MEROMORPHIC FORMS ON RIEMANN SURFACE STRUCTURES

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ABSTRACT. A Riemann surface is a geometric object that is locally modeled on the complex plane. In this paper we extend the notions of holomorphic and meromorphic 1-forms from the complex plane to general Riemann surfaces. In doing so we prove the 1-form case of the general Hodge decomposition theorem, which allows us to decompose the space of 1-forms with compact support into exact, co-exact, and harmonic forms. We then show that this decomposition is a crucial step in proving the remarkable result that every Riemann surface admits a non-constant meromorphic function. From this, we look at linear subspaces of meromorphic 0-forms and 1-forms generated on Riemann surfaces and prove the Riemann-Roch theorem, which gives us an algebraic description of meromorphic functions from Riemann surfaces. We non-trivially apply this theorem to classify compact, simply-connected Riemann surfaces.

CONTENTS

1. INTRODUCTION

A Riemann surface is a topological surface with a fixed conformal structure. Since a Riemann surface locally behaves like the complex plane we can extend definitions from complex analysis in the complex plane to Riemann surfaces. Notably, the definitions for holomorphic and meromorphic functions between subsets of the complex plane have natural extensions to Riemann surfaces. One reason for this is that holomorphicity and meromorphicity are defined locally and so their extension to Riemann surfaces implicitly preserves that notion. In section 2.1 we formalize this extension.

Once we define holomorphic and meromorphic functions it is easy to define holomorphic and meromorphic 1-forms because they are also expressed locally. We

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conclude section 2.1 by defining these notions. We begin section 2.2 by looking at the set of 1-forms with compact support, denoted by $\Omega^1_{\text{comp}}(M;\mathbb{C})$ and define the Hodge operator. We consider an inner product on this space using the Hodge operator and we write $\Omega^1_2(M;\mathbb{C})$ as the completion of this space under the inner product. We find that this space has a nice decomposition as the direct sum of exact, co-exact, and harmonic 1-forms, all of which are defined in 2.3. This is the 1-form case of the Hodge decomposition theorem and we provide a proof in section 2.3. This decomposition is essential to proving the main result of section 3, namely that every Riemann surface admits a non-constant meromorphic function, because it allows us to construct specific harmonic forms.

In section 4 we examine linear subspaces of meromorphic functions generated on Riemann surfaces. More specifically, given a Riemann surface of genus g, and divisor D, we generate subspaces $L(D)$ of meromorphic functions and $\Omega(D)$ of meromorphic 1-forms. We then show that there is an algebraic relationship between them and the topological genus g of the Riemann surface, namely that

$$\text{dim}L(-D) = \text{deg}D + 1 - g + \text{dim}\Omega(D).$$

This result is known as the Riemann-Roch theorem and we conclude section 4 by applying the Riemann-Roch theorem to classify compact, simply-connected Riemann surfaces. We do this by showing that they are conformally isomorphic to the Riemann sphere. (Example 2.2)

In this paper we assume basic knowledge of complex analysis. The reader is expected to be familiar with holomorphic and meromorphic functions on the complex plane. We also assume knowledge of basic differential topology such as integration over manifolds and differential forms. For a more in depth understanding of required complex analysis see [1] and [2]. The differential topology is well explained in [5]. Schlag's Notes on Complex Analysis was a great reference for the Hodge decomposition theorem and the existence of a non-constant meromorphic function.

2. Riemann Surfaces and Complex Manifolds

In this section we begin by defining a Riemann surface and by presenting a few examples, most notably the Riemann sphere. We then talk about differential forms on Riemann surfaces and introduce notions of holomorphic and meromorphic forms. This leads us to the discussion of the Hodge $*$ operator and the completion of the compactly supported harmonic 1-forms. We conclude this section by presenting the statement and proof of Hodge's theorem.

Definition 2.1. A **Riemann surface** M is a two dimensional, orientable, connected, Hausdorff topological manifold with a countable base for the topology and with conformal transition maps between charts. Specifically, there exists a family of open sets $\{U_\alpha\}_{\alpha\in\mathbb{A}}$ covering M and homeomorphisms $\phi_\alpha : U_\alpha \to V_\alpha$ where $V_\alpha \subset \mathbb{R}^2$ is some open set so that

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

is a conformal homeomorphism.

Since the transition maps in a Riemann surface are conformal homeomorphisms, we can really think of the homeomorphism maps as going into $\mathbb{C}$ instead of $\mathbb{R}^2$. This allows us to extend many definitions and theorems from complex analysis in the complex plane to general Riemann surfaces. There are a few trivial examples of Riemann surfaces, namely any open $\Omega \subset \mathbb{C}$. In this case, we take only one chart $\{(\Omega, z)\}$. Now we will present two non-trivial Riemann surfaces.

Example 2.2. The Riemann Sphere, denoted by $\mathbb{C}_\infty$ is $\mathbb{C} \cup \{\infty\}$. One set of charts is $\left\{(\mathbb{C}, z), (\mathbb{C}_\infty - \{0\}, \frac{1}{z})\right\}$. Thus, the transition map is just $z \to \frac{1}{z}$.

Example 2.3. $S^2 \subset \mathbb{R}^3$ is a Riemann surface. Our set of charts is the following:

$$(S^2 \backslash (0,0,1), \phi_+), (S^2 \backslash (0,0,-1), \phi_-)$$

where $\phi_\pm$ are the stereographic projections

$$\phi_+(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3}.$$
$$\phi_-(x_1, x_2, x_3) = \frac{x_1 - ix_2}{1 + x_3}.$$

If $p = (x_1, x_2, x_3) \in S^2$ with $x_3 \neq \pm 1$, then

$$\phi_+(p)\phi_-(p) = 1.$$

This shows that the transition map between the two charts is $z \to \frac{1}{z}$ just like in the Riemann Sphere.

The two surfaces in the above examples are isomorphic via the projection map. This is because it takes every point on the sphere and maps it injectively into the complex plane. Now the inverse map for the positive projection is

$$\phi_+^{-1}(x + iy) = \frac{(2x, 2y, x^2 + y^2 - 1)}{1 + x^2 + y^2}.$$

In this projection every point in $z \in \mathbb{C}$ gets mapped onto S^2. However the inverse map is not surjective because we do not recover $(0,0,1)$. If we take $\mathbb{C} \cup \{\infty\}$ and set $\infty = (0,0,1)$ then we have an isomorphism. This is why $\mathbb{C} \cup \infty$ is referred to as the Riemann sphere.

2.1. **Holomophic and Meromorphic Forms.** We introduce holomorphic and meromorphic maps in the Riemann surface in the following definition and state two theorems and a corollary that have direct analogues in complex analysis over the complex plane.

Definition 2.4. A continuous map $f : M \to N$ between Riemann surfaces is said to be **holomorphic** (analytic) if and only if it is holomorphic in charts. In other words, if $p \in M$ is arbitrary and $p \in U_\alpha$, $f(p) \in V_\beta$ where (U_α, z_α) is a chart of M and (V_β, w_β) is a chart of N, respectively, then $w_\beta \circ f \circ z_\alpha^{-1}$ is holomorphic where it is defined. Similarly a map $f : M \to N$ is **meromorphic** if it is meromorphic in its charts.

There are several theorems that follow easily from the above definition and their analogous ones in complex analysis. For a proof of these theorems see [1]. Let M and N be Riemann surfaces.

Theorem 2.5. *(Uniqueness Theorem) Let $f, g : M \to N$ be analytic. Then either $f = g$ or $\{p \in M | f(p) = g(p)\}$ is discrete in M.*

Theorem 2.6. *(Open Mapping Theorem) A non constant analytic map between Riemann surfaces is an open map.*

Corollary 2.7. *Let M be compact and $f : M \to N$ analytic. If f is not constant, then $f(M) = N$.*

As a smooth manifold, M carries k-forms which we denote by $\Omega^k(M;\mathbb{C})$ (see [3] or [5]) for definition. That being said, $\Omega^0(M;\mathbb{C})$ are simply C^∞ functions on M. Now, any $\omega \in \Omega^1(M;\mathbb{C})$ defines $\omega_p : T_p(M) \to \mathbb{C}$. Now we have a theorem that is very simple but useful.

Theorem 2.8. *If $T : V \to W$ is a $\mathbb{R}$-linear map between complex vector spaces , then there is a unique representation $T = T_1 + T_2$ where T_1 is complex linear and T_2 is complex anti-linear.*

Proof. The proof is straightforward. Take $T_1 = \frac{1}{2}(T - iTi)$ and $T_2 = \frac{1}{2}(T + iTi)$. $\square$

This lets us write $\omega \in \Omega(M;\mathbb{C})$ as follows:

$$\begin{aligned}\omega &= udx + vdy\\ &= \frac{1}{2}(a - ib)dz + \frac{1}{2}(a + ib)d\bar{z}\\ &= udz + vd\bar{z}.\end{aligned}$$

Now suppose $\omega = df$ where $f \in \Omega(M;\mathbb{C})$,then

$$df = \partial_z f dz + \partial_{\bar{z}} f d\bar{z}. \tag{2.9}$$

At this point we have a natural way of defining holomorphic 1-forms. Recall that if f is holomorphic, then $\partial_{\bar{z}} f = 0$. That being said, if $\partial_{\bar{z}} f = 0$ for f in (2.9) then df is holomorphic. The below definition formalizes this.

Definition 2.10. The **holomorphic forms** on a Riemann surface M, denoted by $\mathfrak{H}\Omega(M)$, are those $\omega \in \Omega^1(M;\mathbb{C})$ so that $\omega = udz$ where u is holomorphic. The **meromorphic forms** on a Riemann surface M are defined similarly except u is meromorphic instead of holomorphic. The space of meromorphic differentials is denoted by $\mathfrak{M}\Omega(M)$

2.2. **The Hodge $*$ operator and harmonic forms.**

Definition 2.11. To every $\omega \in \Omega^1(M;\mathbb{C})$ we associate a one-form $*\omega$ defined as follows: if $\omega = adz + vd\bar{z} = fdx + gdy$ in local coordinates then $*\omega = -iadz + ivd\bar{z} = -gdx + fdy$. Moreover, if $\omega, \eta \in \Omega^1_{\text{comp}}(M;\mathbb{C})$(the forms with compact support), we set

$$\langle \omega, \eta \rangle = \int_M \omega \wedge \overline{*\eta}. \tag{2.12}$$

This defines an inner product on $\Omega^1_{\text{comp}}(M;\mathbb{C})$. The completion of this space is denoted by $\Omega^1_2(M;\mathbb{C})$. Note that we can

We see that $*\omega$ is well defined because it follows from the change in coordinates $z = z(w)$. Then, $\omega = udz + vd\bar{z} = uz'd\omega + v\overline{z'}d\bar{\omega}$. Setting $uz'd\omega = -iuz$ and $u\overline{z'}d\overline{\omega} = ivd\bar{z}$ we get $*\omega$.

The following are some properties of the operator.

Lemma 2.13. *For any $\omega, \eta \in \Omega^1_2(M)$ we have*

$$*\overline{\omega} = \overline{*\omega}$$
$$* * \omega = -\omega$$
$$\langle *\omega, *\eta \rangle = \langle \omega, \eta \rangle .$$

We now come to the topic of harmonic forms and we start with the following definition

Definition 2.14. We say that $f \in \Omega^0(M;\mathbb{C})$ is **harmonic** if f is harmonic in every chart. In other words if (U_α, ϕ_α) is a chart in M and (V_α, ψ_α) is a chart in $\mathbb{C}$ then $\triangle(\phi \circ f \circ \psi^{-1}) = 0$, where $\triangle$ is the Laplacian. We say that $\omega \in \Omega^1(M;\mathbb{C})$ is harmonic if and only if $d\omega = d * \omega = 0$. We denote the harmonic forms on M by $h(M;\mathbb{R})$ and $h(M;\mathbb{C})$ if they are real or complex valued respectively.

Now we will give the statement and application of the Hodge decomposition theorem.

Theorem 2.15 (Hodge decomposition for 1-forms)**.** $\Omega^1_2(M;\mathbb{R}) = E \bigoplus *E \bigoplus h_2(M;\mathbb{R})$ *where* $h_2(M;\mathbb{R}) = h(M;\mathbb{R}) \cap \Omega^1_2(M;\mathbb{R})$. $E = \overline{\{df | f \in \Omega^0_{comp}(M;\mathbb{R})\}}$ *and* $*E = \overline{\{*df | f \in \Omega^0_{comp}(M;\mathbb{R})\}}$. *$E$ is referred to as the set of exact forms with compact support and $*E$ is the set of co-exact forms with compact support.*

One interesting application is with $M = \mathbb{C}$. Pick any $\omega \in h_2(M;\mathbb{C})$. Then we can write $w = adx + bdy$ where a, b are harmonic and L^2 bounded. Thus,

$$\int\int_{\mathbb{R}^2} (|a|^2 + |b|^2) dxdy < \infty.$$

We claim that $a = b = 0$. By the mean-value theorem we know that

$$|a(z)|^2 = \left| \frac{1}{D(z,r)} \int\int_{D(z,r)} (|a(\psi)|) d\eta d\alpha \right|^2 < \infty.$$

Since a is L^2-bounded we have:

$$\left| \frac{1}{D(z,r)} \int\int_{D(z,r)} (|a(\psi)|) d\eta d\alpha \right|^2 < \frac{1}{|D(z,r)|^2} \int\int_{D(z,r)} (|a(\psi)|^2) d\eta d\alpha < \infty.$$

This tells us,

$$|a(z)|^2 < \frac{||a||_2^2}{|D(z,r)|}.$$

As $r \to \infty$ we have $a(z) \to 0$. Therefore $a = 0$ everywhere. By a similar process we have that $b = 0$ everywhere. Thus, $\omega = 0$ so $h_2(M;\mathbb{C}) = \{0\}$. In the context of Hodge's theorem, this tells us that every $\alpha \in \Omega^1_2(M;\mathbb{C})$ can be written as the sum of an exact and co-exact form.

2.3. **Proof of Hodge's Theorem.** We begin this section by proving a lemma that is an essential first step in the decomposition. It gives us the intermediate decomposition (2.17). We then show that $h_2(M;\mathbb{R}) = \big(E^\perp \cap (*E)^\perp\big)$, which reduces (2.17) to the decomposition in Theorem 2.14, concluding the proof.

Lemma 2.16. *Let $\alpha \in \Omega^1_2(M;\mathbb{R})$ and smooth. Then $\alpha \in E^\perp$ iff $d * \alpha = 0$ and $\alpha \in (*E)^\perp$ iff $d\alpha = 0$.*

Proof. We see that for $\alpha \in E^{\perp}$, and for an arbitrary f smooth and compactly supported,

$$0 = \langle \alpha, df \rangle = \langle *\alpha, *df \rangle = \int_M df \wedge *\alpha$$

$$= \int_M d(f * \alpha) - f d * \alpha = -\int_M f d * \alpha.$$

Since f was arbitrarily chosen it follows that $d * \alpha = 0$. The other direction follows in the same manner. $\square$

The above lemma tells us that we have $E \subset *E^{\perp}$ and $*E \subset E^{\perp}$. This gives us the following decomposition:

$$\Omega_2^1(M; \mathbb{R}) = E \oplus *E \oplus (E^{\perp} \cap *E^{\perp}). \tag{2.17}$$

To finish the proof of the theorem we just have to show that $h(M; \mathbb{R}) = (E^{\perp} \cap *E^{\perp})$

We use the following lemma in the proof of the Hodge decomposition. For a proof see [3].

Lemma 2.18. *(Weyl's Lemma) Let $V \subset \mathbb{C}$ be open and $u \in L^1(V)$. Suppose u is weakly harmonic, i.e.,*

$$\int_V u \triangle \phi dx dy = 0$$

for all $\phi \in C^{\infty}_{comp}(V)$. Then u is harmonic and $\triangle \phi = 0$.

Theorem 2.19. $h_2(M; \mathbb{R}) = \left(E^{\perp} \cap *E^{\perp}\right)$

Proof. For $\omega \in E^{\perp} \cap *E^{\perp}$, we must have

$$\langle \omega, df \rangle = \langle \omega, *df \rangle = 0$$

for all $f \in C^{\infty}_{\text{comp}}(M)$ with f complex valued. We let $\omega = u dz + v d\bar{z}$ in local coordinates and $z = x + iy$. Expanding the inner products we find that

$$\langle \omega, df \rangle = 2 \int (u \overline{f_z} + v \overline{f_{\bar{z}}}) dx dy,$$

$$\langle \omega, *df \rangle = -2i \int (u \overline{f_z} - v \overline{f_{\bar{z}}}) dx dy.$$

This is equivalent to

$$\int \bar{u} f_z dx dy = \int \bar{v} f_{\bar{z}} dx dy = 0$$

for all such f. Now take $g \in C^{\infty}_{\text{comp}}(M)$. In one case we take $f = g_{\bar{z}}$ and in the other we take $f = g_z$ where g is supported in U. This yields

$$\int \bar{u} \triangle g dx dy = \int \bar{v} \triangle g dx dy = 0.$$

Weyl's Lemma tells us that ω is both closed and co-closed and hence harmonic, concluding the proof of Hodge's theorem. $\square$

3. Every Riemann Surface admits a non-constant meromorphic function

Recall that in 2.15 we gave a nice decomposition of the closure of the set of 1-forms with compact support. We will use that decomposition to prove a remarkable, non-trivial theorem; namely, that any Riemann surface admits a non-constant meromorphic function to $\mathbb{C}$. It is important to note that the basis for our proof of this theorem is the answer to the following question: Given $x \in M$, can we find a u harmonic on $M - \{x\}$ that has a $\frac{1}{z}$ or $\log|z|$ singularity in some parametric disc centered at x? We begin the section by answering this question in the context of Hodge's theorem. We spend the latter part showing how the answer to this question implies the existence of a non-constant meromorphic function. We begin with the following theorem.

Theorem 3.1. *Let $\bar{N} \subset M$, $\bar{N}$ compact with smooth boundary. Fix $p_0 \in N$ and h harmonic on $N \backslash \{p_0\}$ with $h \in C^1(\bar{N})$ and $\frac{\partial h}{\partial n} = 0$ on ∂N where n is some normal vector field on ∂N that never vanishes.*

Then there exists u harmonic in $M\ \{p_0\}$, $u - h$ harmonic on N, and $u \in \Omega(M \backslash K)$ for any compact neighborhood K of p_0. Moreover, u is unique up to constants.

Proof. We begin by showing existence. Choose a smooth function θ on N which agrees with h on $N \backslash K$ where K is arbitrary but fixed compact neighborhood of p_0. Then we extend θ to M by letting it equal 0 outside N. Since θ is a smooth function with compact support we can use Hodge's theorem to decompose it.

$$d\theta = g + \alpha + \beta, \text{ where } g \in h_2(M; \mathbb{C}), \alpha \in E, \beta \in *E.$$

If we take a $\phi \in C^\infty_{\text{comp}}(M)$, then

$$\langle d\theta, d\phi \rangle = \langle \alpha, d\phi \rangle .$$

Note that since $\beta + g \in E^\perp$ and $\alpha \in E$,

$$\langle \beta, d\phi \rangle = \langle \alpha, *d\phi \rangle = 0.$$

If $\text{supp}(\phi) \subset M \backslash K$. Then from $\frac{\partial h}{\partial n} = 0$ and $d * dh = 0$ on N, we obtain that

$$\langle d\phi, d\theta \rangle = \int_N d\phi \wedge *d\bar{h} = \int_{\partial N} \phi \overline{v^*(*dh)} = 0.$$

Here $v : \partial N \to M$ is the inclusion map on the boundary and v^* is the pullback. The main point to note here is that $v^*(*dh)$ is proportional to $\frac{\partial h}{\partial n} = 0$. Therefore α is harmonic on $M \backslash K$ since it is perpendicular to both $d\phi$ and $*d\phi$. If $\text{supp}(\phi) \subset N$, then

$$\langle d\theta - \alpha, d\phi \rangle = 0, \ \ \langle d\theta - \alpha, *d\phi \rangle = 0.$$

Thus, $\alpha - d\theta$ is harmonic on N. In particular, α is smooth on M and thus $\alpha = df$ with f smooth. Now we set

$$u = f - \theta + h.$$

To show existence, we suppose there exists v that has the same properties as u. Then $u - v$ would be harmonic on M and $d(u - v) \in \Omega^1_2(M; \mathbb{C})$. In conclusion, $d(u - v) \in E \cap h = \{0\}$, so $u - v$ is constant. □

We note that the same exact proof allows for several exceptional points $x_0, x_1, \cdots, x_k \in N$. Keep in mind that we still need to find a harmonic function that has either a $\frac{1}{z}$ or $\log|z|$ singularity. The following corollaries are critical to finding such a harmonic function.

Corollary 3.2. *Given $n \geq 1$ and a coordinate chart (D, ϕ) around $x \in M$ with $\phi(x) = 0$ there is u harmonic on $M - \{x\}$ with $u - z^{-n}$ harmonic on U and $du \in \Omega_2^1(M - K)$ for any compact neighborhood K of x.*

Proof. Without loss of generality we can take $\phi(D) = \mathbb{D}$ and have $\phi(x) = 0$ by the Riemann mapping theorem. We define $h(z) = z^{-n} + \bar{z^n}$ for every $z \in \mathbb{D}$. Take $N = \bar{D}$. Now h is harmonic on N. By the preceding theorem we have that there exists a harmonic ω on $D - \{x\}$ with $\omega - h$ harmonic on D. We see that this implies that $\omega - z^{-n}$ is harmonic on D since $\omega - h$ effectively gets rid of the z^{-n} term. □

Corollary 3.3. *Let $x_0, x_1 \in M$ be distinct and suppose z and λ are local coordinates around x_0 and x_1, respectively. Then there exists u harmonic on $M - \{x_0, x_1\}$ with $u - log|z|$ and $u + log|\lambda|$ harmonic locally around x_0, x_1 respectively. Moreover, $du \in \Omega_2^1(M - K)$ where K is any compact neighborhood of $\{x_0, x_1\}$.*

Proof. See [3] for proof. □

Corollary 3.4. *Given a Riemann surface M:*

1. *For $n \geq 1$ and $x_0 \in M$ there exists a meromorphic differential ω with $\omega - \frac{dz}{z^{n+1}}$ holomorphic locally around x_0. Moreover, $\omega \in \Omega_2^1(M - K)$ for ever compact neighborhood K of x_0.*
2. *Let x_0, $x_1 \in M$. There exists ω meromorphic on M with $\omega - \frac{dz}{z}$ holomorphic around x_0 and $\omega + \frac{d\lambda}{\lambda}$ holomorphic around x_1, respectively (z, λ are local coordinates). Moreover, $\omega \in \Omega_2^1(M - K)$ for every compact neighborhood K of $\{x_0, x_1\}$.*

Proof. With u in Corollary 3.3 and 3.4 respectively, we set $\alpha = du$. For (1) we take $\omega = \frac{-1}{2n}(\alpha + i{*}\alpha)$, whereas for (2), we take $\omega = \alpha + i{*}\alpha$ □

This leads us to the following theorem

Theorem 3.5. *Let $\{x_j\}_{j=1}^J \subset M$, $J \geq 2$, and $c_j \in \mathbb{C}$ with $\sum_{j=1}^J c_j = 0$. Then there exists a meromorphic differential ω, holomorphic on $M - \{x_1, x_2, \cdots, x_J\}$ so that ω has a simple pole at each x_j with residue c_j.*

Proof. Pick any other point $x_0 \in X$ and let ω_j be meromorphic with simple poles at x_0, x_j and residues $-c_j, c_j$ respectively. We know that it will have residues of this form by Corollary 3.4. The differential $\omega = \sum_{j=1}^J \omega_j$ has the desired properties. □

We now have all the tools necessary to prove that every Riemann surface admits a non-constant meromorphic function.

Theorem 3.6. *Every Riemann surface M carries a non-constant meromorphic function.*

Proof. Take three points $x_0, x_1, x_2 \in M$ and let ω_1 be a meromorphic one-form with simple poles at x_0, x_1 and residues $1, -1$ respectively and holomorphic everywhere else. Similarly, let ω_2 be a meromorphic one-form with simple poles at x_1, x_2 and residues $-1, 1$ respectively and holomorphic everywhere else. We cant do this by Theorem 3.6. Now set $f = \frac{\omega_1}{\omega_2}$ where the division is well-defined in local coordinates and this defines a meromorphic function. Since ω_1 and ω_2 both have poles at x_1 with residues -1 we see that $f(x_1) = 1$. At x_2, ω_1 is fixed, whereas ω_2 has a simple pole. Thus, $f(x_2) = 0$ and f is not constant. □

4. Riemann-Roch Theorem

In the previous section we answered the following question: does every Riemann surface admit a non-constant meromorphic function?Now we deal with a more abstract question: what sort of meromorphic functions can Riemann surfaces admit? In doing so we prove a powerful theorem called the Riemann-Roch Theorem, which relates a linear subspace of meromorphic functions on a Riemann Surface to the genus of the surface. As a remarkable consequence we get that every compact Riemann surface of genus 0 is conformally isomorphic only to the Riemann-Sphere. In this section M will denote a compact Riemann surface.

Definition 4.1. A **divisor** D on M is a finite formal sum $D = \sum_v s_v p_v$ where $p_v \in M$ are distinct and $s_v \in \mathbb{Z}$. The **degree** of D is the integer

$$\deg(D) = \sum_v s_v.$$

If $s_v \geq 0$ for all v then D is called **integral**. If f is a non-constant meromorphic function on M, then we define the divisor of f as

$$(f) = \sum_v \operatorname{ord}(f; p_v) p_v$$

where the sum runs over the zeros and poles of f with $\operatorname{ord}(f; p_v) > 0$ being the order at a zero p_v of f and $\operatorname{ord}(f; p_v) < 0$ being the order at a pole p_v of f. In the same way we define the divisor of a non-zero meromorphic differential:

$$(\omega) = \sum_v \operatorname{ord}(\omega; p_v) p_v.$$

Given a divisor D we define the $\mathbb{C}$-linear space

$$L(D) = \{f \in \mathfrak{M}(M) | (f) \geq D \text{ or } f = 0\}.$$

Analogously, we define the space

$$\Omega(D) = \{\omega \in \mathfrak{M}\Omega^1(M) | (\omega) \geq D \text{ or } \omega = 0\}.$$

We now give the definition of degree with respect to functions and present a lemma, both of which are crucial to proving that compact simply-connected Riemann surfaces are conformally isomorphic to the Riemann sphere.

Definition 4.2. Let $f : M \to N$ be analytic and non constant. Then the **valency** of f at $p \in M$, denoted by $\nu_f(p)$, is defined to be the unique positive integer with the property that in the charts (U_α, ϕ) around p (with $f(p) = 0$) and (V, ψ) around $f(p)$ (with $\psi(f(p)) = 0$) we have $(\psi \circ f \circ \phi^{-1})(z) = (zh(z))^n$ where $h(0) \neq 0$. If M is compact, then the **degree of** f at $q \in N$ is defined as

$$deg_f(q) = \sum_{p: f(p) = q} \nu_f(p).$$

Note that since M is compact this sum is finite.

Lemma 4.3. *Let $f : M \to N$ be analytic and non constant with M compact. Then $deg_f(q)$ does not depend on q. It is called the* **degree** *of f and is denoted by $deg(f)$. The isomorphisms from M to N are precisely those non constant analytic maps f on M with $deg(f) = 1$.*

Proof. Since M is compact we know that f is onto N by Corollary 2.7.To prove f is one to one we will first prove that $\deg_f(q)$ is locally constant. Let $f(p) = q$ and suppose that $\nu_f(p) = 1$. As remarked before, f is then an isomorphism from a neighborhood of p onto a neighborhood of q. If, on the other hand, $n = \nu_f(p) > 1$, then each q' close but not equal to q has exactly n pre images and $\nu_f\{p_j^{'}\}_{j=1}^n$ at each $1 \leq j \leq n$. This proves that $\deg_f(q)$ is locally constant and therefore globally constant by the connectivity of N. The statement concerning isomorphisms is evident. □

Definition 4.4. The principal part at $z = a$ of a function $f(z) = \sum_{k=-\infty}^{\infty} a_k(z-a)^k$ is the portion of the Laurent series consisting of terms with negative degree. For example, take $g(z) = \frac{1}{z} + z + z^2 + \cdots$. The principal part of $g(z)$ is $\frac{1}{z}$.

Theorem 4.5 (Reverse Residue)**.** *If we have a set of points $\{a_1, \cdots, a_n\}$ on a compact Riemann surface and also a set of principal parts $\{f_1, \cdots, f_n\}$, then the following are equivalent:*

(1) There exists a meromorphic function f that has principal part f_i at each point a_i and has no other poles.
(2) $\sum_{i=1}^n Res_{a_i} f_i\omega = 0$ for all holomorphic 1-forms ω on our Riemann surface.

Proof. See [3]. □

Remark 4.6. Recall that the meromorphic forms are defined locally. So a meromorphic 1-form ω is written as $\omega = gdz$ where g is meromorphic 0-form. The residues of $f_i\omega$ is the residue of $f_i g$.

This leads us to the proof of the Riemann-Roch theorem.

Theorem 4.7 (Riemann-Roch)**.** *Let D be a divisor on M. Then*

$$\dim L(-D) = \deg(D) + 1 - g + \dim(\Omega(D))$$

This proof will take place in two parts. The first is showing the theorem is true for divisors of positive degree. The second is for general divisors. Here we provide a detailed proof of the first part and encourage the reader to look at [3] for the second part.

Proof. Consider a divisor

$$D = \sum_{i=1}^{n} m_i a_i$$

with $D \geq 0$. Let V be the set of tuples $\{f_1, \cdots, f_n\}$ of principal parts of the form

$$f_i = \frac{c_{m_i}}{z^{m_i}} + \cdots + c_{-1}z.$$

We see that V is a linear space over $\mathbb{C}$ of dimension $\deg(D)$. As an example consider the following: Say $D = 3a_1 + 2a_2$. Then

$$V = \left\{ \left(\frac{c_{-3}}{z^3} + \frac{c_{-2}}{z^2} + \frac{c_{-1}}{z}, \frac{d_{-2}}{z^2} + d_{-1}z \right) \right\}$$

and $\dim(V) = 5 = \deg(D)$.

We now create a map $\phi : L(-D) \to V$ that sends $f \in L(-D)$ to the tuple of principal parts of f at the points a_i. Also we consider the kernel of ϕ, which is the set of functions in $L(-D)$ that are sent to 0. Since $D \geq 0$, a function f with

$(f) \geq -D$ that is in the kernel has no principal parts at the points a_i and can have no other poles. This implies that such an f is holomorphic and, therefore, constant. Thus $\dim(\mathrm{Ker}(\phi)) = 1$ since only constant functions are in the kernel. Now if we let $\mathrm{Im}(\phi) = W$, then we have

$$\dim L(-D) = \dim(\mathrm{Ker}(\phi)) + \dim(\mathrm{Im}(\phi)) = 1 + \dim(\mathrm{W}).$$

Next, we need to find out what dim(W) is. We know that W is a set of $\{f_i\}$ principal parts, such that there exists $f \in L(-D)$ with a set of tails equal to $\{f_i\}$. By Theorem 4.5 such an f exists if and only if for all holomorphic 1-forms ω on our Riemann surface $\sum_{i=1}^{n} \mathrm{Res}_{a_i} f_i\omega = 0$. For each holomorphic 1-form ω we will consider the linear map $\lambda_\omega : V \to \mathbb{C}$ defined by

$$\{f_1, \cdots, f_n\} \to \sum_{i=1}^{n} \mathrm{Res}_{a_i} f_i\omega.$$

Now, $W = \bigcap \mathrm{Ker}(\lambda_\omega)$ is the intersection of the kernels of λ_ω for all holomorphic 1-forms ω. It follows that

$$\dim(W) = \dim\left(\bigcap Ker(\lambda_\omega)\right) = \dim(V) - \dim(\{\lambda_\omega\})$$

where $\{\lambda_\omega\}$ is the linear space generated by all the λ_ω.

We know that

$$\dim(V) = \deg(D), \text{ so } \dim(W) = \deg(D) - \dim(\{\lambda_\omega\})$$

and

$$\dim(L(-D)) = 1 + \dim(W) = 1 + \deg(D) - \dim\{\lambda_\omega\}.$$

Now we have to find $\dim(\{\lambda_\omega\})$. We know that the dimension of $\{\lambda_\omega\} \leq g$ since the number of independent 1-forms is g. Thus, one needs to consider the 1-forms that turn all principal parts to zero because they will correspond to maps λ_ω that do not contribute to the space generated by $\{\lambda_\omega\}$.

To make the principal part $\frac{c_{m_i}}{z^{m_i}} + \cdots$ at a_i turn to zero we need to multiply it by an ω that has $\mathrm{ord}_{a_i}(\omega) \geq m_i$. This is true if and only if $(\omega) \geq D$, i.e., if and only if $\omega \in \Omega(D)$. Thus, $\lambda_\omega = 0$ if and only if $\omega \in \Omega(D)$, and so $\dim\lambda_\omega = g - \dim\Omega(D)$. From which it follows,

$$\dim L(-D) = 1 + \deg(D) - g + \dim\Omega(D)$$

as desired. □

Now we conclude this paper by proving the following remarkable result.

Theorem 4.8. *Let M be a compact, simply-connected Riemann Surface. Then $M \cong \mathbb{C}_\infty$*

Proof. Fix $p \in M$ and let $D = p$. By Riemann-Roch we have that $\dim L(-D) \geq 2$. In order for $(f) \geq -1$, f could be holomorphic. Since M is compact we know that all holomorphic functions will be constant. Therefore there must be a f such that $(f) = -1$. This tells us that there is a meromorphic function $f : M \to \mathbb{C}$ such that f has only one pole at p which is a simple pole. Thus, f is a meromorphic function. However, a meromorphic function on $\mathbb{C}$ is holomorphic on $\mathbb{C}_\infty$ Since f has one pole, and it is simple we see that $\deg(f) = 1$, and by Lemma 4.3, f is 1-1. We know that f is surjective by Corollary 2.7. Therefore $M \cong \mathbb{C}_\infty$. □

Note that Theorem 4.8 classifies compact, simply connected Riemann surfaces. In fact, there is a general classification theorem for Riemann surfaces called the Uniformization Theorem, but a proof of this is beyond the scope of the paper.

Remark 4.9. Throughout this paper we have described the relationship between Riemann surfaces and meromorphic forms by proving the preliminary, albeit non-trivial, result that every Riemann surface admits a non-constant meromorphic function. We prefaced the section about the Riemann-Roch theorem by saying that it gives a partial answer to the question of what types of meromorphic functions are admitted. However, it is interesting to note that the Riemann-Roch theorem doesn't implicitly assume the existence of a non-constant meromorphic function. Note in Theorem 4.8 we force the existence of one through Riemann-Roch. The very same technique we used to prove Theorem 4.8 can be used to show that Riemann-Roch implies the existence of non-constant meromorphic functions for compact Riemann surfaces, and this avoids circularity. Given a compact Riemann surface M of genus g, pick a divisor D consisting of $g+1$ distinct points and make $\deg D = g+1$. Then, $\dim L(-D) \geq 2$. Note that in this case

$$L(-D) = \{f \in \mathfrak{M}(M) | (f) \geq -(g+1)\}.$$

We know that $\{f | (f) \geq 0\}$ corresponds to the set of holomorphic functions. By the open mapping theorem we see that every such f must be constant, which means the holomorphic functions account for exactly 1 dimension. Since $\dim L(-D) \geq 2$, we know there must exist $f \in L(-D)$ that is linearly independent of all the holomorphic functions. Therefore such an f has to have a pole which means it is locally non-constant. Hence, it is globally non-constant.

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