THE DISCRETE-TIME BINOMIAL ASSET-PRICING MODEL

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Abstract. This expository paper is meant to be a very brief primer in financial mathematics. In the first half, we will begin by defining some relatively basic probability concepts. Our journey will lead us to some properties related to our definitions as well as a proof of the Optional Stopping Theorem. After establishing a mathematical groundwork, we will enter the second half, in which we investigate the realm of finance. In this part, the reader will encounter our mathematical model and its assumptions, as well as some simple financial terms. Finally, we will investigate an application of financial mathematics: the optimal pricing and exercise of a perpetual American options contract.

Contents

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1. Introduction

It is only relatively recently that a discipline has surfaced to bring together calm and measured mathematics with the cutthroat world of finance. This discipline is aptly named financial mathematics. Its advent has marked a significant cultural shift in Wall Street. Businesses such as algorithmic-trading hedge funds have sprouted to take advantage of computerized formulae to engage in trades. Firms are now attracting many bright mathematical minds to become quantitative analysts, traders, or fund heads. And far from throwing advanced mathematics out the window in search of more elementary applicable arithmetic, financial mathematics proves to be a challenging and stimulating career for many. The world of finance is a captivating one. I hope that by the end of this paper, some readers may come to share in my enthusiasm.

2. The Probability Space

Definition 2.1. A *sample space* Ω is a set comprised of elements ω. Each ω in Ω is an *outcome*.

Definition 2.2. A σ-algebra S of subsets Y of a set X is a collection of subsets that satisfies the following requirements:

- S contains X.
- S is closed under complementation, meaning for any element $Y \in S$, it is true that its complement Y^C is an element of S.
- S is closed under countable union.

Remark 2.3. For our purposes, we define the σ-algebra $\mathcal{F}$ of Ω as comprised of elements E, where each E is an event.

Definition 2.4. The *probability function* $\mathbb{P}\colon \mathcal{F} \longrightarrow [0,1]$ is a function that satisfies the following:

- $\mathbb{P}(\Omega) = 1$
- Consider events $E_1, E_2, \ldots \in \mathcal{F}$ such that the events are all disjoint. Then
$$\mathbb{P}\left[\bigcup_{j=1}^{\infty} E_j\right] = \sum_{j=1}^{\infty} \mathbb{P}\left[E_j\right]$$

As mathematicians, we use $\mathbb{P}(E)$ to express the *probability of an event* E *occurring*.

These terms help bring us to our notion of a probability space:

Definition 2.5. A *probability space* is a sample space Ω taken with the σ-algebra $\mathcal{F}$ and the function $\mathbb{P}$, each with the respective properties outlined above. It is written as $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 2.6. A *Borel set*, denoted B, is any set in an arbitrary topological space that can be formed through the countable union, countable intersection, or relative complement of open sets.

Definition 2.7. A *random variable* is a measurable function $X : \Omega \longrightarrow \mathbb{R}$ such that for every Borel set B on the standard topology on $\mathbb{R}$, we may write
$$X^{-1}(B) = \{\omega \in \Omega \mid X(\omega) \in B\} \in \mathcal{F}$$

3. Expected Value

Definition 3.1. Consider some nonnegative random variable X. We define the function

$$\mathbb{E}(X) = \int X d\mathbb{P}$$

expressed with the Lebesgue integral, as the *expected value function*. If X is a random variable and $\mathbb{E}(|X|) < \infty$, then we define $\mathbb{E}(X)$ as above. If X takes positive and negative values and $\mathbb{E}(|X|) = \infty$, however, $\mathbb{E}(X)$ is undefined.

A simpler definition applies when considering discrete random variables:

Definition 3.2. Consider a discrete random variable X that takes on the values $a_1, a_2, \ldots$. The *expected value* $\mathbb{E}$ is a function defined as

$$\mathbb{E}(X) = \sum_{j=1}^{\infty} (a_j \mathbb{P}\{X = a_j\})$$

defined on variables X for which the sum on the right-side of the equation is absolutely convergent.

There is a more specific kind of expected value, namely the conditional expected value. First we give the formal mathematical definition. Then, for those who do not understand some of the terms at hand (outside of the scope of this paper), a simplified definition will be given, which is more suitable for our purposes anyway.

Definition 3.3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{G} \subset \mathcal{F}$ be a σ-algebra. Let X be an integrable random variable on the probability space. There exists a unique $\mathcal{G}$-measurable random variable Y such that if $A \in \mathcal{G}$,

$$\int_A Y d\mathbb{P} = \int_A X d\mathbb{P}$$

This random variable Y is the *conditional expected value*, and it is written as $\mathbb{E}[X \mid \mathcal{G}]$, read as "the expected value of X given the information in $\mathcal{G}$."

Now we will simplify considerably for a new definition that works well with a binomial assumption for a probability space, an assumption that we will indeed end up taking.

Definition 3.4. Consider some random variable X that depends on the first N outcomes. Consider some n such that $1 \leq n \leq N$. Define

$$X_n(\omega_1\omega_2\ldots) = \omega_n = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } q \end{cases}$$

Fix outcomes $\omega_1, \omega_2, \ldots, \omega_n$. The sequence $\omega_1\omega_2\ldots\omega_n$ is known as a *continuation*. We also say that X_n represents a *binomial model* because it may only take on two values for any given outcome. We define an *N-period* binomial model as one that considers two possibilities for each of its N inputs.

Let $\uparrow$ denote the number of outcomes w_i in a continuation such that $\omega_i = 1$, and define $\downarrow$ analogously, where $\omega_i = -1$. The *conditional expected value* of the random variable X based on information of outcomes $\omega_1\omega_2\ldots\omega_n$ known at time n is written as $\mathbb{E}_n[X](\omega_1\omega_2\ldots\omega_n)$. It is defined as

$$\mathbb{E}_n[X](\omega_1\omega_2\ldots\omega_n) = \sum_{\omega_{n+1}\ldots\omega_N} p^{\uparrow} q^{\downarrow} X(\omega_1\ldots\omega_n\omega_{n+1}\ldots\omega_N)$$

Remark 3.5. Conventionally, we truncate the continuation when considering conditional expected value. That is, we write $\mathbb{E}_n[X]$ rather than $\mathbb{E}_n[X](\omega_1 \dots \omega_n)$.

Properties of Conditional Expected Value. The conditional expected value possesses many interesting properties. These properties will be outlined in the following assumed proposition.

Proposition 3.6. *Let N be a positive integer. Let X and Y be random variables that depend on the first N coin tosses. Fix some integer n such that $0 \leq n \leq N$. Then all of the following properties hold true:*

- Linearity: *For all constants c_1 and c_2, it is true that*
$$\mathbb{E}_n\left[c_1 X + c_2 Y\right] = c_1 \mathbb{E}_n[X] + c_2 \mathbb{E}_n[Y]$$
- Independence: *If X depends only on the first n coin tosses rather than the first N coin tosses, then it is true that*
$$\mathbb{E}_n[XY] = X \cdot \mathbb{E}_n[Y]$$
If X depends only on latter tosses $n+1$ through N rather than all of the first N tosses, then it is true that
$$\mathbb{E}_n[X] = \mathbb{E}[X]$$
- Iterated Conditioning: *Fix some integer m such that $0 \leq n \leq m \leq N$. Then*
$$\mathbb{E}_n\left[[\mathbb{E}_m[X]\right] = \mathbb{E}_n[X]$$
and
$$\mathbb{E}[\mathbb{E}_m[X]] = \mathbb{E}[X]$$
- Trivial: $\mathbb{E}_0[X] = \mathbb{E}[X]$ *and* $\mathbb{E}_N[X] = X$

4. Martingales

Definition 4.1. For our purposes, an *adapted process* is a sequence of random variables $P_0, P_1 \dots, P_N$ such that P_0 is constant and each P_n depends only on the first n coin tosses.

Definition 4.2. Consider the adapted process $M_0, M_1, \dots, M_N$. If $M_n = \mathbb{E}_n[M_{n+1}]$ for all n such that $0 \leq n \leq N-1$, then this process is a *martingale*. If, rather than an $=$ sign, we have a $\geq$ sign, this process is a *supermartingale* and may have a tendency to decrease. An analogous statement regarding a $\leq$ sign classifies a process as a *submartingale*, which may have a tendency to increase.

Properties Of Martingales. As with conditional expected value, we present a couple of assumed propositions. They state some properties of martingales.

Proposition 4.3 (Multistep-Ahead Property)**.** *Consider the martingale $M_n; n = 0, 1, \dots, N$. Then $M_m = \mathbb{E}_m[M_n]$ whenever $0 \leq m \leq n \leq N$. Analogous statements hold for supermartingales and submartingales, with the respective $\geq$ and $\leq$ signs.*

Proposition 4.4 (Constant Expectancy)**.** *The expected value of a martingale is constant over time. This may be written as $M_0 = \mathbb{E}M_n$ for all values of n such that $0 \leq n \leq N$. Analogous statements hold for supermartingales and submartingales, with the respective $\geq$ and $\leq$ signs.*

5. Random Walk and Stopping Times

Definition 5.1. Recall the random variable X from Definition 3.4. Define

$$M_n = \sum_{j=1}^{n} X_j$$

for all $n \geq 1$, where $M_0 = 0$. The adapted process $M_0, M_1, \ldots$ is called a *random walk on* $\mathbb{Z}$.

Remark 5.2. If $p = q = \frac{1}{2}$, we say that the random walk is *symmetric.* If $p \neq q$, we say that the random walk is *asymmetric*.

Definition 5.3. Consider a random variable τ that takes values $0, 1, \ldots, N$ or ∞ and satisfies the following condition: If

$$\tau(\omega_1\omega_2 \ldots \omega_n\omega_{n+1} \ldots \omega_N) = n$$

then

$$\tau(\omega_1\omega_2 \ldots \omega_n\omega'_{n+1} \ldots \omega'_N) = n$$

for all $\omega'_{n+1} \ldots \omega'_N$, an arbitrary sequence of the continuation. Such a variable τ is called a *stopping time.*

Definition 5.4. Fix an integer m. Define

$$\tau_m = \min\{n \mid M_n = m\}$$

If the random walk never reaches m, τ_m is defined to be ∞. Such a variable τ_m is the *first passage time* of the random walk to level m.

Remark 5.5. Note that a first passage time is a special kind of stopping time.

6. Miscellaneous Mathematical Definitions

We now give the definition of a Bellman equation and an accompanying definition of a boundary condition. A Bellman equation is somewhat difficult to mathematically define with our background. Hence, the below definition is rather informal but paints a good picture of the concept.

Definition 6.1. In the context of economics, finance, and programming, a *Bellman equation* is one that greatly simplifies the analysis of a problem. Its input is the result of some initial choices. Its output computes both the value at a certain point in time based on those choices and the value of the remaining problem at hand. Here, value is not necessarily monetary or anything similar. It is just some number, variable, or expression that can be used for comparing with other numbers, variables, or expressions determined by the Bellman equation.

Definition 6.2. A *boundary condition* is an equation or statement that expresses what happens at the boundary.

Finally, we include a couple of functions we shall use.

Definition 6.3. The positive part of a function is defined as $f^+(x) = \max\{f(x), 0\}$.

Definition 6.4. The *minimum* function $\wedge\colon \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is defined as $x \wedge y = \min\{x, y\}$, where x and y are real numbers. We analogously define the *maximum* function $\vee$.

The Optional Stopping Theorem.

Theorem 6.5 (Optional Stopping Theorem)**.** *A martingale stopped at a stopping time is a martingale. A supermartingale stopped at a stopping time is a supermartingale. A submartingale stopped at a stopping time is a submartingale.*

Proof. Consider the martingale $M_n; n = 0, 1, \ldots, N$. We know that $M_m = \mathbb{E}_m[M_n]$, where $0 \leq m \leq n \leq N$, by the "multistep-ahead" property. Consider a stopping time τ. If we let $m = \tau \wedge n$, then we have $M_{\tau \wedge n} = \mathbb{E}_{\tau \wedge n}[M_n]$, which is both a martingale and a martingale stopped at a stopping time. A similar proof may be given for supermartingales and submartingales. □

The Optional Stopping Theorem is easily proved, yet it is still called a theorem because of its importance to martingale theory and a number of other applications. In an application to finance, the theorem asserts that if one buys an asset whose price is a martingale, one cannot hope to make money or lose money off of sheer market timing as a strategy. This is because the expected value is the same as the price paid for it. Analogous statements may be made for assets that have a tendency to go up in price (submartingale) or down (supermartingale). [3]

The Optional Stopping Theorem caps off our discussion of pure mathematics. Now we may begin our foray into a new realm, where we will come to see the marriage of finance and mathematics.

7. An Introduction to the Model and Finance

For the purpose of this discussion, we will define a model in which there are four assumptions. The first assumption is that it is possible to purchase or sell subdivided shares of a security; that is, securities do not necessarily need to be bought or sold in integer quantities. This assumption is generally acceptable because securities tend to be traded in large volume. The second assumption is that the interest rate r for investing and the interest rate r' for borrowing are equal: $r = r'$. This assumption, while not always true, is true for the large institutions that apply the results of financial mathematics. The third assumption is that at a given time, a given security is bought and sold at the same price. This is known as having *zero bid-ask spread*. Realistically, this assumption is rarely satisfied. When not much trading is taking place, however, the bid-ask spread really can be considered negligible.

The fourth assumption is more complicated. For our model, define times

$$t_0, t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots$$

where t_0 is the initial time and where

$$t_0 < \cdots < t_{i-1} < t_i < t_{i+1} < \cdots$$

Then there does not exist some time k between t_{i-1} and t_i, where $i > 0$. We are thus considering a *discrete* model of time. Furthermore, we assume that a security may only take on only two different values in the next time period. Expressed mathematically, define the security price as S_i at time t_i, where $i \geq 0$. The security at time t_{i+1} will cost either $S^{\uparrow}_{i+1}$ or $S^{\downarrow}_{i+1}$, where $S^{\downarrow}_{i+1} \leq S_i \leq S^{\uparrow}_{i+1}$. This is the *binomial* assumption to our asset-pricing. It is our fourth assumption that causes us to term this approach as the *discrete-time binomial asset-pricing model.*

A reader might imagine that geometric Brownian motion would more accurately depict changes in the market, but interestingly enough this is not the case in practice. Nevertheless, a qualm with the binomial assumption is a valid one. There are no claims here that our model works for most or all cases. It is its basic nature that renders it suitable for our discussion. True, it's not the "best" model out there. But in some situations, it really does shine, and it still is worthy of consideration. So let's consider it.

Financial Definitions.

Definition 7.1. A *security* is a negotiable financial instrument that represents a type of financial value. A *derivative* is a kind of security that is so named because it derives its price from one or more underlying assets.

Definition 7.2. An *options contract* is a type of derivative. It is sold by the option writer to the option holder. An options contract may be either a call or a put. A *call* provides the holder the right but not the obligation to purchase a specified quantity of a financial instrument at a *strike price* K from the writer at some specified time on or prior to the options contract's expiration. An analogous definition of a *put* may be given, but as a right to sell. The time of purchase in a call – or, analogously, the time of sale in a put – is known as the *exercise time*. An options contract may only be exercised when the underlying financial instrument costs more than the strike price (in a call) or less (in a put). This is when the options contract is said to be *in-the-money*.

Definition 7.3. An *American* options contract is one that does not have a specified exercise time; it may be exercised at any time prior to its expiration. A *perpetual* options contract is one that does not have an expiration date.

Definition 7.4. The *intrinsic value* of an options contract for the option holder is a function $g\colon \mathbb{R} \longrightarrow \mathbb{R}$ that reflects the difference in the strike price K and the underlying stock price s of the option. For a call, $g(s) = s - K$. For a put, $g(s) = K - s$.

Definition 7.5. Fix N and consider a binomial model. Let $u = \dfrac{S_i^{\uparrow}}{S_{i-1}}$ and let $d = \dfrac{S_i^{\downarrow}}{S_{i-1}}$ for all numbers i such that $0 < i \leq N$, where S_i is the price of the security at time i. We call u the *up factor* and d the *down factor*.

Definition 7.6. The *interest rate* r is a quantified property of the money market that yields $1 + r$ dollars at time one for a dollar invested in the money market at time zero. Similarly, a dollar borrowed at time zero from the money market results in a debt of $1 + r$ dollars.

Remark 7.7. As stated earlier in our assumptions for the model, we require that the interest rate r for the money market is the same as the interest rate for investing. We also require that $r > -1$.

Definition 7.8. *Arbitrage* is a trading strategy that arises in inefficient markets. An investor taking advantage of arbitrage can start with no money, have zero probability of losing money, and have positive probability of making money.

Remark 7.9. Sometimes arbitrage is present in the real world, but with computers and professionals tracking the markets so closely, arbitrage may be taken advantage of quite quickly. We assume an efficient market in our analyses, meaning that we necessitate $0 < d < 1 + r < u$ to make sense.

Definition 7.10. *Hedging* is the process whereby an investor minimizes risk by investing in a way so as to control for market fluctuations.

Someone who hedges attempts to control for risk. Thus, when dealing with equations for determining how to construct a hedge, financial mathematicians don't use actual probabilities p and q. Rather, risk-neutral probabilities $\tilde{p}$ and $\tilde{q}$ are considered. Similarly, $\tilde{\mathbb{E}}$ denotes that risk-neutral expected value is being considered.

Remark 7.11. It is not required – and indeed quite rare – that $\tilde{p} = \tilde{q} = \frac{1}{2}$. "Risk-neutral" doesn't mean these two probabilities are equal. The actual way risk-neutral probabilities are related under the discrete-time binomial asset-pricing model is as follows:

$$\tilde{p} = \frac{1 + r - d}{u - d}$$

$$\tilde{q} = \frac{u - 1 - r}{u - d}$$

For our purposes, we take these equations as assumed.

8. Deriving the Appropriate Bellman Equation

Now begins the fun part: applying our knowledge. In this section, we will prepare an application of a Bellman equation that takes some stated values and gives us a specified optimal stopping time for exercising a perpetual American options contract. To get to that step, though, we're going to need a Bellman equation in the first place. The derivation of our Bellman equation is not very difficult, so long as we keep in mind the core properties that it needs to satisfy. Our Bellman equation should ideally price the options contract at the traded value that represents the "fairest" value $v(S_n)$, where $v \colon \mathbb{R} \longrightarrow \mathbb{R}$ is a function computing the value of the options contract based on the underlying stock price at time n. We have to make sure that the seller does not consider the price to be too low, and we have to make sure that the buyer does not consider the price too high.

The seller is satisfied if he or she has a hedging portfolio that has enough value to pay off the options contract when it is exercised and that has enough value to match the value of the options contract at any given time. In order to pay off a call when exercised, the seller must have a hedging portfolio valued at $(S_n - K)^+$. Analogously, for a put, the portfolio should be valued at $(K - S_n)^+$. An astute reader may recall that these expressions represent the intrinsic value of the options contract, $g^+(S_n)$. Finally, in order to have a portfolio that matches the value of the options contract at any given time, we must compute a function v such that

$$(1 + r)v(S_n) \geq \tilde{\mathbb{E}}_n \left[v(S_{n+1}) \right]]$$

We include the interest rate in there because the value of the money invested is itself changing, and so a hedging portfolio must invest into the money market and take into account the interest rate. This may be rewritten as

$$v(S_n) \geq \left(\frac{1}{1+r}\right) \tilde{\mathbb{E}}_n \left[\tilde{p}v(uS_n) + \tilde{q}v(dS_n)\right]$$

An astute reader may be thinking of the term "supermartingale" right now. We are indeed checking to see that the value of the stock has a tendency to go down, so that the seller can construct a hedge. To satisfy the seller, we have so far demonstrated that these properties need to be met by our function. So, we need a function v such that

$$v(S_n) \geq \max\left\{g^+(S_n), \left(\frac{1}{1+r}\right) \tilde{\mathbb{E}}_n \left[\tilde{p}v(uS_n) + \tilde{q}v(dS_n)\right]\right\}$$

An options contract valued as such certainly meets the seller's demands. The buyer, however, rightfully wishes for the options contract he or she is purchasing to be priced as low as possible. Thus, the buyer requires that the function v is the "smallest" function from a pointwise perspective; that is, for a function y that satisfies the seller, at each time $n \geq 0$, we have $y(S_n) \geq v(S_n)$. We may continue to find a smaller and smaller function that satisfies the Bellman inequality above up to a point: equality. Thus, the appropriate Bellman equation is

$$v(S_n) = \max\left\{g^+(S_n), \left(\frac{1}{1+r}\right) \tilde{\mathbb{E}}_n \left[\tilde{p}v(uS_n) + \tilde{q}v(dS_n)\right]\right\}$$

where g^+ is defined as earlier. Yet while this Bellman equation is correct, it is not in simplest form. Any values of S_n that would make g negative cause our function v to take on the value

$$v(S_n) = \left(\frac{1}{1+r}\right) \tilde{\mathbb{E}}_n \left[\tilde{p}v(uS_n) + \tilde{q}v(dS_n)\right]\}$$

So we may simplify our equation to our desired form:

$$v(S_n) = \max\left\{g^+(S_n), \frac{1}{1+r}\tilde{\mathbb{E}}_n \left[\tilde{p}v(uS_n) + \tilde{q}v(dS_n)\right]\right\}$$

We do, however, require some boundary conditions to be put in place to avoid extraneous solutions. For a call, the boundary conditions are

$$\lim_{S_n \downarrow 0} v(S_n) = 0, \lim_{S_n \to \infty} \frac{v(S_n)}{S_n} = 1$$

For a put, the boundary conditions are

$$\lim_{S_n \downarrow 0} v(S_n) = K, \lim_{S_n \to \infty} v(S_n) = 0$$

9. Solving Our Bellman Equation

Time to apply.

Suppose as a budding "quant," you determine that the market you are currently dealing with satisfies assumptions that allow you to use the discrete-time binomial asset-pricing model. An option writer has sold to you, the option holder, a perpetual American *put* for the price of $S_0 = \$4$. When exercised, you have the right but not the obligation to sell it for the strike price $K = \$4$. You surmise that at each time n, the options contract may be very closely approximated as either doubling in price or as falling to one half of its current price. So you take $u = 2$ and $d = \frac{1}{2}$. You also know that the interest rate r is $\frac{1}{4}$ and shows no sign of changing any time soon. Quick computation yields the risk-neutral values $\tilde{p} = \tilde{q} = \frac{1}{2}$.

We plug into our Bellman equation and simplify to get

$$
\begin{aligned}
v(S_n) &= \max\left\{4 - S_n, \frac{1}{\left(1+\frac{1}{4}\right)}\tilde{\mathbb{E}}_n\left[\frac{1}{2}v(2S_n) + \frac{1}{2}v\left(\frac{1}{2}S_n\right)\right]\right\} \\
&= \max\left\{4 - S_n, \left[\frac{1}{\left(\frac{5}{4}\right)}\right]\left(\frac{1}{2}\right)\tilde{\mathbb{E}}_n\left[v(2S_n) + v\left(\frac{1}{2}S_n\right)\right]\right\} \\
&= \max\left\{4 - S_n, \left(\frac{2}{5}\right)\tilde{\mathbb{E}}_n\left[v(2S_n) + v\left(\frac{1}{2}S_n\right)\right]\right\}
\end{aligned}
$$

The goal is to find a function v that satisfies all of the properties detailed in our derivation. Suppose, after some mathematics outside of the scope of this paper – or simply skilled guesswork – we decide that using the following function would be a good idea:

$$
v(2^j) = \begin{cases} 4 - 2^j, \text{if } j \leq 1 \\ \frac{4}{2^j}, \text{if } j \geq 1 \end{cases}
$$

If we can prove that this function v satisfies all of the properties for all stock prices S_n that are of the form 2^j, then all that is left to do is find the maximum value of v. We will use a more rigorous, proof-based approach:

Satisfying The Seller, Part One.

Theorem 9.1. *$v(S_n) \geq (4 - S_n)^+$ for all $n \geq 0$.*

Proof. For the case where $j \leq 1$ and $S_n = 2^j$, then $4 - S_n > 0$, meaning

$$v(S_n) = 4 - S_n \geq (4 - S_n)^+$$

Similarly, for the case where $j \geq 2$ and $S_n = 2^j$, then $4 - S_n < 0$, meaning

$$v(S_n) \geq 0 = (4 - S_n)^+$$

□

Satisfying The Seller, Part Two. So our function satisfies the first half of our Bellman equation, whether as an equality or as an appropriate inequality. We will now proceed to show that it satisfies the second half of our Bellman equation, and for this we divide our proof into three theorems. The reason why shall become apparent later.

Theorem 9.2. $\tilde{\mathbb{E}}_n\left[v(S_{n+1})\right] < \left(\frac{5}{4}\right) v(S_n)$ *for all values of S_n that correspond to some value 2^j, where $j \leq 0$.*

Proof.

$$\begin{aligned}
\tilde{\mathbb{E}}_n\left[v\left(S_{n+1}\right)\right] &= \left(\frac{1}{2}\right) v\left(2^{j+1}\right) + \frac{1}{2} v\left(2^{j-1}\right) \\
&= \left(\frac{1}{2}\right)\left(4 - 2^{j+1} + 4 - 2^{j-1}\right) \\
&= \left(\frac{1}{2}\right)\left(8 - 2^j\left[2 + \frac{1}{2}\right]\right) \\
&= 4 - \left(\frac{1}{2}\right)\left(2^j\left[\frac{5}{2}\right]\right) \\
&= 4 - \left(2^j\right)\left(\frac{5}{4}\right) \\
\left(\frac{4}{5}\right)^{n+1} \tilde{\mathbb{E}}_n\left[v\left(S_{n+1}\right)\right] &= \left(\frac{4}{5}\right)^{n+1}\left(4 - 2^j\left[\frac{5}{4}\right]\right) \\
&= \left(\frac{4}{5}\right)^{n}\left(\frac{16}{5} - 2^j\right) \\
&< \left(\frac{4}{5}\right)^{n}\left(4 - 2^j\right) \\
\tilde{\mathbb{E}}_n\left[v\left(S_{n+1}\right)\right] &< \left(\frac{5}{4}\right) v\left(S_n\right)
\end{aligned}$$

□

Theorem 9.3. $\tilde{\mathbb{E}}_n\left[v\left(S_{n+1}\right)\right] = \left(\frac{5}{4}\right) v\left(S_n\right)$ *for all values of* S_n *that correspond to some value* 2^j*, where* $j \geq 2$.

Proof.

$$\begin{aligned}
\tilde{\mathbb{E}}_n\left[v\left(S_{n+1}\right)\right] &= \left(\frac{1}{2}\right) v\left(2^{j+1}\right) + \left(\frac{1}{2}\right) v\left(2^{j-1}\right) \\
&= \left(\frac{1}{2}\right)\left(\frac{4}{2^{j+1}} + \frac{4}{2^{j-1}}\right) \\
\left(\frac{4}{5}\right)^{n+1} \tilde{\mathbb{E}}_n\left[v\left(S_{n+1}\right)\right] &= \left(\frac{4}{5}\right)^{n+1}\left(\frac{1}{2}\right)\left(\frac{4}{2\cdot 2^j} + \frac{16}{2\cdot 2^j}\right) \\
&= \left(\frac{4}{5}\right)^{n}\left(\frac{2}{5}\right)\left(\frac{20}{2\cdot 2^j}\right) \\
&= \left(\frac{4}{5}\right)^{n}\left(\frac{4}{2^j}\right) \\
\tilde{\mathbb{E}}_n\left[v\left(S_{n+1}\right)\right] &= \left(\frac{5}{4}\right) v\left(S_n\right)
\end{aligned}$$

□

Theorem 9.4. $\tilde{\mathbb{E}}_n\left[v\left(S_{n+1}\right)\right] < \left(\frac{5}{4}\right) v\left(S_n\right)$ *for all values of* S_n *that correspond to the value* 2.

Proof.

$$\begin{aligned}
\tilde{\mathbb{E}}_n\left[v\left(S_{n+1}\right)\right] &= \left(\frac{1}{2}\right) v\left(2^{j+1}\right) + \frac{1}{2} v\left(2^{j-1}\right) \\
&= \left(\frac{1}{2}\right) v\left(2^{2}\right) + \frac{1}{2} v\left(2^{0}\right) \\
&= \left(\frac{1}{2}\right)\left(\frac{4}{4} + 4 - 2^0\right) \\
&= \frac{3}{2} \\
\left(\frac{4}{5}\right)^{n+1} \tilde{\mathbb{E}}_n\left[v\left(S_{n+1}\right)\right] &= \left(\frac{4}{5}\right)^{n+1}\left(\frac{3}{2}\right) \\
&= \left(\frac{4}{5}\right)^{n}\left(\frac{6}{5}\right) \\
&< \left(\frac{4}{5}\right)^{n} \cdot 2 \\
&< \left(\frac{4}{5}\right)^{n} v\left(S_n\right) \\
\tilde{\mathbb{E}}_n\left[v\left(S_{n+1}\right)\right] &< \left(\frac{5}{4}\right) v\left(S_n\right)
\end{aligned}$$

□

Satisfying the Buyer. Theorems 9.2 - 9.4, together with Theorem 9.1, prove that the seller is satisfied. Now we will prove that the buyer is satisfied by demonstrating that our function not just satisfies a Bellman inequality, but may be correctly written as the Bellman *equation* derived earlier. This is because our function is the smallest such function.

Theorem 9.5. *Consider an adapted process Y_n that satisfies the seller for all $n \geq 0$. Then $Y_n \geq v(S_n)$ for all n.*

Proof. Fix n. First we will consider the case where $j \leq 1$. In this case, $S_n \leq 2$, and we have $v(S_n) = 4 - S_n \leq Y_n$ by definition of Y_n. Next we will consider the case where $j \geq 2$. Define τ as the first time after time n that the price of the stock falls to \$2. Because $4 - S_\tau = 2 > 0$, we get the following equation

$$v(S_n) = \tilde{\mathbb{E}}_n\left[\left(\frac{4}{5}\right)^{\tau-n}(4 - S_\tau)\right] = \tilde{\mathbb{E}}_n\left[\left(\frac{4}{5}\right)^{\tau-n}(4 - S_\tau)^+\right] \tag{9.6}$$

This equation will come in handy now that we turn our attention to Y_n. If we recall the Optional Stopping Theorem as well as a property of Y_n that allows it to satisfy the seller, we know that for all $k \geq n$

$$\begin{aligned}\left(\frac{4}{5}\right)^n Y_n &= \left(\frac{4}{5}\right)^{\tau\wedge n} Y_{\tau\wedge n}\\ &\geq \tilde{\mathbb{E}}_n\left[\left(\frac{4}{5}\right)^{\tau\wedge k} Y_{\tau\wedge k}\right]\\ &\geq \tilde{\mathbb{E}}_n\left[\left(\frac{4}{5}\right)^{\tau\wedge k}(4 - S_{\tau\wedge k})^+\right]\end{aligned}$$

If we let k approach infinity, we get

$$\begin{aligned}\left(\frac{4}{5}\right)^n Y_n &\geq \tilde{\mathbb{E}}_n\left[\left(\frac{4}{5}\right)^{\tau}(4 - S_\tau)^+\right]\\ Y_n &\geq \tilde{\mathbb{E}}_n\left[\left(\frac{4}{5}\right)^{\tau-n}(4 - S_\tau)^+\right]\end{aligned}$$

Together with Equation 9.6, we have $Y_n \geq v(S_n)$ for all n. □

The Optimal Stopping Time. Thus, our function v satisfies the Bellman equation we derived earlier. All that is left for us to do is to find the maximum value $v([0, \infty])$ takes on. Recall that in Theorem 9.5 we calculated the first passage time for the stock to fall to the price of \$2. This was no accident. In fact, recall further back to Theorems 9.2 - 9.4 regarding the supermartingale/martingale properties of v. Because v shows the martingale property for all values of $j \geq 2$ and the supermartingale property for all values of $j \leq 1$ – namely, beginning at the value $S_n = 2$ – we recognize that the stock is at its highest value for the holder of the put at the time when the stock falls to \$2. Having demonstrated that our function obeys our Bellman equation, we were ultimately able to demonstrate that the optimal strategy for the holder of this perpetual American put in market conditions as stated is to sell it when it first falls to \$2.

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