more detailed discussion of the epsilon algebra, as well as proofs of its relationship to the tangent space of the Lie group at the identity, see chapter twelve of [1]. As an example, let us work through the calculation for $SL_2(\mathbb{C})$.

Example 2.1.2. We compute

$$I + A\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \epsilon = \begin{pmatrix} 1 + a\epsilon & b\epsilon \\ c\epsilon & 1 + d\epsilon \end{pmatrix}.$$

So, we can calculate the determinant:

$$\begin{aligned}\det(I + A\epsilon) &= (1 + a\epsilon)(1 + d\epsilon) - bc\epsilon^2 \\ &= 1 + (a + d)\epsilon.\end{aligned}$$

It is easy to see that if we want $I + A\epsilon$ to be in $SL_2(\mathbb{C})$ we need to require the trace of A to be zero.

Throughout this paper I will use a standard basis for $\mathfrak{sl}_2(\mathbb{C})$, the Lie algebra associated with $SL_2(\mathbb{C})$:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It is easy to check, using matrix multiplication, that the following relations hold:

$$[H, E] = 2E, \tag{1}$$

$$[H, F] = -2F, \tag{2}$$

$$[E, F] = H. \tag{3}$$

These relations will be important later in this paper when we try to classify the irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$.

There is also a more general definition of a Lie algebra:

Definition 2.1.3. A *Lie algebra* $\mathfrak{g}$ is a vector space V over a field F together with a bilinear operation $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that obeys the following properties for all x, y, z in $\mathfrak{g}$:

$$[x, y] = -[y, x] \text{ (anticommutativity)}$$
$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \text{ (Jacobi Identity).}$$

Therefore, we can define $\mathfrak{sl}_2(\mathbb{C})$ to be the complex matrices with zero trace, and the bracket operator to be the commutator $[A, B] = AB - BA$. Checking these properties for this definition of $\mathfrak{sl}(2, \mathbb{C})$ is a simple computation.

Definition 2.1.4.
(1) A subspace $\mathfrak{h}$ of $\mathfrak{g}$ is a *sub-algebra* of $\mathfrak{g}$ if $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$.
(2) A sub-algebra $\mathfrak{h}$ of $\mathfrak{g}$ is an *ideal* if $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$
(3) A Lie Algebra is said to be *abelian* if $[x, y] = 0$ for any x and y.
(4) We say a non-abelian Lie Algebra is *simple* if it has no non-trivial proper ideals.
(5) A Lie Algebra is *semisimple* if it has no non-trivial abelian ideals.

It is clear from these definitions that simplicity implies semisimplicity.

Lemma 2.1.5. *The Lie algebra* $\mathfrak{sl}_2(\mathbb{C})$ *is simple.*

Proof. Let $\mathfrak{h}$ be a non-zero ideal of $\mathfrak{sl}(2,\mathbb{C})$. Let $Z = \alpha E + \beta F + \gamma H \neq 0$ be some element of $\mathfrak{h}$ written in our usual basis. Then let

$$X = [H,[H,Z]] = [H, 2\alpha E - 2\beta F] = 4\alpha E + 4\beta F.$$

This element X is in $\mathfrak{h}$ because $\mathfrak{h}$ is an ideal. Using the fact that ideals are closed under addition and scalar multiplication, we can say that $Z - \frac{1}{4}X = \gamma H \in \mathfrak{h}$. If $\gamma \neq 0$ then $H \in \mathfrak{h}$. Because $\mathfrak{h}$ is an ideal, it follows that $[H,E] = 2E \in \mathfrak{h}$ implies that $E \in \mathfrak{h}$. Similarly we can deduce that $F \in \mathfrak{h}$. We then conclude that $\mathfrak{h} = \mathfrak{sl}(2,\mathbb{C})$. If $\gamma = 0$ we note that $[E,Z] = \beta H \in \mathfrak{h}$, and $[F,Z] = \alpha H \in \mathfrak{h}$. We can then use the fact that $Z \neq 0$ to conclude that H is in $\mathfrak{h}$. We can now proceed as in the case where $\gamma \neq 0$, and conclude that $\mathfrak{h} = \mathfrak{sl}(2,\mathbb{C})$. □

For readers familiar with the representation theory of Lie groups, just as a Lie algebra is the tangent space of the Lie group at the identity, a representation of a Lie algebra is the derivative of a Lie group representation at the identity.

Definition 2.1.6. A *representation* of a Lie algebra $\mathfrak{g}$ is a vector space V together with a function $\rho : \mathfrak{g} \to \mathrm{End}(V)$ such that $\rho([x,y]) = [\rho(x),\rho(y)] = \rho(x)\rho(y) - \rho(y)\rho(x)$. A representation is said to be *irreducible* if there is no non-trivial, proper subspace W such that $\rho(x)W \subset W$ for all $x \in \mathfrak{g}$.

We will often drop the $\rho(X)v$ notation, and work with V as a $\mathfrak{g}$-module where $\mathfrak{g}$ acts on V by $X\dot{V} = \rho(X)v$. Note that the condition for irreducibility as a representation is identical to the condition that V is irreducible as a $\mathfrak{g}$-module.

One of the most important representations of a given Lie algebra is the adjoint representation. If $\mathfrak{g}$ is a Lie algebra with operator $[,]$ then for x in $\mathfrak{g}$ the function $(y) \mapsto [x,y]$ is a linear function on $\mathfrak{g}$. We can then define a representation where $V = \mathfrak{g}$ and $\rho(x) = \mathrm{ad}(x)$. Checking that this map preserves the bracket is a straightforward computation.

2.2. **Nilpotent and Solvable.**

Definition 2.2.1. We define $\mathcal{D}_1 = [\mathfrak{g},\mathfrak{g}]$. We then inductively define $\mathcal{D}_n = [\mathfrak{g},\mathcal{D}_{n-1}]$ to be the *lower central series* of $\mathfrak{g}$. We define $\mathcal{D}^1 = [\mathfrak{g},\mathfrak{g}]$, and $\mathcal{D}^n = [\mathcal{D}^{n-1},\mathcal{D}^{n-1}]$ to be the *upper central series.* If the *lower central series* eventually vanishes, we say that $\mathfrak{g}$ is *nilpotent.* If the *upper central series* eventually vanishes, we say that $\mathfrak{g}$ is *solvable.*

Lemma 2.2.2. *A Lie algebra* $\mathfrak{g}$ *is semisimple if and only if* $\mathfrak{g}$ *has no non-trivial solvable ideals.*

Proof. Suppose $\mathfrak{g}$ is semisimple and $\mathfrak{h}$ is a non-trivial solvable ideal of $\mathfrak{g}$. By definition the upper series of $\mathfrak{g}$ eventually vanishes. In particular, $\mathcal{D}^n = 0$ for some n. Therefore $[\mathcal{D}^{n-1}, \mathcal{D}^{n-1}] = 0$, and $\mathcal{D}^{n-1}$ is a non-trivial abelian ideal. The converse is trivial, because abelian sub-algebras are obviously solvable. $\square$

There are two major theorems about nilpotent and solvable Lie algebras: Lie's Theorem and Engel's Theorem. The following proofs of these theorems are similar in substance to the proofs presented in chapter nine of [2].

Theorem 2.2.3 (Engel's Theorem)**.** *Let $\mathfrak{g} \subset \mathrm{End}(V)$ be a Lie algebra such that for any X in $\mathfrak{g}$, X is a nilpotent endomorphism of V. Then there exists a v in V such that $Xv = 0$ for all X in $\mathfrak{g}$.*

Proof. The first major claim in this theorem is that if $X \in \mathfrak{g}$ is nilpotent, then $\mathrm{ad}(X)$ is also nilpotent. Associate to X two functions from $\mathrm{End}(V)$ to $\mathrm{End}(V)$, $\lambda_X(Y) = XY$ and $\rho_X(Y) = YX$. Where XY is X composed with Y as endomorphisms. Because X is nilpotent, it is obvious that λ_X and ρ_X are both also nilpotent. Let m be such that $\lambda_X^m = \rho_X^m = 0$. I claim that these two functions commute:

$$\lambda_X(\rho_X(Y)) = XYX = \rho_X(\lambda_X(Y)).$$

Therefore, $\mathrm{ad}(X) = \lambda_X - \rho_X$ and we can apply the binomial formula to calculate

$$\mathrm{ad}(X)^{2m+1} = \sum_{i=0}^{2m+1} (-1)^i \rho_X^i \lambda_X^{2m+1-i}.$$

I claim that every term in this sum will always be zero. Any term where $i > m$ contains $\rho_X^m = 0$, which means the entire term is zero. We can therefore now assume that $i \leq m$. But if $i \leq m$ then $2m + 1 - i \leq m$ which implies that $\lambda_X^{2m+1-i} = 0$, so the term must actually be equal to zero. Therefore $\mathrm{ad}(X)^{2m+1} = 0$, implying $\mathrm{ad}(X)$ is nilpotent.

We will now proceed by induction on the dimension of $\mathfrak{g}$. If $\dim(\mathfrak{g}) = 1$, the proof is trivial because anticommutitivity of the bracket implies that $\mathfrak{g}$ is abelian. If $\dim(\mathfrak{g}) > 1$, we will show that there exists an ideal $\mathfrak{h} \subset \mathfrak{g}$ with co-dimension equal to one. I claim that any maximal proper sub-algebra will do the trick. To prove this claim look at the adjoint representation of $\mathfrak{g}$. The fact that $\mathfrak{h}$ is a sub-algebra tells us that $\mathrm{ad}(\mathfrak{h})$ acting on $\mathfrak{g}$ preserves the subspace $\mathfrak{h}$. We can therefore think of $\mathrm{ad}(\mathfrak{h})$ as acting on the quotient space $\mathfrak{g}/\mathfrak{h}$. By the first claim, we can then conclude that any for $X \in \mathfrak{h}$, $\mathrm{ad}(X)$ acts nilpotently on that quotient space. We can now apply our inductive hypothesis to conclude that for some $\bar{Y} \neq 0$ in that quotient space $\mathrm{ad}(X)(\bar{Y}) = 0$ in $\mathfrak{g}/\mathfrak{h}$ for all X in $\mathfrak{h}$. Therefore we can take a pre-image of $\bar{Y}$ in $\mathfrak{g}$ and conclude that there is a non-zero Y in $\mathfrak{g}$ such that $\mathrm{ad}(X)(Y)$ is in $\mathfrak{h}$ for all X in $\mathfrak{g}$. Then there is a sub-algebra spanned by $\mathfrak{h}$ and Y. It is clear that $\mathfrak{h}$ is an ideal of this algebra, and it has dimension equal to $\dim(\mathrm{span}(\mathfrak{h}, Y)) - 1$. However, we declared $\mathfrak{h}$ to be a maximal proper sub-algebra. Therefore $\mathrm{span}(\mathfrak{h}, Y) = \mathfrak{g}$, which proves the claim.

We now return to $\mathfrak{g}$. We apply our inductive hypothesis to the ideal from the previous paragraph. So we have a vector v such that $Xv = 0$ for any $X \in \mathfrak{h}$. Now let W be the subspace of all vectors with this property. Also let $Y \in \mathfrak{g}$ be any element not in $\mathfrak{h}$. Finally let w be a vector in W and let X be any vector in $\mathfrak{h}$, and then calculate:

$$X(Y(w)) = Y(X(w)) + [X, Y](w).$$

Now it is clear that $Y(X(w)) = 0$ by the definition of W. Also the fact that $\mathfrak{h}$ is an ideal tells us that $[X, Y] \in \mathfrak{h}$ and that the second term is also zero. Therefore $X(Y(w)) = 0$ for any $X \in \mathfrak{h}$. We can then conclude that $Y(w) \in W$. This tells us that Y preserves W. However, we know that Y acts nilpotently on V. Therefore there must be some v in W such that $Y(v) = 0$. Because Y together with $\mathfrak{h}$ spans $\mathfrak{g}$ we can conclude that $T(v) = 0$ for any T in $\mathfrak{g}$. The proof follows by induction. □

Corollary 2.2.4. *If $\mathfrak{g} \subset \mathrm{End}(V)$ is a Lie algebra such that X is a nilpotent endomorphism of V for any X in $\mathfrak{g}$, then there exists a basis of V such that each element is strictly upper triangular.*

Proof. We will induct on $\dim(V)$. If $\dim(V) = 1$, then the proof follows immediately from Engel's Theorem. Now suppose the corollary is true for any V such that $\dim(V) \leq n - 1$. Let $\dim(V) = n$. Now we know there exists some v such that $Xv = 0$ for any X. Let $W = \mathrm{span}(v)$, and consider the quotient space V/W. This space has dimension $n - 1$. Apply the inductive hypothesis to get a basis $\bar{v_1}, \bar{v_2}, \bar{v_3}, \ldots, \bar{v_{n-1}}$ for V/W such that all X are strictly upper triangular. Take the pre-image of those vectors, together with v, to get a basis for V, such that all X are strictly upper triangular. □

Lie's Theorem is a similar statement for solvable Lie algebras: that they can always be put in upper triangular form.

Theorem 2.2.5 (Lie's Theorem)**.** *Let $\mathfrak{g} \subset \mathrm{End}(V)$ be a solvable Lie algebra. Then there is a vector $v \in V$ such that v is an eigenvector for all $X \in \mathfrak{g}$.*

Proof. The proof of this theorem will follow a process similar to that used in the previous proof of Engel's Theorem. The first step is to find $\mathfrak{h} \subset \mathfrak{g}$ an ideal with a co-dimension of 1. To find one, consider $\mathcal{D}_1 = [\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$. Because $\mathfrak{g}$ is solvable, we can consider the algebra $\mathfrak{g}/\mathcal{D}_1$. This is a non-zero abelian algebra. Let $\mathfrak{a}$ be any sub-algebra of $\mathfrak{g}/\mathcal{D}_1$ with co-dimension one . Because $\mathfrak{g}/\mathcal{D}_1$ is an abelian algebra, $\mathfrak{a}$ will automatically be an ideal. Therefore its pre-image in $\mathfrak{g}$ will also be an ideal with co-dimension one.

We now apply our inductive hypothesis to this $\mathfrak{h}$. There is some v such that for all $X \in \mathfrak{h}$ we know v is an eigenvector of X. Let $\lambda(X)$ be the X eigenvalue of v. As before we will define

$$W = \{v : Xv = \lambda(X)v \text{ for all } X \in \mathfrak{h}\},$$

and show that for some Y not in $\mathfrak{h}$, $Y(W) \subset W$. Let $w \in W$, $X \in \mathfrak{h}$, and $Y \in \mathfrak{g}$, Y not in $\mathfrak{h}$. As before we will calculate:

$$X(Y(w)) = Y(X(w)) + [X,Y](w) = \lambda(X)Y(w) + \lambda([X,Y])w. \tag{4}$$

If the second term were 0, then it would be obvious that $Y(w) \in W$. However, unlike in the previous theorem, it is not immediately obvious that the second term is zero. We will show this by proving that $\lambda([X,Y]) = 0$ for any X, Y in $\mathfrak{h}$. Define a subspace $U \subset V$:

$$U = \mathrm{span}(w, Y(w), Y^2(w), \dots, Y^k(w), \dots).$$

It is clear that $Y(U) \subset U$. I claim that $X(U) \subset U$ as well. To prove this I will use induction. It is clear from (4) that X takes $Y(w)$ to a linear combination of $Y(w)$ and w, both of which are elements of U. Now suppose $X(Y^{k-1}(w)) \in U$ for any X in $\mathfrak{h}$. Then we can calculate:

$$X(Y^k(w)) = Y(X(Y^{k-1}(w))) + [X,Y](Y^{k-1}(w)).$$

Because we assumed that $X(Y^{k-1}(w)) \in U$, we know the first term is in U. I will now show that the second term must also be in U. The fact that $\mathfrak{h}$ is an ideal ensures that $[X,Y] \in \mathfrak{h}$, and by our inductive hypothesis the second term must also be in U. By induction we see that $X(U) \subset U$.

In fact, the action of X is upper triangular on U with respect to the basis $w, Y(w), Y^2(w), \dots$. To show this I will prove that $X(Y^k(w))$ is a linear combination of Y^is where $i \leq k$. We will also prove this by induction on k. It is obvious from the definition that $X(w) = \lambda(X)w$. Now suppose $X(Y^{k-1}(w)) = \sum_{i=0}^{k-1} a_i(X)Y^i(w)$ for any $X \in \mathfrak{h}$ where $a_i(X)$ are scalars that depend on X. Let $y = Y(w)$. Then

$$\begin{aligned}
X(Y^k(w)) &= Y(X(Y^{k-1}(w))) + [X,Y](Y^{k-1}(w)) \\
&= Y\left(\sum_{i=0}^{k-1} a_i(X)Y^i(w)\right) + [X,Y](Y^{k-1}(w)) \\
&= \sum_{i=0}^{k-1} a_i(X)Y^{i+1}(w) + \sum_{i=0}^{k-1} a_i([X,Y])Y^i(w) \\
&= a_0([X,Y])(w) + \sum_{i=1}^{k}(a_{i-1}(X) + a_i([X,Y]))Y^i(w).
\end{aligned}$$

We can therefore see that $X(Y^k(w))$ for any $X \in \mathfrak{h}$ is a linear combination of $Y^i(w)$ for $i \leq k$. This completes the proof of the claim. In light of this fact, we may conclude that $\mathrm{Tr}(X|_U) = \lambda(X)\dim(U)$. Furthermore, $[X,Y] \in \mathfrak{h}$ acting on U is a commutator and therefore has trace equal to zero. Because $\dim(U) \neq 0$ the only possibility is that $\lambda([X,Y]) = 0$. This completes the proof. $\square$

Corollary 2.2.6. *If $\mathfrak{g} \subset \mathrm{End}(V)$ is a solvable Lie algebra, there exists a basis of V such that each X is upper triangular with respect to that basis.*

Proof. The proof of this almost identical to the proof of Corollary 2.2.4. A more detailed discussion may be found in chapter 9 of [2]. □

3. Representations of Semisimple Lie Algebras

Unfortunately, in the cases of Lie groups and Lie algebras, many of the facts that were useful in the representation theory of finite groups are no longer generally true. Complete reducibility does not necessarily hold, and we cannot even be sure that a given representation preserves diagonalizability. Fortunately, these statements are true in the narrower case of semisimple algebras. Because $\mathfrak{sl}_2(\mathbb{C})$ is simple, these theorems are vital in understanding its finite dimensional representations. The proofs in this section are modeled after those in appendix C of [2].

Lemma 3.0.7. *The adjoint representation preserves the Jordan canonical form. In other words if $X \in \mathfrak{g} \subset \mathrm{End}\, V$ then $\mathrm{ad}(X_s) = \mathrm{ad}(X)_s$, and $\mathrm{ad}(X_n) = \mathrm{ad}(X)_n$, where X_s is the semisimple part of X and X_n is the nilpotent part of X.*

Proof. We have already shown that if X is nilpotent, $\mathrm{ad}(X)$ is nilpotent. It is clear that we will be done if we show that the same is true for diagonalizable matrices. This simple computation is omitted, but is a good test of one's understanding of the adjoint representation. □

3.1. Complete Reducibility and the Killing Form.

Theorem 3.1.1. *Let V, ρ over F be a representation of a semisimple Lie algebra $\mathfrak{g}$, and let $W \subset V$ be a subrepresentation. Then there exists a subrepresentation W' that is complementary to W.*

To prove this, we will need to define the Killing form B_V for a vector space V.

Definition 3.1.2. Let X, Y be elements of $\mathrm{End}(V)$. Then we define the Killing form B_V to be

$$B_V(X, Y) = \mathrm{Tr}(XY).$$

The Killing form $B(X, Y)$ (if the subscript is omitted) is understood to refer to the vector space $\mathfrak{g}$ together with the adjoint representation, such that

$$B(X, Y) = B_{\mathfrak{g}}(\mathrm{ad}(X), \mathrm{ad}(Y)).$$

The relation

$$\begin{aligned} B_V(X, [Y, Z]) &= \mathrm{Tr}(XYZ - XZY) \\ &= \mathrm{Tr}(XYZ - YXZ) \\ &= B_V([X, Y], Z) \end{aligned} \tag{5}$$

holds for any X, Y, Z.

Lemma 3.1.3 (Cartan's Criterion)**.** *Let $\mathfrak{g} \subset \mathrm{End}(V)$ be a Lie algebra. If $B_V(X,Y) = 0$ for all X and Y then $\mathfrak{g}$ is solvable.*

Proof. For our proof we will show that every element of $\mathcal{D}_\mathfrak{g}$ is nilpotent. Then we can use Engel's Theorem to conclude that $\mathcal{D}_\mathfrak{g}$ is nilpotent, and therefore $\mathfrak{g}$ is solvable. Let $X \in \mathcal{D}_\mathfrak{g}$ be arbitrary. Let $\lambda_1 \dots \lambda_r$ be the eigenvalues of X with multiplicity. We want to show that all the eigenvalues are zero. It is clear that $B_V(X,X) = Tr(X \circ X) = \sum \lambda_i \lambda_i = 0$. If we were to show that $|\lambda_1|^2 + |\lambda_2|^2 + \cdots + |\lambda_r|^2 = 0$ the proof would be done. Choose a basis such that X is in Jordan canonical form, and let $D = X_s$ be the diagonal part of the matrix.

$$\begin{aligned} |\lambda_1|^2 + |\lambda_2|^2 + \cdots + |\lambda_r|^2 &= \lambda_1 \bar{\lambda_1} + \lambda_2 \overline{\lambda_2} + \cdots + \lambda_r \overline{\lambda_r} \\ &= \mathrm{Tr}(\overline{D} \circ X). \end{aligned}$$

Therefore it suffices to show that $\mathrm{Tr}(\overline{D} \circ X) = 0$. We can write $X \in \mathcal{D}_\mathfrak{g}$ as a sum of commutators $[Y,Z]$ for $Y, Z \in \mathfrak{g}$. Therefore $\mathrm{Tr}(\overline{D} \circ X)$ can be written as a sum of $\mathrm{Tr}(\overline{D} \circ [Y,Z]) = \mathrm{Tr}([\overline{D}, Y] \circ Z)$. Now all that remains is to show that $[\overline{D}, Y] \in \mathfrak{g}$, because then, by our hypothesis, $\mathrm{Tr}([\overline{D}, Y] \circ Z) = B_V([\overline{D}, Y], Z) = 0$.

To show that $[\overline{D}, Y] \in \mathfrak{g}$ we will show that $\mathrm{ad}(\overline{D})$ can be written as a polynomial in $\mathrm{ad}(X)$. It is trivial that $\mathrm{ad}(X)(Y)$ is in $\mathfrak{g}$ if X and Y are in $\mathfrak{g}$. By Lemma 3.0.7 we know that $\mathrm{ad}(D) = \mathrm{ad}(X_s) = \mathrm{ad}(X)_s$. We know from algebra that $\mathrm{ad}(X)_s$ is a polynomial in $\mathrm{ad}(X)$. A simple calculation will show that $\mathrm{ad}(\bar{(}D)) = \bar{(}\,\mathrm{ad}(D))$.

I claim that for any diagonal matrix A over $\mathbb{C}$ the following is true: $\overline{A}$ is polynomial in A. Applying this statement to $\mathrm{ad}(D)$ will complete the proof. To prove the statement let $s_1, s_2, \dots, s_k$ be the distinct eigenvalues of A such that $A = \mathrm{diag}(s_1, \dots, s_1, \dots, s_k, \dots, s_k)$. For $0 < i \leq k$ and $0 < j \leq k$ define

$$\sigma_i \in \mathbb{C} = \prod_{j \neq i} (s_i - s_j).$$

Note that for all i, $\sigma_i \neq 0$. Finally we define

$$B_i = \sigma_i^{-1} \prod_{j \neq i} (A - s_j).$$

A simple calculation shows that B_i is the diagonal matrix with ones where A takes the value s_i and zeros everywhere else. It is clear then that each B_i is a polynomial in A and

$$A = \sum_{i=1}^{k} s_i B_i.$$

Taking the complex conjugate of both sides we get

$$\overline{A} = \sum_{i=1}^{k} \overline{s_i} \cdot \overline{B_i}.$$

However, B_i is a matrix with real entries, so $\overline{B_i} = B_i$. Therefore,

$$\overline{A} = \sum_{i=1}^{k} \overline{s_i} \cdot B_i = \sum_{i=1}^{k} \overline{s_i} \left(\sigma_i^{-1} \prod_{j \neq i} (A - s_j) \right)$$

which is clearly a polynomial in A, completing the proof of the claim. □

Lemma 3.1.4. *A Lie algebra $\mathfrak{g}$ is semisimple if and only if its Killing form B is non-degenerate.*

Proof. The relation described in (5) tells us that

$$\mathfrak{s} = \{X \in \mathfrak{g} : B(X, Y) = 0 \text{ for all } Y \in \mathfrak{g}\}$$

is an ideal. Now suppose that $\mathfrak{g}$ is semisimple. That implies that $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is injective, because $\mathrm{Ker}(\mathrm{ad})$ would be an abelian ideal. Then $\mathrm{ad}(\mathfrak{g})$ is isomorphic to $\mathfrak{g}$. We apply Cartan's criterion to $\mathrm{ad}(\mathfrak{s})$ and conclude that $\mathrm{ad}(\mathfrak{s})$ is a solvable ideal in $\mathrm{ad}(\mathfrak{g})$. Because $\mathfrak{s}$ is isomorphic to $\mathrm{ad}(\mathfrak{s})$ and $\mathfrak{g}$ is isomorphic to $\mathrm{ad}(\mathfrak{g})$ we can conclude that $\mathfrak{s}$ is solvable in $\mathfrak{g}$. But $\mathfrak{g}$ is semisimple, so $\mathfrak{s}$ must be trivial.

Now assume that $\mathfrak{s} = 0$. Let $\mathfrak{a}$ be an abelian ideal of $\mathfrak{g}$. Let $X \in \mathfrak{a}$ and $Y \in \mathfrak{g}$, and consider $A = \mathrm{ad}(X) \circ \mathrm{ad}(Y)$. Then $A(\mathfrak{g}) \subset \mathfrak{a}$, and $A(\mathfrak{a}) = 0$. This means that $A^2(\mathfrak{g}) = 0$, or A is nilpotent of degree two. We can therefore conclude that $\mathrm{Tr}(A) = \mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)) = B(X, Y) = 0$. This tells us that $X \in \mathfrak{s}$, so $\mathfrak{a} = 0$, therefore $\mathfrak{g}$ is semisimple, which completes the proof. □

The final step in this puzzle is Schur's lemma:

Lemma 3.1.5 (Schur's Lemma)**.** *Let V and W be irreducible $\mathfrak{g}$-modules. Then if $\phi : V \to W$ is a homomorphism of modules, then ϕ is either zero or multiplication by a scalar.*

Proof. The proof of this lemma, which is non-trivial but straightforward, is omitted. It may be found in chapter 1 of [2]. □

We are now ready to prove Theorem 3.1.1.

Proof of Theorem 3.1.1. My first claim is that the Killing form B_V is non-degenerate on $\rho(\mathfrak{g})$. We know $\rho(\mathfrak{g})$ must be semisimple because $\mathfrak{g}$ semisimple implies that ρ is injective (its kernel would be an abelian ideal of $\mathfrak{g}$), which makes ρ an isomorphism onto $\rho(g)$. Let $\mathfrak{s} = \{Y : \text{ such that } B_V(\rho(X), \rho(Y)) = 0\}$. We know that ρ preserves ideals. That together with (5) implies that $\rho(\mathfrak{s})$ is an ideal in $\rho(\mathfrak{g})$. By Cartan's criterion it is solvable, implying that $\rho(\mathfrak{s}) = 0$. Therefore we may conclude that the Killing form is non-degenerate.

Let $U_1, U_2, \ldots, U_r$ be a basis for $\rho(\mathfrak{g})$, and define $U'_1, U'_2, \ldots, U'_r$ to be a dual basis with respect to the Killing form. This is possible because the Killing form is non-degenerate. We can now define C_V an endomorphism of V:

$$C_V(v) = \sum_{i=1}^{r} U_i(U_i'(v)).$$

We can then calculate its trace:

$$\mathrm{Tr}(C_V) = \sum_{i=1}^{r} \mathrm{Tr}(U_i \circ U_i') = \sum_{i=1}^{r} B_V(U_i, U_i') = r = \dim(\rho(\mathfrak{g})).$$

The endomorphism C_V is a generalization of the Casimir operator. A detailed discussion of this operator is outside the scope of this paper; it is enough for us to know that C_V commutes with the ρ action of $\mathfrak{g}$ on V. For more information on the Casimir operator and a proof of this property, see chapter 25 of [2].

We can easily see that C_V preserves W because W is preserved by any $\rho(\mathfrak{g})$ action and C_V is a sum of compositions of elements of $\rho(\mathfrak{g})$. Additionally, we note that all one-dimensional representations are trivial.

We will first prove the special case that W is irreducible of co-dimension one. In this case, because C_V preserves W we have that C_V acts by zero on V/W. We can then apply Schur's Lemma to conclude that C_V is multiplication by a scalar on W. Because $\mathrm{Tr}(C_v) > 0$, this scalar must be non-zero. We can then conclude that $V = W \oplus \mathrm{Ker}(C_V)$ as $\mathfrak{g}$-modules, which completes the proof in this special case.

Suppose W has co-dimension one but is not irreducible. We will prove this case by inducting on the dimension of V. In the case where $\dim(V) = 1$ the proof is trivial. Now assume it is true for $\dim(V) = n - 1$. Then find L a non-zero proper subrepresentation of W. By the inductive hypothesis we can find a complementary space Y/L to $W/L \subset V/L$. We can use induction again to find a U such that $Y = L \oplus U$. We can then conclude that $V = W \oplus U$.

It remains to handle the case where W does not have co-dimension one. An argument similar to the one in the above paragraph will reduce the proof to the case where W is irreducible. In this case, we consider the restriction map:

$$\nu : \mathrm{Hom}(V, W) \to \mathrm{Hom}(W, W),$$

where $\mathrm{Hom}(V, W)$ and $\mathrm{Hom}(W, W)$ are $\mathfrak{g}$-modules under the following action. Let $x \in \mathfrak{g}$ and $v \in V$, and let x act on $f \in \mathrm{Hom}(V, W)$ by $(xf)(v) = xf(v) - f(xv)$. In that case ν is a surjective homomorphism of $\mathfrak{g}$-modules. We omit the proof of this claim, as it is a straightforward calculation. Now consider

$$\mathrm{Hom}_{\mathfrak{g}}(W, W) = \{f \in \mathrm{Hom}(W, W) : f(xw) = xf(w) \text{ for all } x \in \mathfrak{g}\}.$$

A quick application of Schur's Lemma indicates that $\nu^{-1}(\mathrm{Hom}_{\mathfrak{g}}(W, W))$ is precisely those homomorphisms $V \to W$ that restrict to multiplication by a scalar on W. Define h to be the homomorphsim that is the identity on W but zero everywhere else. It is obvious that any $f \in \nu^{-1}(\mathrm{Hom}_{\mathfrak{g}}(W, W))$ can be written as the sum of some element of $\text{Ker}(\nu)$ plus ah where a is a scalar. Therefore $\text{Ker}(\nu)$ is a $\mathfrak{g}$ submodule in $\nu^{-1}(\text{Hom}_{\mathfrak{g}}(W,W))$ with co-dimension one.

We can then apply our preceding special case to conclude that $\nu^{-1}(\text{Hom}_{\mathfrak{g}}(W,W)) = \text{Ker}(\nu) \oplus U$ for some U. We now note that because ν is surjective, it maps U surjectively onto $\text{Hom}_{\mathfrak{g}}(W,W)$. We can therefore choose a ψ in U such that $\nu(\psi) = I$. Note that this implies $\text{Ker}(\psi) \cap W = 0$. Because U is one-dimensional, we know that the $\mathfrak{g}$ acts on U by zero.

Recall the action of $\mathfrak{g}$ on $\text{Hom}(V,W))$ for $x \in \mathfrak{g}$, $v \in V$, $f \in \text{Hom}(V,W)$ as

$$(xf)(v) = x(f(v)) - f(xv).$$

In the case of ψ, we can say that

$$0 = (x\psi)(v) = x(f(v)) - f(xv),$$

or,

$$x(\psi(v)) = \psi(xv).$$

Therefore, if $v \in \text{Ker}(\psi)$ we may conclude that $\psi(xv) = x(\psi(v)) = 0$, or $xv \in \text{Ker}(\psi)$. This allows us to conclude that $\text{Ker}(\psi)$ is a $\mathfrak{g}$ submodule of V. Recall that we have also showed that $\text{Ker}(\psi) \cap W = 0$. Together these statements prove that $V = \text{Ker}(\psi) \oplus W$. □

Corollary 3.1.6. *A semisimple Lie algebra is a direct sum of simple Lie algebras.*

Proof. This follows from Theorem 3.1.1 by induction. □

Theorem 3.1.1 is hugely important for understanding the irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$. Because $\mathfrak{sl}_2(\mathbb{C})$ is a simple Lie algebra, by this theorem, we only need to identify the irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$; all other representations will be direct sums of the irreducible representations.

3.2. **Preservation of Jordan Canonical Form.** The fact that representations of semisimple Lie algebras preserve Jordan decomposition will be the foundation of our study of the representations of $\mathfrak{sl}_2(\mathbb{C})$. The proof of this theorem relies heavily on Theorem 3.1.1.

Theorem 3.2.1. *If V, ρ is a representation of a semisimple Lie algebra $\mathfrak{g}$, and $X_s + X_n$ is the Jordan form of a matrix X, then $\rho(X_s) + \rho(X_n)$ is the Jordan form of $\rho(X)$.*

Most of the work for the above theorem is done in the following lemma.

Lemma 3.2.2. *Let $\mathfrak{g} \subset \text{End}(V)$ be a Lie algebra. If $\mathfrak{g}$ is semisimple, then for any $X \in \mathfrak{g}$, X_s and X_n are in $\mathfrak{g}$ as well.*

Proof. To prove this we will consider $\mathfrak{g}$ as an intersection of sub-algebras of $\text{End}(V)$. We will then prove that the claim is true for those sub-algebras. If $\mathfrak{g}$ is semisimple then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. Because each X is a commutator, each X has zero trace. It is clearly true that $\mathfrak{g} \subset \mathfrak{sl}(V)$. It is also trivial that X_s and X_n have zero trace if X has zero trace. Therefore $X_n, X_s \in \mathfrak{sl}(V)$.

If V is not irreducible, for any W a subrepresentation of V, let:

$$\mathfrak{s}_W = \{Y \in \mathrm{End}(V) : Y(W) \subset W, \mathrm{Tr}(Y|_W) = 0\}.$$

It is obvious that $\mathfrak{g} \subset \mathfrak{s}_W$, and that if $X \in \mathfrak{s}_W$ then $X_s, X_n \in \mathfrak{s}_W$. Now, X_s and X_n can be written as polynomial forms of X. It is also easy to see that $p(X)\mathfrak{g} \subset \mathfrak{g}$ for any polynomial $p(T)$. Together, these statements imply that $X_s\mathfrak{g} \subset \mathfrak{g}$ and similarly $X_n\mathfrak{g} \subset \mathfrak{g}$. Therefore X_s and X_n are elements of $\mathfrak{n} = \{X \in \mathrm{End}(V) : [X, \mathfrak{g}] \subset \mathfrak{g}\}$. I claim that

$$\mathfrak{g} = \left(\bigcap_{W \subset V} \mathfrak{s}_W\right) \bigcap \mathfrak{n}.$$

Let $\mathfrak{g}'$ be the right hand side of that equation. Then $\mathfrak{g}$ is an ideal in $\mathfrak{g}' \subset \mathfrak{n}$. By Theorem 3.1.1 there exists U a sub-algebra of $\mathfrak{g}'$ such that $\mathfrak{g}' = \mathfrak{g} \oplus U$. We must have that $[\mathfrak{g}', U] = 0$. We would be finished if we could show that $U = 0$. We will show this by proving that for any $Y \in U$, $Y|_W = 0$ for any irreducible $W \subset V$. By our definition of $\mathfrak{g}'$ we know that $Y \in \mathfrak{s}_W$. This implies that Y preserves W, and then Schur's lemma tells us that $Y|_W$ is multiplication by a scalar λ, and $\mathrm{Tr}(Y|_W) = 0$ implies $\lambda = 0$, completing the proof. □

We will now prove Theorem 3.2.1.

Proof of Theorem 3.2.1. As we have stated earlier, the fact that $\mathfrak{g}$ is semisimple implies that ad embeds $\mathfrak{g}$ in $\mathrm{End}(\mathfrak{g})$. Therefore it suffices to prove the theorem for the case $\mathfrak{g} \subset \mathrm{End}(W)$. By the previous lemma for any $X \in \mathfrak{g}$ X_s, X_n in $\mathfrak{g}$,

Now co-restrict ρ to the image $\rho(\mathfrak{g}) \subset \mathrm{End}(V)$, making it a surjective homomorphism of Lie algebras. I claim that $\rho(X_s)$ is semisimple. I begin by proving the special case when $\mathfrak{g}$ is simple. This implies that $\rho(\mathfrak{g})$ is also simple. We can therefore conclude that ρ is injective, because the kernel would be a non-trivial ideal of $\mathfrak{g}$. This means that ρ is an isomorphism from $\mathfrak{g}$ to $\rho(\mathfrak{g})$. In this case it is trivial that $\rho(X_s)$ is still semisimple and $\rho(X_n)$ is still nilpotent.

If $\mathfrak{g}$ is semisimple, by Corollary 3.1.6 we know that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k$, a direct sum of simple Lie algebras. Therefore

$$\rho(\mathfrak{g}) = \rho(\mathfrak{g}_1) \oplus \rho(\mathfrak{g}_2) \oplus \cdots \oplus \rho(\mathfrak{g}_k),$$

where ρ must either be the zero map, or injective on each $\mathfrak{g}_i$. Because all ρ must map simple algebras in $\mathfrak{g}$ to simple algebras in $\rho(\mathfrak{g})$, it is clear that $\rho(X_s)$ is still semisimple in $\rho(\mathfrak{g})$. Similarly we may conclude that $\rho(X_n)$ is nilpotent.

It is clear that $\rho(X) = \rho(X_n) + \rho(X_s)$ where $\rho(X_n)$ is nilpotent and $\rho(X_s)$ is semisimple. We recall that Jordan decomposition is unique and conclude that $\rho(X_n) = \rho(X)_n$ and $\rho(X_s) = \rho(X)_s$. □

4. Irreducible Representations of $\mathfrak{sl}_2(\mathbb{C})$

Let V, ρ be an arbitrary n dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$. Our aim is to classify the irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$. We know that $\rho(H)$ is diagonalizable by Theorem 3.2.1, and therefore we let $\lambda_1 \dots \lambda_k$ be the distinct eigenvalues of H ordered by their real parts in a non-decreasing order. Finally let W_λ be the eigenspace that corresponds to λ, and if λ is not an H eigenvalue, then $W_\lambda = 0$.

4.1. **E and F acting on Eigenspaces of H.** We begin by recalling our basis E, F, and H from section 2.1. The first step in understanding representations of $\mathfrak{sl}_2(\mathbb{C})$ is understanding how E and F act on the eigenspaces of H. To do that we will begin by studying how these linear operators act on W_{λ_k}, an eigenspace with the maximal real value.

Lemma 4.1.1. *If $v \in W_{\lambda_k}$ then $Ev = 0$. And in the general case, $EW_{\lambda_i} \subset W_{\lambda_i+2}$ regardless of i.*

Remark. Note that I will be dropping the $\rho(E)$ for the rest of this discussion for the sake of clarity; when I write $Ev = 0$ what I mean is $(\rho(E))v = 0$.

Proof. Recalling our original basis for $\mathfrak{sl}_2(\mathbb{C})$ described in 2.1, we notice that $[H, E] = 2E$. Because our representation must preserve the Lie bracket, we can conclude that

$$\begin{aligned} HEv - EHv &= 2Ev \\ HEv - \lambda_k Ev &= 2Ev \\ H(Ev) &= (\lambda_k + 2)Ev. \end{aligned}$$

Therefore Ev must be an eigenvector of H, and its corresponding eigenvalue must be $\lambda_k + 2$. But we ordered the λ_k by real value, and therefore there is no non-zero eigenvector with corresponding to $\lambda_k + 2$. The only other possibility is that Ev is the zero vector. □

The above lemma shows exactly how E acts on the eigenspaces of H. The linear operator E sends each eigenspace W_{λ_i}into W_{λ_i+2}. And as one might expect, F sends each eigenspace W_{λ_i} into W_{λ_i-2}.

Lemma 4.1.2. *For all i, $FW_{\lambda_i} \subset W_{\lambda_i-2}$.*

Proof. The proof is almost identical to the calculation in Lemma 4.1.1. □

Corollary 4.1.3. *There exists an N such that $F^N v = 0$ for all v in V.*

Proof. We showed in the above that $FW_{\lambda_i} \subset W_{\lambda_i-2}$. Because there are only finitely many eigenvalues, and F will strictly lower the real part of the eigenvalue with each application, we can conclude that for any v an eigenvalue of H, there is some minimal N_v such that $F^{N_v} v = 0$. Combining this with the fact that we have an eigenbasis of V with respect to H, we can take N to be the maximum of all N_{v_i} for v_i in the eigenbasis. Then $F^N v = 0$ for all v in V. □

4.2. **Composing E, F, and H.** Now that we understand how E and F act on the eigenspaces of H, we need to start to understand what happens when we compose E and F. The majority of this subsection will involve the proof of the following lemma, which by itself is not very illuminating but will be invaluable as we study the relationships between the H eigenspaces of V.

Lemma 4.2.1. *Let v be in* $\mathrm{Ker}(E)$. *Then there exists a polynomial $P_k(H)$ such that $E^k F^k v = P_k(H)v$. Moreover,*

$$P_k(H) = \prod_{j=1}^{k} j(H - (j-1)I).$$

Proof. We will first prove the special case where v is an eigenvector of H. In the case of $k = 1$ the proof is easy:

$$Hv = EFv - FEv = EFv.$$

In this case $P_1(H) = H$. I will prove the rest by induction. But first there is a claim that needs to be proved:

$$E^{k+1}F = FE^k + (k)(HE^{k-1} - (k-1)E^{k-1}). \tag{6}$$

In the case $k = 1$ the proof is trivial, by the relation stated above. Now suppose the relation is true for some k. Then by the inductive hypothesis:

$$E^{k+1}F = E(E^kF) = EFE^k + (k)(EHE^{k-1} - (k-1)E^k).$$

Using (3) and (1),

$$EFE^k + (k)(EHE^{k-1} - (k-1)E^k) = (FE+H)E^k + (k)((HE-2E)E^{k-1} - (k-1)E^k).$$

We can then simplify the equation,

$$\begin{aligned}(FE + H)E^k &+ (k)((HE - 2E)E^{k-1} - (k-1)E^k) \\ &= FE^{k+1} + HE^k + (k)HE^k - 2(k)E^k - (k)(k-1)E^k \\ &= FE^{k+1} + (k+1)HE^k - (k+1)(k)E^k \\ &= FE^{k+1} + (k+1)(HE^k - (k)E^{k-1}).\end{aligned}$$

The proof of (6) follows by induction on k. We also prove a similar statement for use later:

$$EF^k = F^{k-1}(H - 2(k-1)). \tag{7}$$

The proof of this statement follows the pattern of our proof of (6).

We will now assume that the special case of the lemma is true for some k and proceed by induction.

$$E^{k+1}F^{k+1} = E(E^k F)F^k$$

First we use (6):

$$= E(FE^k + (k)(HE^{k-1} - (k-1)E^{k-1}))F^k v$$

then we regroup terms,

$$= (EF)E^k F^k v + (k)(EH)E^{k-1}F^k v - (k)(k-1)(E^k F^k v).$$

Next, we plug in (1), and use the inductive hypothesis,

$$= (EF)P_k(H)v + k(HE - 2E)E^{k-1}F^k v - k(k-1)P_k(H)v.$$

Now we use the fact that v is a λ eigenvector of H

$$= (EF)P_k(\lambda)v + kHE^kF^kv - 2kE^kF^kv - k(k-1)P_k(H)v.$$

Then we apply the inductive hypothesis and plug in (3)

$$= P_k(\lambda)(EF)v + kHP_k(H)v - 2kP_k(H)v - k(k-1)P_k(H)v.$$

Finally, we simplify the equation:

$$\begin{aligned}
&= HP_k(H)v + kHP_k(H)v - 2kP_k(H)v - k(k-1)P_k(H)v \\
&= (H + kH - 2k - k^2 + k)P_k(H)v \\
&= ((k+1)H - k(k+1))P_k(H)v \\
&= (k+1)(H-k)\prod_{j=1}^{k} j(H - (j-1)I) \\
&= \prod_{j=1}^{k+1} j(H - (j-1)I).
\end{aligned}$$

The proof of the special case v an H eigenvector follows by induction on k.

Now, suppose v is an arbitrary vector in $\mathrm{Ker}(E)$. We can choose an H eigenbasis for $\mathrm{Ker}(E)$. We have shown the lemma to be true for each vector in this basis, therefore we may conclude that it is true for all v in $\mathrm{Ker}(E)$. □

We will use this polynomial in conjunction with some carefully designed subspaces of V to learn more about the relationships among the eigenspaces of H.

4.3. **The Eigenspaces and Eigenvalues of H.** In this section we will begin to study the relationships between the eigenspaces of H as well as discuss exactly what the possible eigenvalues of H are.

Definition 4.3.1. Let λ_i be an eigenvalue of H such that $EW_{\lambda_i} = 0$. Then, recalling Corollary 4.1.3, let N_{λ_i} be minimal such that $F^{N_{\lambda_i}} W_{\lambda_i} = 0$. Then S_{λ_i} is defined to be the subspace

$$W_{\lambda_i} + FW_{\lambda_i} + F^2 W_{\lambda_i} \cdots + F^{N_{\lambda_i}-1} W_{\lambda_i},$$

which, by Lemma 4.1.2 is a subset of

$$W_{\lambda_i} + W_{\lambda_i - 2} + W_{\lambda_i - 4} + \cdots + W_{\lambda_i - 2(N_{\lambda_i} - 1)}.$$

.

Note that in the definition above all W_{λ_j} are non-zero.

Lemma 4.3.2. *For λ_k an eigenvalue of maximal real value, $S_{\lambda_k} = V$.*

Proof. I will prove this by showing that S_{λ_k} is preserved under the actions of H, F, and E, making it a $\mathfrak{sl}_2(\mathbb{C})$ submodule of V. However, we declared V to be irreducible as a $\mathfrak{sl}_2(\mathbb{C})$ module, and therefore $S_{\lambda_k} = V$.

It is trivial to see that S_{λ_k} is preserved under the action of F.

It is easy to show that S_{λ_k} is preserved under H. Each $F^i(W_{\lambda_k})$ is a subset of $W_{\lambda_k - 2i}$, which is a $\lambda_k - 2i$ eigenspace of H. We may therefore conclude that

$$HF^i(W_{\lambda_k}) = (\lambda_k - 2i)F^i(W_{\lambda_k}) = F^i(W_{\lambda_k}).$$

Because this is true for each i, we may conclude that H preserves S_{λ_k}.

We will now show that $E(S_{\lambda_k}) \subset S_{\lambda_k}$. First we calculate:

$$\begin{aligned} E(S_{\lambda_k}) &= E(W_{\lambda_k} + FW_{\lambda_k} + F^2 W_{\lambda_k} \cdots + F^{N_{\lambda_k}-1} W_{\lambda_k}) \\ &= EW_{\lambda_k} + EFW_{\lambda_k} + EF^2 W_{\lambda_k} \cdots + EF^{N_{\lambda_k}-1} W_{\lambda_k}. \end{aligned}$$

Now we recall (7) from Lemma 4.2.1 and simplify:

$$\begin{aligned} &= 0 + HW_{\lambda_k} + F(H-2)W_{\lambda_k} \cdots + F^{N_{\lambda_k}-2}(H - 2(N_{\lambda_k} - 2))W_{\lambda_k} \\ &\subset W_{\lambda_k} + FW_{\lambda_k} + \cdots + F^{N_{\lambda_k}-2} W_{\lambda_k} \\ &\subset S_{\lambda_k}. \end{aligned}$$

This completes the proof. □

Corollary 4.3.3. *The H eigenspaces W_{λ_i} are precisely $W_{\lambda_k}, FW_{\lambda_k} \dots, F^{N_{\lambda_k}-1} W_{\lambda_k}$.*

Proof. This follows directly from the definition of S_{λ_k} and the above lemma. □

Lemma 4.3.4. *If λ_k is the eigenvalue of H with maximal real part, then the unique eigenvalues of H are $\lambda_k, \lambda_k - 2, \lambda_k - 4, \dots, \lambda_k - 2(k-1)$.*

Proof. Recall the statement of Lemma 4.1.2, that for any i, $FW_{\lambda_i} \subseteq W_{\lambda_i - 2}$. The above corollary allows us to make that an equality. This allows us to calculate the possible H eigenvalues. This simple calculation yields the desired result. □

Lemma 4.3.5. *Let W_{λ_k} be as above. Then* $\dim W_{\lambda_k} = 1$.

Proof. Let v be a non-zero vector in W_{λ_k}. Then consider the subspace $U = \operatorname{span}(v) + F \operatorname{span}(v) + \cdots + F^{N_{\lambda_k}-1} \operatorname{span}(v)$. By an argument like that used in Lemma 4.3.2, we can show that U is preserved under the actions of E, H, and F. Then we can conclude that U is a submodule of V under $\mathfrak{sl}_2(\mathbb{C})$, and because we assumed that V was an irreducible representation we can see that $U = V$. Therefore $W_{\lambda_k} = \operatorname{span}(v)$. □

Corollary 4.3.6. *Let W_{λ_i} be any non-zero H eigenspace. Then* $\dim(W_{\lambda_i}) = 1$.

Proof. By Corollary 4.3.3 we know that $W_{\lambda_i} = F^j W_{\lambda_k}$ for some j. Now, W_{λ_k} is one-dimensional, therefore its image under the linear operator F can have one or zero dimensions, but no more. We said that W_{λ_i} is non-zero, and therefore it must have dimension one. □

From this lemma, if we know λ_k, we know all the eigenvalues of H exactly. The previous lemma tells us each eigenvalue of H, and this lemma tells us that there are no repeated eigenvalues So there are n distinct eigenvalues and they take the values $\lambda_k, \lambda_k - 2, \ldots, \lambda_k - 2(n-1)$. There is actually one more thing we can say about these eigenvalues, and that is the subject of our next lemma.

Lemma 4.3.7. *If* $\dim(V) = n$ *then* $\lambda_k = n - 1$.

Proof. To get this result, we use the fact that $\operatorname{Tr}(H) = \sum_{i=0}^{n} \lambda_i = 0$ because H is in $\mathfrak{sl}_2(\mathbb{C})$.

$$\begin{aligned} 0 &= \lambda_k + \lambda_k - 2 + \cdots + \lambda_k - 2(n-1) \\ &= n\lambda_k - 2n(n-1)/2 \\ &= \lambda_k - (n-1) \end{aligned}$$

□

Theorem 4.3.8. *There exists only one isomorphism class of n-dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$.*

Proof. The three previous lemmas do the bulk of the work for this theorem. By Lemma 4.3.5 we can choose a v such that $W_{\lambda_k} = \operatorname{span}(v)$. Then by Corollary 4.3.6 together with Lemma 4.3.7 we have that $v, Fv, F^2v, \ldots, F^{n-1}v$ is a basis for V that is also an eigenbasis of H. Under this basis, H is a diagonal matrix with eigenvalues $n-1, n-3, n-5, \ldots, 3-n, 1-n$, and F is the matrix of ones below the diagonal and zeros elsewhere. Our only remaining question then is how E acts on this basis. We have already shown that $Ev = 0$. Now consider EF^i for some $0 < i < n$. Recall (7) which tells us that

$$EF^iv = F^{i-1}(H - 2(i-1))v = F^{i-1}((n-1) - 2(i-1)v) = (n-2i)F^{i-1}v.$$

This means that E acts on this basis as the matrix with $n-2, n-4, n-6, \ldots, 2-n$ on the line directly above the diagonal, with zeros elsewhere. Therefore given the dimension for the representation, we have completely determined the action of H, E, and F on all the vectors in the vector space, up to a change of basis. $\square$

With this theorem we have classified all irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$. However, we have also done much more. Because $\mathfrak{sl}_2(\mathbb{C})$ is simple, we know any representation will have complete reducibility. This means that any representation of $\mathfrak{sl}_2(\mathbb{C})$ will be the direct sum of irreducible representations. Now we have classified all representations of $\mathfrak{sl}_2(\mathbb{C})$.

Acknowledgments

I would like to thank the University of Chicago Math Department for funding this program, and Professor Peter May for running it. Special thanks are also due to Robin Walters for his guidance and assistance during the writing of this paper.

References

[1] M. Artin; *Algebra*. Upper Saddle River, NJ: 1991.
[2] W. Fulton, J. Harris; *Representation Theory: A First Course*. New York, NY: 1991.
[3] J.E. Humphries; *Introduction to Lie Algebras and Representation Theory*. New York, NY: 1970.