THE GENERALIZED STOKES' THEOREM

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Abstract. This paper will prove the generalized Stokes Theorem over k-dimensional manifolds. We will begin from the definition of a k-dimensional manifold as well as introduce the notion of boundaries of manifolds. Using these, we will construct the necessary machinery, namely tensors, wedge products, differential forms, exterior derivatives, and integrals over manifolds, in order to prove the main result of this paper. Please note that, unless otherwise noted, all material is provided by [1].

Contents

1. Manifolds and Other Preliminaries

Manifolds are the fundamental setting in which the Generalized Stokes' Theorem will be constructed. We begin by defining the idea of a smooth map.

Definition 1.1. A map f of an open set $U \subset \mathbb{R}^n$ into $\mathbb{R}^m$ is called *smooth* if it has continuous partial derivatives of all orders. Let X be an arbitrary open subset of $\mathbb{R}^n$. Then we say that a map $f : X \to \mathbb{R}^m$ is smooth if it may be locally extended to a smooth map on open sets; that is, if around each point $x \in X$ there is an open set $U \subset \mathbb{R}^n$ and a smooth map $F : U \to \mathbb{R}^m$ such that F equals f on $U \cap X$.

Using this idea of smoothness, we now define the most important type of function that we will need to understand manifolds: a diffeomorphism.

Definition 1.2. Let X and Y be open sets in $\mathbb{R}^n$. We say $f : X \to Y$ is a *diffeomorphism* if it is bijective and smooth, and if the inverse map $f^{-1} : Y \to X$ is also smooth.

Intuitively, manifolds are sets that may be locally described using Euclidean space. In other words, we are able to construct a bijective continuously smooth map, whose inverse is smooth as well, between the local region of the manifold and

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a representative region of Euclidean space. More formally, we have the following definition from [1] and [3].

Definition 1.3. Let $\{V_i\}_{i\in I}$ be an open cover for X, where X is a subset of some big, ambient Euclidean space $\mathbb{R}^N$ such that for all $p \in X$, there is a set V_i containing p that is open in X, a set U_i that is open in $\mathbb{R}^k$, and a continuous map $\alpha_i : U_i \to V_i$ carrying U_i onto V_i in a one-to-one fashion, such that α_i is a local diffeomorphism. Then X is a k-dimensional manifold, and (V_i, α_i) is called a *coordinate chart*.

In addition to a smooth manifold, we have the notion of a manifold with boundary. Just as the name suggests, a manifold with boundary is exactly like the smooth manifolds we have already discussed but with additional structure known as the boundary. In order to rigorously define such an object, we introduce an analog to $\mathbb{R}^k$.

Definition 1.4. The upper half space, denoted H^k, is the set of all points with nonnegative final coordinate. Its boundary is $\mathbb{R}^{k-1}$.

Definition 1.5. The *boundary* of a k-dimensional manifold X under one or more local diffeomorphisms between an open set of X and an open set H^k used to parametrize the manifold is defined as follows with the following notation used henceforth: $\partial X = \{x : x \in Im(\partial H^k)\}$, where Im(A) denotes the image of the set A.

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With the idea of a manifold with boundary defined rigorously, we take a closer look at what exactly this new structure, the boundary, is. Not very surprisingly, the boundary is itself a manifold; however, it is a manifold without boundary.

Fact 1.7. *Let X be a k-dimensional manifold with boundary. Then ∂X is a $(k-1)$ dimensional manifold without boundary.*

A few important definitions and theorems will be introduced for later use.

Definition 1.8. Let x be a point on a manifold X in $\mathbb{R}^N$. Moreover, let $U \subset \mathbb{R}^k$ with $\phi : U \to X$ being a local diffeomorphism around x, and let $d\phi : \mathbb{R}^k \to \mathbb{R}^N$ be the derivative map, so it should map from k-space to the tangent space of the manifold. Then we say that the image of $d\phi : \mathbb{R}^k \to \mathbb{R}^N$ is the tangent space of X at x. We denote the tangent space of X at x as $T_x(X)$. In $\mathbb{R}^N$, the map $d\phi$ resembles the Jacobian matrix.

Linear transformations from a tangent space of one manifold to another are well-defined and mimic the desired properties of derivatives.

Besides tangent spaces that can be placed on manifolds, orientations can be assigned. We first look at orientations for vector spaces and then for manifolds.

Definition 1.9. Given an ordered basis of a finite-dimensional real vector space V, an orientation is the arbitrary assignment of either a positive or negative sign to equivalence classes that result from the partitioning based on the sign of the determinant of a linear transformation $A : V \to V$. In other words, whether the determinant is positive or negative will determine the orienation.

From [3], we have the following definition that is necessary for oriented manifolds.

Definition 1.10. Let X be a topological n-manifold. If $(U, \phi), (V, \psi)$ are two coordinate charts such that $U \cap V \neq \emptyset$, the composite map $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is called the *transition map* from ϕ to ψ. Two charts (U, ϕ) and (V, ψ) are said to be *smoothly compatible* if either $U \cap V = \emptyset$ or the transition map $\psi \circ \phi^{-1}$ is a diffeomorphism.

Definition 1.11. Given a manifold with boundary X, its orientation is a smoothly compatible choice of orientations at each of its tangent spaces $T_x(X)$.

Remarks 1.12. First, if the choice is of orientation for a manifold satisfies smooth compatibility, we say that the coordinate charts are *orientation-preserving*. Second, every orientation induces an orientation on its boundary, which is either directed outward or inward on the boundary. Thus, we shall refer to these as the outward and inward normals.

We state a useful fact about orientations without proof.

Fact 1.13. *A connected, orientable manifold with boundary admits exactly two orientations.*

One concept remains to be constructed, namely a partition of unity. Loosely speaking, a partition of unity is a set of locally finite functions that sum to unity. Its power lies in that it allows us to extend local topological properties to global settings. We state the following theorem without proof for later use.

Theorem 1.14. *Let X be a smooth manifold in $\mathbb{R}^N$. For any covering of X by (relatively) open subsets $\{U_\alpha\}$, there exists a sequence of smooth functions $\{\theta_i\}$, called a partition of unity subordinate to the open cover $\{U_\alpha\}$, with the following properties:*
(a) $0 \leq \theta_i(x) \leq 1$ for all $x \in X$ and all i.
(b) Each $x \in X$ has a neighborhood on which all but finitely many functions θ_i are identically zero.
(c) Each function θ_i is identically zero except on some closed set contained in one of the U_α.
(d) For each $x \in X$, $\sum_i \theta_i(x) = 1$. (Note that according to (b), this sum is always finite.)
Note: (relatively) open subsets are to be interpreted in the following manner: if X is a subset in $\mathbb{R}^n$, then a subset Y of X is (relatively) open in X if it can be written as the intersection of X with an open subset of $\mathbb{R}^n$.

2. Exterior Algebra: Tensors and Wedge Products

We now take a detour away from manifolds in order to develop the tools necessary to discuss differential forms. Our building blocks come in the form of tensors. For our purposes, tensors will refer to multilinear functions.

Definition 2.1. A p-tensor on V is a real-valued function T on the cartesian product

$$V^p = \underbrace{V \times \cdots \times V}_{p \text{ times}},$$

which is separately linear in each variable, or *multilinear*.

Tensors tend to be generalized versions of various functionals we know. For example, a real-valued functional is a 1-tensor, while the dot product in Euclidean space is a 2-tensor.

One way of combining tensors is known as the tensor product.

Definition 2.2. Let T be a p-tensor and S a q-tensor. We define a $(p+q)$-tensor $T \otimes S$ by the formula

$$(T \otimes S)(v_1, \ldots, v_p, v_{p+1}, \ldots, v_{p+q}) = T(v_1, \ldots, v_p) \cdot S(v_{p+1}, \ldots, v_{p+q})$$

We call $T \otimes S$ the tensor product of T with S.

Remark 2.3. The set $\mathfrak{J}^p(V^*)$ of all p-tensors forms a vector space.

Fact 2.4. *Tensor product is associative; that is,* $(T \otimes S) \otimes R = T \otimes (S \otimes R)$.

Proof. Take $T \in \mathfrak{J}^p(V^*), S \in \mathfrak{J}^q(V^*)$, and $R \in \mathfrak{J}^s(V^*)$. By definition and associativity of the reals, we have

$$\begin{aligned}
&((T \otimes S) \otimes R)(v_1, \ldots, v_p, v_{p+1}, \ldots, v_{p+q}, v_{p+q+1}, \ldots, v_{p+q+s}) \\
&= [T(v_1, \ldots, v_p) \cdot S(v_{p+1}, \ldots, v_{p+q})] \cdot R(v_{p+q+1}, \ldots, v_{p+q+s}) \\
&= T(v_1, \ldots, v_p) \cdot [S(v_{p+1}, \ldots, v_{p+q}) \cdot R(v_{p+q+1}, \ldots, v_{p+q+s})] \\
&= (T \otimes (S \otimes R))(v_1, \ldots, v_p, v_{p+1}, \ldots, v_{p+q}, v_{p+q+1}, \ldots, v_{p+q+s}).
\end{aligned}$$

□

While we will not be using the tensor product directly, a special case, known as the wedge product, will be necessary for our purposes. To that end, we introduce a special type of tensor.

Definition 2.5. A tensor T is alternating if the sign of T is reversed whenever two variables are transposed:

$$T(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_p) = -T(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_p).$$

It may appear that alternating tensors are rare, so it is important to produce a method for constructing alternating tensors from an arbitrary tensor T. If we let $\pi \in S_p$ be a permutation, where S_p denotes the symmetric group of p elements, we denote $(-1)^\pi = +1$ or $(-1)^\pi = -1$ if π is even or odd, respectively. Using this notation, we have an immediate and obvious result: given that T is an alternating tensor, $T^\pi = (-1)^\pi T$, where $T^\pi(v_1, \ldots, v_p) = T(v_{\pi(1)}, \ldots, v_{\pi(p)})$. Moreover, we claim, without proof, that $(T^\pi)^\sigma = T^{\pi \circ \sigma}$.

Proposition 2.6. *Let T be a p-tensor. Construct a p-tensor Alt(T)* $= \frac{1}{p!}\sum_{\pi \in S_p}(-1)^\pi T^\pi$. *Then Alt(T) is an alternating tensor.*

Proof. Take a p-tensor as constructed above, and let $\sigma \in S_p$. Then we have the following:

$$[\text{Alt(T)}]^\sigma = \left[\frac{1}{p!}\sum_{\pi \in S_p}(-1)^\pi T^\pi\right]^\sigma = \frac{1}{p!}\sum_{\pi \in S_p}(-1)^{\pi\circ\sigma\circ\sigma}(T^\pi)^\sigma = \frac{1}{p!}(-1)^\sigma \sum_{\pi \in S_p}(-1)^{\pi\circ\sigma}T^{\pi\circ\sigma}.$$

Now let $\tau = \pi \circ \sigma$. This yields

$$[\text{Alt(T)}]^\sigma = (-1)^\sigma\left[\frac{1}{p!}\sum_{\tau \in S_p}(-1)^\tau T^\tau\right] = (-1)^\sigma[\text{Alt(T)}].$$

□

Remark 2.7. The set $\Lambda^p(V^*)$ of alternating p-tensors forms a subspace of $\mathfrak{J}^p(V^*)$.

Now that we are able to construct an alternating tensor out of any tensor, we put it to use by combining it with the tensor product. Using the tensor product, we can apply it to two arbitrary tensors, say T and S, and we get a new tensor $T \otimes S$. Applying the algorithm we developed above, we are now able to turn $T \otimes S$ into an alternating tensor, which we proceed to define below.

Definition 2.8. Let $T \in \Lambda^p(V^*)$ and $S \in \Lambda^q(V^*)$. We call $T \wedge S \in \Lambda^{p+q}(V^*)$ the wedge product, and we define it by the formula $T \wedge S = \mathrm{Alt}(T \otimes S)$.

One especially important property of the wedge product is associativity. In order to prove that it possess this property, however, we need to prove the following lemma.

Lemma 2.9. *If Alt(T) = 0, then* $T \wedge S = 0 = S \wedge T$.

Proof. S_{p+q} carries a natural copy of S_p: the subgroup G consisting of all permutations of $(1, \ldots, p+q)$ that fix $p+1, \ldots, p+q$. The correspondence between G and S_p assigns to each $\pi \in G$ the permutation π' caused by limiting π to $(1, \ldots, p)$. Note that $(T \otimes S)^\pi = T^{\pi'} \otimes S$, and $(-1)^\pi = (-1)^{\pi'}$. Thus

$$\sum_{\pi \in G} (-1)^\pi (T \otimes S)^\pi = \left[\sum_{\pi' \in S_p} (-1)^{\pi'} T^{\pi'} \right] \otimes S = Alt(T) \otimes S = 0.$$

Now a subgroup G decomposes S_{p+q} into a disjoint union of right cosets $G \circ \sigma = \{\pi \circ \sigma : \pi \in G\}$. But for each coset

$$\sum_{\pi \in G} (-1)^{\pi \circ \sigma} (T \otimes S)^{\pi \circ \sigma} = (-1)^\sigma \left[\sum_{\pi \in G} (-1)^\pi (T \otimes S)^\pi \right]^\sigma = 0.$$

Since $T \wedge S = \mathrm{Alt}(T \otimes S)$ is the sum of these partial summations over the right cosets of G, then $T \wedge S = 0$. Similarly, $S \wedge T = 0$. ☐

Theorem 2.10. *Wedge product is associative,*

$$(T \wedge S) \wedge R = T \wedge (S \wedge R),$$

justifying the notation $T \wedge S \wedge R$.

Proof. We claim that $(T \wedge S) \wedge R$ equals $\mathrm{Alt}(T \otimes S \otimes R)$. By definition,

$$(T \wedge S) \wedge R = Alt((T \wedge S) \otimes R),$$

so the linearity of Alt implies

$$(T \wedge S) \wedge R - Alt(T \otimes S \otimes R) = Alt([T \wedge S - T \otimes S] \otimes R).$$

Since $T \wedge S$ is alternating,

$$Alt(T \wedge S - T \otimes S) = Alt(T \wedge S) - Alt(T \otimes S) = T \wedge S - T \wedge S = 0.$$

So the lemma implies

$$Alt([T \wedge S - T \otimes S] \otimes R) = 0,$$

as needed. Using the linearity of Alt as well as the alternating properties of the wedge product, we can follow a similar set of steps to show that

$$T \wedge (S \wedge R) = Alt(T \otimes S \otimes R).$$

Since $(T \wedge S) \wedge R$ and $T \wedge (S \wedge R)$ equal the same thing, we conclude that they are equal to each other. □

We state, without proof, the following fact about the wedge product.

Fact 2.11. *The wedge product is anticommutative in the following sense:*

$$T \wedge S = (-1)^{pq} S \wedge T,$$

where T and S are alternating p- and q-tensors, respectively.

Given a linear transformation $A : V \to W$, we define the transpose map $A^* : \Lambda^p(W^*) \to \Lambda^p(V^*)$ as follows. If $T \in \Lambda^p(W^*)$, then $A^*T(v_1, \ldots, v_p) = T(Av_1, \ldots, Av_p)$ and $A^*T \in \Lambda^p(V^*)$. Since it will require us to use some definitions and ideas beyond the scope of this paper, we will simply state the following theorem.

Theorem 2.12. *(Determinant Theorem) If $A : V \to V$ is a linear map, then $A^*T = (\det A)T$ for every $T \in \Lambda^k(V)$, where $k = dim\ V$. In particular, if $\phi_1, \ldots, \phi_k \in \Lambda^1(V^*)$, then $A^*\phi_1 \wedge \cdots \wedge A^*\phi_k = (\det A)\phi_1 \wedge \cdots \wedge \phi_k$..*

3. Differential Forms and Exterior Derivatives

In this section, we use the tools we developed from exterior algebra and begin to apply them to our setting of manifolds. The result is known as the differential form, or a p-form, and we construct our first operator that can be use on a differential form: the exterior derivative.

Definition 3.1. Let X be a smooth manifold. A p-form on X is a function ω that assigns to each point $x \in X$ an alternating p-tensor $\omega(x)$ on the tangent space of X at x; more tersely, $\omega(x) \in \Lambda^p[T_x(X)^*]$.

Examples 3.2. Real-valued functions on a manifold X are known as 0-forms.
If you have an arbitrary smooth function $\phi : X \to \mathbb{R}$, then you can construct a 1-form $d\phi_x : T_x(X) \to \mathbb{R}$, known as the differential, which is linear.

Definition 3.3. Let X and Y be smooth manifolds with or without boundary, and let $df_x : T_x(X) \to T_y(Y)$ be the derivative map from one tangent space of a manifold to another tangent space of another manifold. Then we define the *transpose map* which maps forms on Y to X by $f^*\omega(x) = (df_x)^*\omega[f(x)]$. If ω is a 0-form, $f^*\omega = \omega \circ f$.

Proposition 3.4. *Suppose that $f : V \to U$ is a diffeomorphism of two open sets in $\mathbb{R}^k$. Then $f^*(dx_1 \wedge \cdots \wedge dx_k) = \det(df) dy_1 \wedge \cdots \wedge dy_k$.*

Proof. If $f(y) = x$, let the standard basis of linear functions on $\mathbb{R}^k$ be written as $dy_1(y), \ldots, dy_k(y)$ for $T_y(V)$ and $dx_1(x), \ldots, dx_k(x)$ for $T_x(U)$. By the Determinant Theorem,

$$f^*\omega\,(y) = (df_y)^*\,dx_1(x) \wedge \cdots \wedge (df_y)^* dx_k(x) = \det(df_y)\,dy_1(y) \wedge \cdots \wedge dy_k(y).$$

□

Now that we understand the idea of a differential form, our next objective is to generate differential forms from ones that we already have. While this is possible to do in multiple ways, we make this idea precise via the exterior derivative. Given a p-form, the exterior derivative is an operator that transforms this form into a $(p + 1)$-form. More precisely, we have the following definition.

Definition 3.5. If we ω is a 0-form, we know that that the differential is a 1-form that can be expressed as $df = \sum \frac{\partial f}{\partial x_i} dx_i$. Now, let $\omega = \sum a_I dx_I$ is a smooth p-form on an open subset of $\mathbb{R}^k$. Call the *exterior derivative* of ω the $(p+1)$-form $d\omega = \sum da_I \wedge dx_I$.

Theorem 3.6. *The exterior differentiation operator defined for forms on arbitrary manifolds with boundary has the following properties:*

1. $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$.
2. $d(\omega \wedge \theta) = (d\omega) \wedge \theta + (-1)^k \omega \wedge d\theta$, *where ω is a k-form.*
3. $d(d\omega) = 0$.

Proof. 1. Let $\omega_1 = \sum_I a_I dx_I, \omega_2 = \sum_I b_I dx_I$, where a_I and b_I are functions. Then

$$d(\omega_1 + \omega_2) = d\left(\sum_I a_I\, dx_I + \sum_I b_I\, dx_I\right) = d\left(\sum_I (a_I\, dx_I + b_I\, dx_I)\right)$$
$$= d\left(\sum_I (a_I + b_I)\, dx_I\right) = \left(\sum_I d(a_I + b_I) \wedge dx_I\right) = \left(\sum_I (da_I + db_I) \wedge dx_I\right)$$
$$= \left(\sum_I da_I \wedge dx_I + db_I \wedge dx_I\right) = \left(\sum_I da_I \wedge dx_I\right) + \left(\sum_I db_I \wedge dx_I\right) = d\omega_1 + d\omega_2.$$

2. From [2], we have the following inductive proof. Take $\omega_1 = f$ and $\omega_2 = g$ to be 0-forms. Then

$$d(\omega_1 \wedge \omega_2) = \sum_I d(f \cdot g)\, dx_I = \sum_I (df \cdot g + f \cdot dg)\, dx_I = \sum_I df \cdot g\, dx_I$$
$$+ \sum_I f \cdot dg\, dx_I = df \wedge g + f \wedge dg = d\omega_1 \wedge \omega_2 + \omega_1 \wedge d\omega_2.$$

Now let ω and θ be k- and ℓ-forms, respectively. Using the base case above as well as the anticommutativity of the wedge product, we have the following:

$$\begin{aligned} d(\omega \wedge \theta) = d(f \cdot g\, dx_I \wedge dx_J) &= (d(f \cdot g) \wedge dx_I \wedge dx_J) \\ &= (df \wedge g + f \wedge dg) \wedge dx_I \wedge dx_J \\ &= (df \wedge g \wedge dx_I \wedge dx_J) + (f \wedge dg \wedge dx_I \wedge dx_J) \\ &= (d \wedge dx_I) \wedge (g \wedge dx_J) + (-1)^k\, (f \wedge dx_I) \wedge (dg \wedge dx_J) \\ &= d\omega \wedge \theta + (-1)^k\, \omega \wedge d\theta. \end{aligned}$$

3. Let $\omega = \sum_I a_I\, dx_I$ be a k-form. By definition, $d\omega = \sum_I da_I \wedge dx_I$. We then apply the operator in question a second time:

$$\begin{aligned} d(d\omega) &= d\left(\sum_I a_I \wedge dx_I\right) = d\left(\sum_I \left(\sum_i \frac{\partial a_I}{\partial x_i}\right) \wedge dx_I\right) \\ &= \sum_I \sum_i \left(\sum_j \frac{\partial^2 a_I}{\partial x_i \partial x_j} dx_J\right) \wedge dx_i \wedge dx_I. \end{aligned} \tag{3.7}$$

Using a basic property of iterated differentiation, with the anticommutativity of the wedge product (i.e., $dx_j \wedge dx_i = -dx_i \wedge dx_j$), all of the terms in (3.6) cancel, yielding the desired conclusion. □

4. Integration on Manifolds

One tool remains to the construct: the integral over manifolds. We open this section by recalling a familiar theorem from analysis—the change of variables in $\mathbb{R}^k$—and restating it using the tools we have learned thus far. From there, we construct the integral of an arbitrary compactly supported smooth k-form ω on a k-dimensional manifold.

Recall 4.1. In analysis, we have the following theorem for the change of variables in $\mathbb{R}^k$: assume that $f : U \to V$ is a diffeomorphism of open sets in $\mathbb{R}^k$ and that a is an integrable function on U. Then

$$\int_U a\, dx_1 \cdots dx_k = \int_V (a \circ f)|\det(df)| dy_1 \cdots dy_k.$$

Theorem 4.2. *(Change of Variables in $\mathbb{R}^k$) Assume that $f : U \to V$ is an orientation-preserving diffeomorphism of open sets in $\mathbb{R}^k$ or H^k, and let ω be an integrable k-form on U. Then*

$$\int_U \omega = \int_V f^* \omega.$$

Proof. Let $\omega = a\, dx_1 \cdots dx_k$ be a k-form. Using Proposition 3.3, we know that $f^*(\omega) = (a \circ f) \det(df)\, dy_1 \wedge \cdots \wedge dy_k$. Since we assumed that f is orientation-preserving, $\det(df) > 0$, which implies $|\det(df)| = \det(df)$. Substituting this into the above change of variables formula, we get the desired result. □

We formally define the support of a differential to be the set of all points x on a manifold X such that the form evaluated at x is nonzero.

Definition 4.3. Let ω be a smooth k-form on X, a k-dimensional manifold with boundary. The support of ω is defined as the closure of the set of points where $\omega(x) \neq 0$; we say that ω is compactly supported if the support is compact.

Definition 4.4. Let ω be an arbitrary, compactly supported, smooth k-form on a manifold X. Take a subordinate partition of unity $\{\rho_i\}$ for a given open cover U_α. Then we define the integral of ω on X to be the following:

$$\int_X \omega = \sum_i \int \rho_i \omega.$$

Proposition 4.5. *$\int_X \omega$ is well-defined.*

Proof. Suppose that $\{\rho'_j\}$ is another partition of unity. Then for each i,

$$\int_X \rho_i \omega = \sum_j \int_X \rho'_j \rho_i \omega;$$

similarly, for each j,

$$\int_X \rho'_j \omega = \sum_i \int_x \rho_i \rho'_j \omega.$$

Combining these results gives us the following:

$$\sum_i \int_X \rho_i \omega = \sum_i \sum_j \int_X \rho'_j \rho_i \omega = \sum_j \sum_i \int_X \rho_i \rho'_j \omega = \sum_j \int_X \rho'_j \omega,$$

which implies that $\int_X \omega$ is well-defined. □

Using our definition of the integral over manifolds, we extend our earlier change of variables theorem to manifolds in $\mathbb{R}^N$.

Theorem 4.6. *If $f : X \to Y$ is an orientation-preserving diffeomorphism, then*

$$\int_X \omega = \int_Y f^* \omega$$

for every compactly supported, smooth k-form on $X (k = dimX = dimY)$.

Proof. Let $\{\rho_i\}$ be a partition of unity of a compactly supported, smooth k-form on X subordinate to a relatively open cover $\{U_\alpha\}$. Furthermore, for all α, let $f^{-1}(V_\alpha) = U_\alpha$, where $\{V_\alpha\}$ is an open cover of Y. Applying our modified change of variables theorem, we have the following consequence:

$$\int_X \omega = \sum_i \int_{U_\alpha} \rho_i \omega = \sum_i \int_{V_\alpha} f^*(\rho_i \omega) = \sum_i \int_{V_\alpha} (\rho_i \circ f) f^* \omega = \int_Y f^* \omega.$$

□

5. The Generalized Stokes' Theorem

Utilizing all of the machinery we have developed, we now prove the main result of this paper.

Theorem 5.1. *Suppose that X is any compact oriented k-dimensional manifold with boundary, so ∂X is a $k-1$ dimensional manifold with the boundary orientation. If ω is any smooth $(k-1)$-form on X, then*

$$\int_{\partial X} \omega = \int_X d\omega$$

Proof. Both sides of the equation are linear in ω, so we may assume ω to have compact support contained in the image of a local diffeomorphism h: $U \to X$, where U is an open subset of $\mathbb{R}^k$ or H^k.

The rest of the proof proceeds by considering the theorem by cases. Our first case will consist of a neighborhood of X locally diffeomorphic to an open subset of $\mathbb{R}^k$, where we expect both sides of the theorem to evaluate to 0 since there is no boundary over which to evaluate the differential form. Our second case will consist of a neighborhood of X locally diffeomorphic to an open subset of H^k.

First, assume U is open in $\mathbb{R}^k$. Then

$$\int_{\partial X} \omega = 0 \quad \textit{and} \quad \int_X d\omega = \int_U h^*(d\omega) = \int_U d\nu,$$

where $\nu = h^* \omega$. Since ν is a $(k-1)$-form in k-space, it can be expressed as

$$\nu = \sum_{i=1}^{k} (-1)^{i-1} f_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k$$

Here the $\widehat{dx_i}$ means the term dx_i is ommitted. Then

$$d\nu = \left(\sum_i \frac{\partial f_i}{\partial x_i} \right) dx_1 \wedge \ldots \wedge dx_k$$

and

$$\int_{\mathbb{R}^k} d\nu = \sum_i \int_{\mathbb{R}^k} \frac{\partial f_i}{\partial x_i} dx_1 \ldots dx_k. \tag{5.2}$$

The integral over $\mathbb{R}^k$ is computed via iterated integrals over $\mathbb{R}^1$, which may be taken in any order due to Fubini's Theorem. Integrate the ith term first with respect to x_i:

$$\int_{\mathbb{R}^{k-1}} \left(\int_{-\infty}^{\infty} \frac{\partial f_i}{\partial x_i} \right) dx_1 \cdots \widehat{dx_i} \cdots dx_k.$$

Of course,

$$\int_{-\infty}^{\infty} \frac{\partial f_i}{\partial x_i} dx_i$$

is the function of $x_1, \ldots, \widehat{x_i}, \ldots, x_k$ that maps to any $(k-1)$-tuple $(b_1, \ldots, \widehat{b_i}, \ldots, b_k)$ the number $\int_{-\infty}^{\infty} g'(t)dt$, where $g(t) = f_i(b_1, \ldots, t, \ldots, b_k)$. Since ν has compact support, g vanishes outside any sufficiently large interval $(-a, a)$ in $\mathbb{R}^1$. Therefore the Fundamental Theorem of Calculus implies

$$\int_{-\infty}^{\infty} g'(t)\, dt = \int_{-a}^{a} g'(t)\, dt = g(a) - g(-a) = 0 - 0 = 0.$$

Thus $\int_X d\omega = 0$, as desired because this implies that the theorem holds since both sides of the equation are evaluated to 0.

We now take a look at our second case of the theorem. When $U \subset H^k$, we repeat the above process for every term of (5.2) except the last term. Since the boundary of H^k is the set where $x_k = 0$, the last integral is

$$\int_{\mathbb{R}^{k-1}} \left(\int_0^{\infty} \frac{\partial f_k}{\partial x_k}\, dx_k \right) dx_1 \cdots dx_k.$$

Now compact support implies that f_k vanishes if x_k is outside some large interval $(0, a)$, but although $f_k(x_1, \ldots, x_{k-1}, a) = 0$, $f_k(x_1, \ldots, x_{k-1}, 0) \neq 0$. Thus applying the Fundamental Theorem of Calculus as we did above, we obtain

$$\int_X d\omega = \int_{\mathbb{R}^{k-1}} -f_k(x_1, \ldots, x_{k-1}, 0) dx_1 \cdots dx_{k-1}.$$

On the other hand,

$$\int_{\partial X} \omega = \int_{\partial H^k} \nu.$$

Since $x_k = 0$ on ∂H^k, $dx_k = 0$ on ∂H^k as well. Consequently, if $i < k$, the form $(-1)^{i-1} f_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx^k$ restricts to 0 on ∂H^k. So the restriction of ν to ∂H^k is $(-1)^{k-1} f(x_1, \ldots, x_k, 0) dx_1 \wedge \cdots \wedge dx_{k-1}$, whose integral over ∂H^k is therefore $\int_{\partial X} \omega$.

Now ∂H^k is diffeomorphic to $\mathbb{R}^{k-1}$ under the map $(x_1, \ldots, x_{k-1}) \to (x_1, \ldots, x_{k-1}, 0)$, but this diffeomorphism does not always carry the usual orientation of $\mathbb{R}^{k-1}$ to the boundary orientation of ∂H^k. Let $e_1, \ldots, e_k$ be the standard ordered basis for $\mathbb{R}^k$, so $e_1, \ldots, e_{k-1}$ is the standard ordered basis for $\mathbb{R}^{k-1}$. Since H^k is the upper half-space, the outward unit normal to ∂H^k is $-e_k = (0, \ldots, 0, -1)$. Thus in the boundary orientation of ∂H^k, the sign of the ordered basis $\{-e_k, e_1, \ldots, e_{k-1}\}$ is the standard orentation of H^k. The latter sign is easily seen to be $(-1)^k$, so the usual diffeomorphism $\mathbb{R}^k \to \partial H^k$ changes orientation by the factor $(-1)^k$.

The result is the following formula.

$$\begin{aligned}\int_{\partial X} \omega &= \int_{\partial H^k} (-1)^{k-1} f_k(x_1, \ldots, x_{k-1}, 0) dx_1, \cdots, dx_{k-1} \\ &= (-1)^k \int_{\mathbb{R}^{k-1}} (-1)^{k-1} f_k(x_1, \ldots, x_{k-1}, 0) dx_1 \cdots dx_{k-1}.\end{aligned}$$

Since $(-1)^k(-1)^{k-1} = -1$, it is exactly the formula we derived for $\int_X d\omega$. Since both sides of the theorem are evaluated to the same value, they are equivalent, which means the theorem holds for subset of H^k. $\square$

With the Generalized Stokes' Theorem proved, we touch on a few important results. Most notably, the four higher-dimensional analogs of the Fundamental Theorem of Calculus (specifically, the Fundamental Theorem for Line Integrals, Green's Theorem, the Clasical Stokes' Theorem, and the Divergence Theorem) are immediate results of the Generalized Stokes' Theorem; however, it is important to point out that even though we could say the Fundamental Theorem of Calculus is also a result, it would be circular since we used it in our proof. Moreover, our result is applicable in the study of complex analysis and the study of the cohomology of forms.

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