Elementary Topos Theory and Intuitionistic Logic

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August 28, 2012

Abstract

A topos is a particular kind of category whose definition has rich and striking consequences in various contexts. In this expository paper, the role that topoi play in intuitionistic logic is explored through Heyting algebras. In particular, I examine first the relationship between the axioms of intuitionistic propositional calculus and the structure of a Heyting algebra, and then the relationship between the structure of a Heyting algebra and that of a topos. In the third section I derive propositions necessary for the main result. The main result is the proof of the existence of a natural, internal Heyting algebra in any topos.

Contents

1 Categorical necessities

We first introduce the requisite categorical concepts for discussing topos theory.

Definition 1.1. A *category* $\mathcal{C}$ consists of the following data:

- A collection of objects $\mathrm{Ob}(\mathcal{C})$;
- For any objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a collection $\mathcal{C}(X, Y)$, called the morphisms from X to Y. We write $f\colon X \to Y$ for a morphism f from X to Y.

- For any objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a binary operation *composition*, $- \circ -\colon \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$, such that for all morphisms $f\colon W \to X, g\colon X \to Y$ and $h\colon Y \to Z$, the following holds:

$$(h \circ g) \circ f = h \circ (g \circ f).$$

- For each object $X \in \mathrm{Ob}(\mathcal{C})$, an identity morphism $1_X\colon X \to X$ such that, for each morphism $f\colon X \to Y$, we have $f \circ 1_X = f = 1_Y \circ f$.

The most familiar example of a category is **Set**, in which the objects are sets and the morphisms are functions.

Definition 1.2. A covariant *functor* $F\colon \mathcal{C} \to \mathcal{D}$ between two categories $\mathcal{C}$ and $\mathcal{D}$ is an assignment to each object $X \in \mathrm{Ob}(\mathcal{C})$ of an object $FX \in \mathrm{Ob}(\mathcal{D})$, and to each morphism $f\colon X \to Y \in \mathrm{Mor}(\mathcal{C})$ of a morphism $Ff\colon FX \to FY \in \mathrm{Mor}(\mathcal{D})$ such that $F(1_X) = 1_{FX}$ for all $X \in \mathrm{Ob}(\mathcal{C})$ and $F(g \circ f) = F(g) \circ F(f)$ for all $f, g \in \mathrm{Mor}(\mathcal{C})$.

Definition 1.3. For any category $\mathcal{C}$, one can construct the *dual* category $\mathcal{C}^{\mathrm{op}}$ by, for all objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, interchanging the source with the target of each morphism in $\mathcal{C}(X, Y)$; i.e., if $f\colon X \to Y$ in $\mathcal{C}$ then $f\colon Y \to X$ in $\mathcal{C}^{\mathrm{op}}$.

We say a covariant functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is a *contravariant* functor $F\colon \mathcal{C} \to \mathcal{D}$. For contravariant fuctors we have $F(gf) = F(f) \circ F(g)$ for all $f\colon X \to Y, g\colon Y \to Z$ in $\mathcal{C}$.

Definition 1.4. A *natural transformation* $\eta : F \to G$ between two functors $F, G : \mathcal{C} \to \mathcal{D}$ is a collection of morphisms $\eta_X : FX \to GX$ for each $X \in \mathrm{Ob}(\mathcal{C})$ such that for any morphism $f : X \to Y \in \mathrm{Morph}(\mathcal{C})$, the following diagram commutes:

$$\begin{array}{ccc} FX & \xrightarrow{\eta_X} & GX \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FY & \xrightarrow{\eta_Y} & GY \end{array} \; .$$

A natural transformation η is called a *natural isomorphism* if each of its components is an isomorphism, where an arrow $f\colon X \to Y$ is an isomorphism if there is some arrow $g\colon Y \to X$ for which $fg = 1_Y$ and $gf = 1_X$. We call the morphism $\eta_X\colon FX \to GX$ the *component* of η at X.

Lemma 1.5. *(Yoneda) For any functor $F\colon \mathcal{C} \to$ **Set** and for any object X of $\mathcal{C}$, there is a bijection*

$$\mathrm{Nat}(\mathcal{C}(X, -), F) \cong FX$$

natural in both X and F, where $\mathcal{C}(X, -)$ denotes the functor from $\mathcal{C}$ into **Set** *which assigns to each object Y of $\mathcal{C}$ the set $\mathcal{C}(X, Y)$, and* $\mathrm{Nat}(\mathcal{C}(X, -), F)$ *denotes the set of natural transformations from $\mathcal{C}(X, -)$ to F.*

Proof. The proof of this lemma rests upon both the diagram

$$\begin{array}{ccc} \mathcal{C}(X,X) & \xrightarrow{\phi_X} & FX \\ \Big\downarrow{\scriptstyle f\circ -} & & \Big\downarrow{\scriptstyle Ff\circ -} \\ \mathcal{C}(X,Y) & \xrightarrow{\phi_Y} & FY \end{array}$$

and the observation that the identity morphism $1_X \colon X \to X$ must live in $\mathcal{C}(X,X)$.

Consider a natural transformation $\phi \colon \mathcal{C}(X,-) \to F$. For any $f \colon X \to Y$, we have that the above diagram is commutative. "Chasing" the obligatory identity morphism around the diagram, we see that $Ff(\phi_X(1_X)) = \phi_Y(f)$. We thus obtain the element $\phi_X(1_X) \in FX$.

Conversely, observe that each element $z \in FX$ determines a unique natural transformation γ^z as follows. For each morphism $f \colon X \to Y$, put $\gamma^z_Y(f) = Ff(z)$. For instance, for the identity morphism $1_X \colon X \to X$, we have $\gamma^z_X(1_X) = F(1_X)(z) = 1_{FX}(z) = z$. It follows from our choice of γ^z that γ^z is indeed a natural transformation. The twofold naturality of the bijections is mechanically verified as well. □

The contravariant version of the Yoneda Lemma is proved analogously and states that $\mathrm{Nat}(\mathcal{C}(-,X),F) \cong FX$. The following will be useful later:

Proposition 1.6. *The Yoneda Lemma respects (component-wise) composition of natural transformations, in the (contravariant) sense that the diagram*

$$\begin{array}{ccc} \mathrm{Nat}(\mathcal{C}(-,B),\mathcal{C}(-,C)) \times \mathrm{Nat}(\mathcal{C}(-,A),\mathcal{C}(-,B)) & \xrightarrow{\circ} & \mathrm{Nat}(\mathcal{C}(-,A),\mathcal{C}(-,C)) \\ \wr\!\downarrow & & \wr\!\downarrow \\ \mathcal{C}(B,C) \times \mathcal{C}(A,B) & \xrightarrow{\qquad\circ\qquad} & \mathcal{C}(A,C) \end{array}$$

commutes.

Proof. Suppose we have natural transformations $\alpha \colon \mathcal{C}(-,B) \to \mathcal{C}(-,C)$ and $\beta \colon \mathcal{C}(-,A) \to \mathcal{C}(-,B)$. By the Yoneda Lemma, we have that α and β correspond to elements $\overline{\alpha} = \alpha_B(1_B) \in \mathcal{C}(B,C)$ and $\overline{\beta} = \beta_A(1_A) \in \mathcal{C}(A,B)$, and, composing these morphisms, we obtain $\overline{\alpha}\overline{\beta} \in \mathcal{C}(A,C)$. This corresponds to the bottom-left path of the diagram.

Alternatively, by the Yoneda Lemma, the natural transformation $\alpha\beta \colon \mathcal{C}(-,A) \to \mathcal{C}(-,C)$ obtained by composing at each component corresponds to an element $\overline{\alpha\beta} = (\alpha\beta)_A(1_A) = \alpha_A(\beta_A(1_A))$. As $\beta_A(1_A)$ is a morphism $A \to B$, we have by naturality of α that the diagram

$$\begin{array}{ccc} \mathcal{C}(B,B) & \xrightarrow{\alpha_B} & \mathcal{C}(B,C) \\ {\scriptstyle -\circ\beta_A(1_A)}\Big\downarrow & & \Big\downarrow{\scriptstyle -\circ\beta_A(1_A)} \\ \mathcal{C}(A,B) & \xrightarrow{\alpha_A} & \mathcal{C}(A,C) \end{array}$$

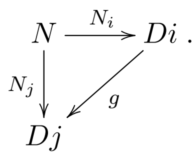
commutes, and so, chasing the identity 1_B, we have that

$$\begin{aligned}\overline{\alpha\beta} &= \alpha_A(\beta_A(1_A)) \\ &= \alpha_A(1_B \circ \beta_A(1_A)) \\ &= \alpha_B(1_B) \circ \beta_A(1_A) \\ &= \overline{\alpha}\overline{\beta},\end{aligned}$$

as required. □

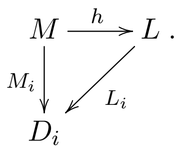
Definition 1.7. A contravariant functor $F\colon \mathcal{C} \to \mathbf{Set}$ is *representable* if it is naturally isomorphic to $\mathcal{C}(-, A)$ for some object A of $\mathcal{C}$. A *representation* of F is a pair (F, ϕ), where $\phi\colon \mathcal{C}(-, A) \to F$ is a natural isomorphism.

Definition 1.8. A *diagram* of shape $\mathcal{S}$ in a category $\mathcal{C}$ is a functor $D\colon \mathcal{S} \to \mathcal{C}$. A *cone* of the diagram D is an object N in $\mathcal{C}$, equipped with morphisms $N_i : N \to Di$ for each object i in $\mathcal{S}$, such that for any $g\colon Di \to Dj$, we have that $gN_i = N_j$; i.e., the following diagram commutes:



We say that the object N is the *vertex* of the cone.

Definition 1.9. The *limit* (unique up to unique isomorphism) of a diagram D of shape $\mathcal{S}$ in a category $\mathcal{C}$ is a cone with vertex $L \in \mathrm{Ob}(\mathcal{C})$ such that for any other cone with some vertex $M \in \mathrm{Ob}(\mathcal{C})$ there exists a unique morphism $h : L \to M$ such that $M_i = hL_i$, for all objects i in the shape category S. Diagrammatically, the following must commute:



Intuitively, we can "factor" the cone with vertex M through the cone with vertex L. This distinguishing property for the limit — that any other cone can be factored through it — is categorically referred to as a *universal property*, in the sense that the limiting cone is universal among such cones.

Definition 1.10. The *equalizer* of two morphisms $f, g\colon X \to Y$ in a category $\mathcal{C}$ is the limit of the diagram

$$X \rightrightarrows Y\ .$$

For any two objects X, Y in a category $\mathcal{C}$, the *product* of X and Y is the limit of the diagram which just has X and Y and their identity morphisms, and is denoted $X \times Y$.

For any morphisms $f : X \to Z$ and $g : Y \to Z$ in some category $\mathcal{C}$, the *pullback* of f along g or, equivalently, of g along f, is the limit of the diagram

$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z \end{array} .$$

Definition 1.11. A colimit in $\mathcal{C}$ is a limit in $\mathcal{C}^{\text{op}}$, a coproduct is a product in $\mathcal{C}^{\text{op}}$, a coequalizer is an equalizer in $\mathcal{C}^{\text{op}}$, and a pushout is a pullback in $\mathcal{C}^{\text{op}}$.

The following exercise will be of use in section 3.

Exercise 1.12. Any finite limit in a category $\mathcal{C}$ can be constructed from finite products and equalizers in $\mathcal{C}$.

Definition 1.13. An *adjunction* between two categories $\mathcal{C}$ and $\mathcal{D}$ is a pair of functors, $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, together with, for each pair of objects $X \in \mathcal{C}, Y \in \mathcal{D}$, a bijection

$$\phi_{X,Y} \colon \mathcal{C}(X, GY) \cong \mathcal{D}(FX, Y),$$

which is natural in X and Y. This twofold naturality requires both that for any morphism $f : X \to X'$ in $\mathcal{C}$ the diagram

$$\begin{array}{ccc} \mathcal{C}(X', GY) & \xrightarrow{\phi_{X',Y}} & \mathcal{D}(FX', Y) \\ \downarrow{\scriptstyle -\circ f} & & \downarrow{\scriptstyle -\circ Ff} \\ \mathcal{C}(X, GY) & \xrightarrow{\phi_{X,Y}} & \mathcal{D}(FX, Y) \end{array}$$

commutes and that the analogous diagram for each map $g : Y \to Y'$ commutes. We say in this situation that F is left adjoint to G, and G is right adjoint to F, and we denote this relation by $F \dashv G$.

Example 1.14. A class of examples of adjunctions concerns the relation between "free" and "forgetful" functors. For instance, the functor $F \colon \mathbf{Set} \to \mathbf{Set}_*$, which adjoins a disjoint basepoint $*_A$ to A, is left adjoint to the "forgetful" functor $U \colon \mathbf{Set}_* \to \mathbf{Set}$, which forgets about the basepoint $*_A$ for each set A. Moreover, the free group functor $F \colon \mathbf{Set} \to \mathbf{Grp}$ is left adjoint to the forgetful functor $U \colon \mathbf{Grp} \to \mathbf{Set}$, which forgets the group structure of groups.

Alternatively,

Definition 1.15. An *adjunction* between two categories $\mathcal{C}$ and $\mathcal{D}$ consists of a pair of functors, $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{C}$, and two natural transformations, $\eta \colon 1_{\mathcal{C}} \to GF$ and $\epsilon \colon FG \to 1_{\mathcal{D}}$, such that $\epsilon F \circ F\eta = 1_F$ and $G\epsilon \circ \eta G = 1_G$; i..e, for any objects X and Y in the categories $\mathcal{C}$ and $\mathcal{D}$ we have $\epsilon_{FX} \circ F(\eta_X) = 1_{FX}$ and $G(\epsilon_Y) \circ \eta_{GY} = 1_{GY}$, respectively. We say that η and ϵ are the unit and counit of the adjunction.

Proposition 1.16. *The two given definitions of adjunction between two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent.*

Proof. By $\phi_{X,Y}$ we mean the bijection between the hom-sets $\mathcal{C}(X, GY)$ and $\mathcal{D}(FX, Y)$. Given such bijections, we construct the unit $\eta\colon 1_{\mathcal{C}} \to GF$ by putting $\eta_X = \phi_{X,FX}(1_{FX})$ for all objects X of $\mathcal{C}$, and $\epsilon\colon FG \to 1_{\mathcal{D}}$ by putting $\epsilon_Y = \phi^{-1}_{GY,Y}(1_{GY})$ for all objects Y of $\mathcal{D}$. The reader may readily verify that the naturality of η and ϵ readily follows from the naturality of the bijections. As $\eta_X\colon X \to GFX$ and $\epsilon_Y\colon FGY \to Y$, the naturality of the bijection shows that

$$\epsilon_{FX} \circ F(\eta_X) = \phi^{-1}_{X,FX}(\eta_X) = \phi^{-1}_{X,FX}(\phi_{X,FX}(1_{FX})) = 1_{FX},$$

and

$$G(\epsilon_Y) \circ \eta_{GY} = \phi_{GY,Y}(\epsilon_Y) = \phi_{GY,Y}(\phi^{-1}_{GY,Y}(1_{GY}) = 1_{GY},$$

as required.

Conversely, suppose we have natural transformations $\eta\colon 1_{\mathcal{C}} \to GF$ and $\epsilon\colon FG \to 1_{\mathcal{D}}$. For any arrow $f\colon X \to GY$, let $\phi_{X,Y}(f) = \epsilon_Y \circ Ff$, and for any $g\colon FX \to Y$ let $\phi^{-1}_{X,Y}(g) = Gg \circ \eta_X$. To verify that this indeed produces a bijection, observe that for $f\colon X \to GY$ and $g\colon FX \to Y$ we have from the naturality of ϵ and η that

$$\phi^{-1}_{X,Y}(\phi_{X,Y}(f)) = \phi^{-1}_{X,Y}(\epsilon_Y \circ Ff) = G(\epsilon_Y \circ Ff) \circ \eta_X$$

$$= G(\epsilon_Y) \circ GFf \circ \eta_X = G(\epsilon_Y) \circ \eta_{GY} \circ f = 1_{GY} \circ f = f$$

and

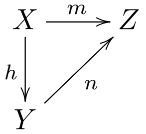
$$\phi_{X,Y}(\phi^{-1}_{X,Y}(g)) = \phi_{X,Y}(Gg \circ \eta_X) = \epsilon_Y \circ F(Gg \circ \eta_X)$$

$$= \epsilon_Y \circ FGg \circ F(\eta_X) = g \circ \epsilon_{FX} \circ F(\eta_X) = g \circ 1_{FX} = g.$$

The naturality of these bijections follows from the naturality of η and ϵ. □

A noteworthy property of adjoints is that every left adjoint preserves limits and every right adjoints preserves colimits.

Definition 1.17. A *monomorphism* or *monic* $m\colon Y \to Z$ in $\mathcal{C}$ is a morphism from Y to Z such that for any two morphisms $f, g\colon X \to Y$, we have that $mf = mg$ implies $f = g$. We say that two monomorphisms $m\colon X \to Z$ and $n\colon Y \to Z$ are *equivalent* if there exists an isomorphism $h\colon X \to Y$ such that the diagram



commutes. A *subobject* of an object Z in $\mathcal{C}$ is an equivalence class of monomorphisms into Z.

Definition 1.18. For any two objects X, Y in a category $\mathcal{C}$, the *exponential* of X and Y consists of an object Y^X and an arrow $e\colon Y^X \times X \to Y$ such that for any other object Z and arrow $g\colon Z \times X \to Y$, there is a unique $f'\colon Z \to Y^X$ such that $e \circ (f' \times 1_X) = f$; i.e., making the diagram

$$\begin{array}{ccc} Y^X \times X & \xrightarrow{\;e\;} & Y \\ {\scriptstyle f'\times 1_X}\,\big\uparrow & \nearrow_{f} & \\ Z \times X & & \end{array}$$

commute. Thus, if we have an exponential Y^X for all objects X, Y in the category $\mathcal{C}$, then we have an isomorphism $\mathcal{C}(Z \times X, Y) \cong \mathcal{C}(Z, Y^X)$, so that the functor $- \times X$ is left adjoint to $-^X$.

Definition 1.19. An *initial object* of a category $\mathcal{C}$ is a limit of the empty diagram, while a *terminal object* is a colimit of the empty diagram. A category $\mathcal{C}$ is *cartesian closed* if it has a terminal object and for all objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ both a product $X \times Y$ and an exponential Y^X.

Definition 1.20. For any object X in a category $\mathcal{C}$, the *slice category* $\mathcal{C}/X$ is the category with morphisms into X from $\mathcal{C}$ as its objects; for any two objects $a\colon A \to X$ and $b\colon B \to X$ of $\mathcal{C}/X$, a morphism $f\colon a \to b$ in $\mathcal{C}/X$ is an arrow $f\colon A \to B$ such that $bf = a$.

2 Algebraic logic

Definition 2.1. A *lattice* L is a partially ordered set which has for all $x, y \in L$ both a supremum (or join) $x \vee y \in L$ and an infinum (or meet) $x \wedge y \in L$. A lattice L is *distributive* if for any $x, y, z \in L$, we have that $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. In lattices with elements 0 and 1 such that $0 \leq x \leq 1$, for all $x \in L$, a *complement* for an element x is an element $a \in L$ such that $x \wedge a = 0$ and $x \vee a = 1$. A *Boolean algebra* B is a distributive lattice with 0 and 1 in which every element has a complement.

For any poset P, we can construct a category $\mathcal{C}_P$ as follows. Let the objects of $\mathcal{C}_P$ consist precisely of the elements of P, and let there be a single morphism $x \to y$ whenever $x \leq y$ in P. If the poset is a lattice, then the category will have binary products and coproducts, and if it is Boolean, then the category will have an initial and a terminal object, namely 0 and 1.

The essential ingredients of propositional logic are *propositional variables.* Intuitively, they are variables which are either true or false. For any set of propositional variables we have the binary operations "or", "and", and "implies", denoted $\vee, \wedge$, and $\to$; the unary operation, negation, denoted $\neg$; and the nullary operations $\top$ and $\perp$ which correspond to true and false.

Definition 2.2. For a set $\mathbb{P}$ of propositional variables, we recursively define formulae, so that

- p is a formula, for all $p \in \mathbb{P}$;
- Given any formula F, the negation $\neg F$ is a formula;
- For any formulae F and G, each of $F \vee G, F \wedge G$ and $F \to G$ is a formula.

Definition 2.3. For a set of propositional variables $\mathbb{P}$ and a Boolean algebra B, a *valuation* is a function $V : \mathbb{P} \to B$ such that

$$V(\top) = 1, V(\bot) = 0, V(P \wedge Q) = V(P) \cap V(Q), V(\neg F) = \neg V(F),$$

$$V(P \vee Q) = V(P) \cup V(Q) \text{ and } V(P \to Q) = V(P) \Rightarrow V(Q),$$

for all formulae P and Q, where $\cap, \cup$, and $\Rightarrow$ are the operations and $\neg$ is complementation in B. A formula F is *valid* if $V(F) = 1$, for all valuations V into $\{0, 1\}$. A formula F is *B-valid* if $V(F) = 1$ for every valuation into B.

The completeness property of classical propositional logic states that a formula F is provably true iff F is valid — i.e., F is always "true." However, noting that the proof of this fact only relies on algebraic properties satisfied by all Boolean algebras, and not just $\{0, 1\}$, we can state instead:

Theorem 2.4. *Any formula F is provable in classical propositional logic if and only if F is B-valid for every Boolean algebra B.*

This is the precise sense in which Boolean algebras serve as models for classical propositional calculus. A hallmark feature of intuitionistic logic is its exclusion of the notorious "law of excluded middles," that $\top = p \vee \neg p$, for all $p \in \mathbb{P}$. A striking consequence of this limitation is the invalidity of proofs by contradiction. Analogously, Heyting algebras — in which neither p nor $\neg p$ are necessarily true — serve as models for intuitionistic propositional calculus (henceforth referred to as IPC).

Definition 2.5. A *Heyting algebra* H is a poset with all finite products and coproducts which, considered as a category, is cartesian closed. Analogously to Definition 2.3 above, a valuation into H is a structure-preserving function $V : \mathbb{P} \to H$ and a formula F is *H-valid* if $V(F) = 1$ for every valuation into H. For a finite set of formulas Γ, we let $\bigwedge \Gamma$ denote the formula obtained by taking the join over Γ. A formula F is an *H-consequence of* Γ if $V(\bigwedge \Gamma) \leq V(F)$ for every valuation into H.

Definition 2.6. For a set Γ of formulae, we write $\Gamma \vdash F$ in IPC if there exists a deduction ending with F that only uses *IPC axioms*, elements of Γ, and the inference rule, modus ponens:

$$\frac{F, (F \to G)}{G},$$

to be read G is deduced from F and $F \to G$.

In particular, the IPC axioms are

$$F \to (G \to F),$$

$$(F \to (G \to H)) \to ((F \to G) \to (F \to H)),$$
$$F \wedge G \to F, F \wedge G \to G, F \to (G \to (F \wedge G)),$$
$$F \to F \vee G, G \to F \vee G,$$
$$(F \to H) \to ((G \to H) \to (F \vee G \to H)),$$
$$\text{and} \perp \to F,$$

for all formulae F, G, and H.

Definition 2.7. A formula F is *provable* in IPC if there exists a derivation $\top \vdash F$, where a derivation from $\top$ is a sequence of applications of modus ponens which concludes with F and uses only axioms of IPC.

We can model logic with a lattice by taking the elements of the lattice to be all formulae, and by putting $F \leq G$ if and only if $\top \vdash F \to G$, for all formulae F and G. A Heyting algebra H can thus be equivalently defined as a lattice with 0 and 1 which has for all elements $a, b \in H$ an exponential b^a, denoted $a \Rightarrow b$, which is characterized — under the exponential adjunction — by $c \leq (a \Rightarrow b)$ if and only if $c \wedge a \leq b$. That is, the element $a \Rightarrow b$ is the least upper bound among all such elements c. Note that every Boolean algebra is also a Heyting algebra, as $c \leq (\neg a \vee b)$ if and only if $c \wedge a \leq b$.

If F is an element of Γ then $\Gamma \vdash F$, trivially. If not, then, by the definition of a formula, we must have that F is the product of some sequence of operations defined on propositional variables. The following proof is motivated by the observation that any derivation $\Gamma \vdash F$ must end with an assertion of the structure of F:

$$F = G \vee J,\ \Gamma \vee \{G\} \vdash F,\ \Gamma \vee \{J\} \vdash F,\ \Gamma \vdash G \vee J;$$
$$F = G \wedge J,\ \Gamma \vdash G,\ \Gamma \vdash J;$$
$$\text{or } F = (G \to J),\ \Gamma \vee \{G\} \vdash J.$$

Lemma 2.8. *If $\Gamma \vdash F$ in IPC then F is an H-consequence of Γ for every Heyting algebra H.*

Proof. We show this by induction by considering all three possibilities for the last step in the deduction $\Gamma \vdash F$, as described above.

If $F = G \wedge J$, for some formulae G and J, then we must have that $\Gamma_1 \vdash G$ and $\Gamma_2 \vdash J$ for some subsets Γ_1 and Γ_2 of Γ. Hence, our induction hypothesis states that $V(\bigwedge \Gamma_1) \leq V(G)$ and $V(\bigwedge \Gamma_2) \leq J$. From the definition of V, we consequently have that

$$V(\bigwedge \Gamma) = V(\bigwedge \Gamma_1 \wedge \bigwedge \Gamma_2) = V(\bigwedge \Gamma_1) \wedge V(\bigwedge \Gamma_2)$$
$$\leq V(G) \wedge V(J) = V(G \wedge J) = V(F)$$

so that F is indeed an H-consequence of Γ.

If $F = G \vee J$ for some formulae G and J, then there are derivations $\Gamma \vdash G \vee J$, $\Gamma \vee \{G\} \vdash F$ and $\Gamma \vee \{J\} \vdash F$. Thus, from our induction hypothesis, we have that

$$V(\bigwedge \Gamma) \leq V(G \vee J) = V(G) \vee V(J),$$

$$V(\bigwedge \Gamma) \vee V(G) \leq V(F),$$

$$\text{and } V(\bigwedge \Gamma) \vee V(J) \leq V(F).$$

Consequently, we obtain

$$V(\bigwedge \Gamma) \leq V(\bigwedge \Gamma) \wedge (V(G) \vee V(J))$$

$$= (V(\Gamma) \wedge V(G)) \vee (V(\Gamma) \wedge V(J)) \leq V(F) \vee V(F) = V(F \vee F) = V(F).$$

Finally, if $F = (G \to J)$ then $\Gamma \vee \{G\} \vdash J$, so we have from the induction hypothesis that

$$V(\bigwedge \Gamma \cup \{G\}) = V(\bigwedge \Gamma) \wedge V(G) \leq V(J).$$

By the definition of $\Rightarrow$ in a Heyting algebra, we therefore have that $V(\bigwedge \Gamma) \leq V(G) \Rightarrow V(J) = V(G \to J) = V(F)$, as desired. □

Theorem 2.9. *(Completeness for IPC) Any formula F is provable in IPC if and only if F is H-valid in every Heyting algebra H.*

Proof. Lemma 2.8 nearly completes the forward direction; if F is provable in IPC, then $\top \vdash F$, and so, by Lemma 2.8, we have $V(\top) = 1 \leq V(F)$, and, as 1 is the top element of any Heyting algebra H, we also have $V(F) \leq 1$. Therefore, we have that $V(F) = 1$, so that F is H-valid in every Heyting algebra H.

For the converse, we must construct a certain Heyting algebra and apply a particular valuation. Denote by $\mathcal{F}$ the set of all formulae. We can then define an equivalence relation $\sim$ in $\mathcal{F}$ by putting $G \sim J$ when and only when $\vdash G \Leftrightarrow J$; that is, when we can deduce both $G \Rightarrow J$ and $J \Rightarrow G$ from the axioms of IPC. Let $\mathcal{H} = \mathcal{F}/{\sim}$. For any elements $[F], [G] \in \mathcal{H}$, we let $[F] \leq [G]$ when and only when $\vdash F \Rightarrow G$. For the top and bottom elements, we let $1 = [\top]$ and $0 = [\bot]$. For operations in the Heyting algebra, we let $[F] \wedge_{\mathcal{H}} [G] = [F \wedge G], [F] \vee_{\mathcal{H}} [G] = [F \vee G]$, and $[F] \Rightarrow_{\mathcal{H}} [G] = [F \Rightarrow G]$. The reader may readily verify that these operations are well-defined, and that they indeed satisfy the axioms for a Heyting algebra. We now let our valuation $V \colon \mathbb{P} \to \mathcal{H}$ be given by $V(J) = [J]$, for all propositional variables $J \in \mathbb{P}$.

Thus, for a formula F valid in $\mathcal{H}$, we have $V(F) = [F] = 1 = V(\top)$ and so, as $\top \in 1$, we have that $F \sim \top$. Therefore, by our definition of $\sim$, we have $\vdash F \Leftrightarrow \top$, and so $\top \vdash F$. □

We are thus licensed to assert that giving a valuation into a Heyting algebra for which $V(F) = 0$ is enough to ensure that F is not provable in IPC.

3 Elementary topos theory

Definition 3.1. A topos $\mathcal{E}$ is a category with

- All finite limits;
- An object Ω and a monic true: $A \to \Omega$ such that for any monic $m\colon S \to X$ there is a unique arrow $\phi\colon X \to \Omega$ in $\mathcal{E}$ rendering the square

$$\begin{array}{ccc} S & \longrightarrow & A \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \text{true}} \\ X & \xrightarrow{\phi} & \Omega \end{array}$$

  a pullback. We call ϕ the *classifiying map* of m and Ω the *subobject classifier* of $\mathcal{E}$.

- For every object X, an object PX and an arrow $\in_X\colon X \times PX \to \Omega$ such that for any arrow $f\colon X \times Y \to \Omega$ there is a unique arrow $f'\colon Y \to PX$ for which the diagram

$$\begin{array}{ccc} X \times Y & \xrightarrow{f} & \Omega \\ {\scriptstyle 1\times f'}\downarrow & & \Vert \\ X \times PX & \xrightarrow{\in_X} & \Omega \end{array}$$

  commutes.

It is an exercise to show that every topos has exponentials, which are constructed from finite limits and power objects. It is a bit more involved to show that every topos has all finite colimits, which is a consequence of the fact that the power-set functor is monadic. Both of these results are illustrated in section IV of [2].

Notice that the second and third requirements of the above definition give us bijections $\mathrm{Sub}_{\mathcal{E}}(Y) \cong \mathcal{E}(Y, \Omega)$ and $\mathcal{E}(X \times Y, \Omega) \cong \mathcal{E}(Y, PX)$, respectively. The fact that $\mathrm{Sub}_{\mathcal{E}}(-)$ and $\mathcal{E}(X \times -, \Omega)$ are representable functors will allow us to thoroughly utilize Yoneda's Lemma and Proposition 1.6.

Proposition 3.2. *In a topos, the source A of the monic true is the terminal object 1.*

Proof. We begin by taking the pullback of true along true, as on the left:

$$\begin{array}{ccc} P & \xrightarrow{p} & A \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \text{true}} \\ A & \xrightarrow{\text{true}} & \Omega \end{array} \quad ; \quad \begin{array}{ccc} X & \xrightarrow{q} & A \\ \Vert & & \downarrow{\scriptstyle \text{true}} \\ X & \xrightarrow{\phi_{1_X}} & \Omega \end{array} \quad .$$

For any object X, note that 1_X is a monic, and so there exists some $q\colon X \to A$ such that the diagram above on the right is a pullback square. To show the uniqueness of this q and that A is therefore the terminal object, suppose that there are two morphisms $q, q\colon X \to A$. Note that both diagrams

$$\begin{array}{ccc} X & \xrightarrow{q} & A \\ \| & & \downarrow{\scriptstyle \mathrm{true}} \\ X & \xrightarrow{\mathrm{true}\, q} & \Omega \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{q'} & A \\ \| & & \downarrow{\scriptstyle \mathrm{true}} \\ X & \xrightarrow{\mathrm{true}\, q'} & \Omega \end{array}$$

are trivially pullbacks, and so, by the uniqueness of the classifying map for the subobject $1_X\colon X \to X$, we have that $\mathrm{true}\, q = \mathrm{true}\, q'$ and so $q = q'$, as true is monic. □

Definition 3.3. The *image* of the arrow $f\colon X \to Y$ is a monomorphism $m\colon M \to Y$ together with an arrow $e\colon X \to M$ such that $f = me$ and if f factors through any other monic k, then m does as well; that is, the image is universal among such factoring monomorphisms.

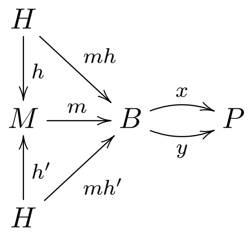
Notice the similarity of universal categorical properties — such as the image — and "universal" poset properties, such as the least upper bound. Indeed, we will need to use the image in defining the supremum of our lattice.

Proposition 3.4. *In a topos, every arrow f has an image m.*

Proof. For any arrow $f\colon A \to B$, consider the pair of arrows $x, y\colon B \to P$ in the push-forward of f along f

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle y} \\ B & \xrightarrow{x} & P \end{array} ,$$

and let $m\colon M \to B$ be the equalizer of x and y. Suppose that $mh = mh'$ for some object H and arrows $h, h'\colon H \to B$. Then the diagram

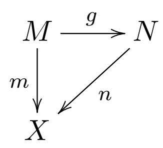


commutes and by the universality of the equalizer m we must have that $h = h'$, so that m is monic. As $f\colon A \to B$ we have also by this universality a unique morphism $e\colon A \to M$ such that $f = me$.

Note that in a topos every monic $m\colon S \to B$ is the equalizer of some pair of arrows — in particular, of $\mathrm{true}\,!_B$ and ϕ_m, where $!_B$ is the unique map $!_B\colon B \to 1$ and $\phi_m\colon B \to \Omega$ is the classifying map for m. The universality of this equalizer is a consequence of the universality of the pullback square of $\mathrm{true}!_B$ along ϕ_m. Hence, for any monic $h\colon H \to B$

through which f factors as, say, $f = hq$ for some $q\colon A \to H$, we have that h is an equalizer of some pair of arrows s, t into some object P'. Thus, $sf = shq = thq = tf$, and so by the universality of the pushforward P we have a unique arrow $j\colon P \to P'$ such that $jx = s$ and $jy = t$. Consequently we have that $sm = jxm = jym = tm$ and so by the universality of the equalizer h there is a unique $l\colon M \to H$ such that $m = hl$; that is, m factors through h, as desired. □

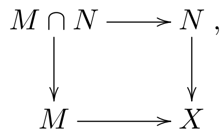
Let X be an object of the topos $\mathcal{E}$. Given two subobjects $m\colon M \to X$ and $n\colon N \to X$ of X, we say that $m \leq n$ (more precisely $[m] \leq [n]$) when there is some arrow $g\colon M \to N$ of $\mathcal{E}$ such that



commutes. Observe that $\leq$ gives a poset $\mathrm{Sub}_{\mathcal{E}}(X)$.

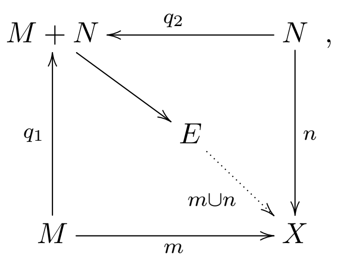
Lemma 3.5. *For every object X in a topos $\mathcal{E}$, the poset $Sub_{\mathcal{E}}(X)$ is a lattice.*

Proof. We must, as per Definition 2.1, show that $\mathrm{Sub}_{\mathcal{E}}(X)$ has all finite meets and joins. Define the meet of any two subobjects $m\colon M \to X$ and $n\colon N \to X$ to be the unique arrow $m \cap n\colon M \cap N \to X$ defined by the pullback



which indeed is a monic into and therefore subobject of X, as pullbacks preserve monics and the composition of monics is a monic. The universal property of the pullback in $\mathcal{E}$ is precisely the requirement in the poset that $m \cap n\colon M \cap N \to X$ is the greatest lower bound of m and n.

For the join of m and n, consider the diagram



where $M + N$ denotes the coproduct of M and N and $m \cup n\colon E \to X$ is the image of the unique arrow $m + n$ from $M + N$ to X for which $(m+n)\,q_1 = m$ and $(m+n)\,q_2 = n$. As the image $m \cup n$ is monic, it is a subobject of X. Its universality in $\mathcal{E}$ is equivalent, in the the poset $\mathrm{Sub}_{\mathcal{E}}(X)$, to the condition of being a least upper bound. □

Definition 3.6. For each $f\colon X \times Y \to \Omega$, the *P-transpose* of f is the unique map $f'\colon Y \to PX$ for which $\in_X (1 \times f') = f$. The *diagonal map* is the unique map $\Delta_X\colon X \to X \times X$ for which $\pi_1 \Delta_X = \pi_2 \Delta_X = 1_X$. It is trivially a monic and thus, as a subobject of $X \times X$, classified by a unique map $\delta_X\colon X \times X \to \Omega$. The P-transpose of δ_X is a map $\{\cdot\}_X\colon X \to PX$ which we call the *singleton map* of X.

Definition 3.7. Given a morphism $f\colon X \to Y$ in a topos $\mathcal{E}$, we define the arrow $Pf\colon PY \to PX$ to be the unique map such that the diagram

$$\begin{array}{ccc} X \times PY & \xrightarrow{f \times 1} & Y \times PY \\ {\scriptstyle 1 \times Pf}\downarrow & & \downarrow{\scriptstyle \in_Y} \\ X \times PX & \xrightarrow{\in_X} & \Omega \end{array}$$

commutes.

Proposition 3.8. *The above assignment* $f \longmapsto Pf$ *defines a contravariant functor* $P(-)\colon \mathcal{E} \to \mathcal{E}$.

Proof. We need to show for any arrows and objects $f\colon X \to Y, g\colon Y \to Z$ in $\mathcal{E}$ that $P(gf) = PfPg$, which is equivalent to the commutativity of the diagram

$$\begin{array}{ccccc} X \times PZ & \xrightarrow{f \times 1} & Y \times PZ & \xrightarrow{g \times 1} & Z \times PZ \\ \downarrow{\scriptstyle 1 \times Pg} & & & & {\scriptstyle \in_Z}\downarrow \\ X \times PY & \xrightarrow{1 \times Pf} & X \times PX & \xrightarrow{\in_X} & \Omega \end{array}.$$

From the definitions of Pf and Pg, we have that

$$\in_X (1 \times Pf)(1 \times Pg) = \in_Y (f \times 1)(1 \times Pg) = \in_Y (f \times Pg)$$

$$= \in_Y (1 \times Pg)(f \times 1) = \in_Z (g \times 1)(f \times 1),$$

and so $P(gf) = PfPg$. □

We are working towards showing that $\mathrm{Sub}_{\mathcal{E}}(X)$ is a Heyting algebra, for any object X. However, it is far easier to show that $\mathrm{Sub}_{\mathcal{E}}(1)$ is a Heyting algebra, where 1 is a terminal object. Given the isomorphism $\mathrm{Sub}_{\mathcal{E}/X}(1_X) \cong \mathrm{Sub}_{\mathcal{E}}(X)$, we can far more readily conclude that $\mathrm{Sub}_{\mathcal{E}}(X)$ is a Heyting algebra if we can show that $\mathcal{E}/X$ is a topos. Interesting in its own right, we therefore need the theorem

Theorem 3.9. *For any object X in a topos $\mathcal{E}$, the slice category $\mathcal{E}/X$ is also a topos.*

Proof. Given two objects $a\colon A \to X$ and $b\colon B \to X$ of $\mathcal{E}/X$, consider the pullback

$$\begin{array}{ccc} A \times_X B & \xrightarrow{p_2} & B \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle b} \\ A & \xrightarrow{a} & X \end{array}$$

in the topos $\mathcal{E}$. Note that the composite $\langle a, b\rangle = ap_1 = bp_2 \colon A \times_X B \to X$ is (up to unique isomorphism) the product of a and b in $\mathcal{E}/X$, as any object q in $\mathcal{E}/X$ with maps $q_1 \colon q \to a$ and $q_2 \colon q \to b$ in $\mathcal{E}/X$ gives rise to a unique map $j \colon q \to \langle a, b\rangle$ from the universality of the pullback in $\mathcal{E}$. Similarly, note that for any pair of arrows $h, h' \colon a \to b$ in $\mathcal{E}/X$, the equalizer $\overline{e}$ is simply the composite $he = h'e$, where $e \colon E \to A$ is the equalizer of h and h' in $\mathcal{E}$. From Proposition 1.12 we thus have that the slice category $\mathcal{E}/X$ has all finite limits.

To see that $\mathcal{E}/X$ has a subobject classifier, note first that 1_X is the terminal object in $\mathcal{E}/X$. As the identity $1_X \colon X \to X$ is monic, it is classified in $\mathcal{E}$ by a unique map v for which $v = \mathrm{true}\,!_X$. Let $\mathrm{true}_X = \mathrm{true}\,!_X \times 1_X \colon X \to \Omega \times X$, and let $\Omega_0 = \pi_2 \colon \Omega \times X \to X$, where π_2 is the second projection. Any monic $\gamma \colon a \to b$ in $\mathcal{E}/X$, where $a \colon A \to X$ and $b \colon B \to X$, is a monic $\gamma \colon A \to B$ in $\mathcal{E}$ and therefore a subobject of B. It is thus classified by a unique map ϕ_γ which renders the diagram

$$\begin{array}{ccc} A & \xrightarrow{\;!_A\;} & 1 \\ {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle \mathrm{true}} \\ B & \xrightarrow{\;\phi_\gamma\;} & \Omega \end{array}$$

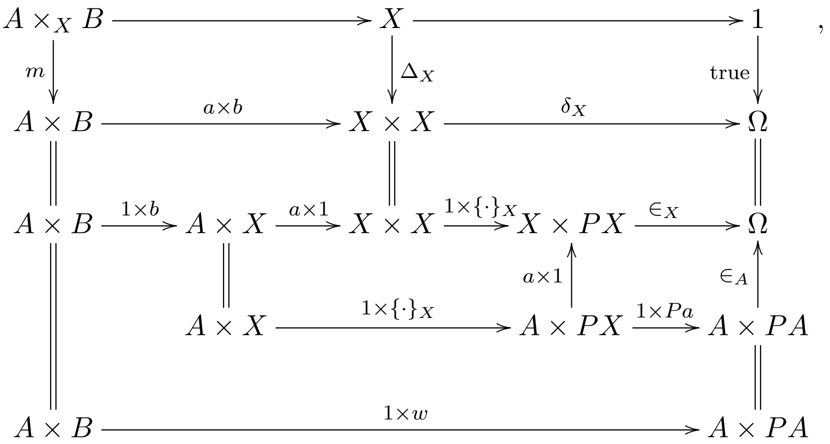
a pullback. Consequently, for any monic $\gamma \colon a \to b$, the map $\overline{\gamma} = \phi_\gamma \times b \colon B \to \Omega \times X$ is a morphism $\overline{\gamma} \colon b \to \Omega_0$ in $\mathcal{E}/X$ such that the diagram

$$\begin{array}{ccc} a & \longrightarrow & 1_X \\ {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle \mathrm{true}_X} \\ b & \xrightarrow{\;\overline{\gamma}\;} & \Omega_0 \end{array}$$

is a pullback, as any map $\eta \colon c \to b$ for some $c \colon C \to X$ in $\mathcal{E}/X$ yields a unique map $\eta' \colon C \to A$ by the universality of the pullback in $\mathcal{E}$. We have therefore shown that Ω_0, together with the monic $\mathrm{true}_X = \mathrm{true}\,!_X \times 1_X$, is a subobject classifier for $\mathcal{E}/X$.

Showing that $\mathcal{E}/X$ has power objects amounts to providing for each object a of $\mathcal{E}/X$ an object $P_X a$ such that $\mathcal{E}/X(\langle a, b\rangle, \Omega_0) \cong \mathcal{E}/X(b, P_X a)$ for each object $b \colon B \to X$ of $\mathcal{E}/X$. Notice first that $A \times_X B$ is a subobject of the product $A \times B$ in $\mathcal{E}$, because the pullback of the monic Δ_X along $a \times b \colon A \times B \to X \times X$ gives a monic $m \colon A \times_X B \to A \times B$.

This subobject is therefore classified by a unique map $A \times B \to \Omega$. The diagram



whose commutativity follows from the definitions of the morphisms involved, shows that the subobject m is classified by $\in_A (1 \times w) =\in_A (1 \times Pa \circ \{\cdot\}_X \circ b)\colon A \times B \to \Omega$.

Hence, there is a bijection $\mathcal{E}/X(\langle a, b\rangle, \Omega_0) \cong \mathcal{E}(A \times_X B, \Omega)$, under which morphisms $z\colon A \times_X B \to \Omega_0$ get sent to $\pi_1 z$, and morphisms $y\colon A \times_X B \to \Omega$ get sent to $y \times \langle a, b\rangle$.

$\mathcal{E}(A \times_X B, \Omega)$ is isomorphic to $\mathrm{Sub}_{\mathcal{E}}(A \times_X B)$, which is in turn isomorphic to $\{s\colon S \to A \times B \,|\, s \in \mathrm{Sub}_{\mathcal{E}}(A \times B) \,\text{and}\, s \leq m\}$, where $s \leq m$ means $s = mn$ for some n. This bijection can be seen by noting that each such n is a subobject of $A \times_X B$ and that any such subobject uniquely yields a subobject $s = mn$ of $A \times B$, as the composition of monics is a monic.

Observe that $s \leq m$ iff $S \cap A \times_X B = S$, where $\cap$ denotes the intersection operator of the lattice $\mathcal{E}(A \times B, \Omega)$. If $S \cap A \times_X B = S$ then we must have that $mn = s1_S = s$ for some $n\colon S \to A \times_X B$, and if $mn = s$ for some n then 1_S must be the pullback of m along s. Consequently, we have a further bijection to the set

$$\{h\colon A \times B \to \Omega \,|\, h \wedge \in_A(1 \times w) = j\},$$

as each subobject of $A \times B$ is uniquely classified by a map $h\colon A \times B \to \Omega$, and $h \wedge \in_A(1 \times w) = h$ if and only if $S \cap A \times_X B = S$.

Taking the P-transpose of each h provides us with yet another bijection, to the set $\{k \,|\, k\colon B \to PA \text{ and } k \wedge w = k\}$. From the diagram above, we have that $w = Pa \circ \{\cdot\}_X \circ b$, and so

$$\begin{aligned} k = k \wedge w = \wedge(k \times w) &= \wedge(k \times (Pa \circ \{\cdot\}_X \circ b)) \\ &= \wedge(1 \times (Pa \circ \{\cdot\}_X))(k \times b) = t(k \times b) \end{aligned}$$

where $t = \wedge(1 \times Pa\{\cdot\}_X)$. This gives a bijection to the set of those arrows $k\colon B \to PA$ for which $\pi_1(k \times b) = k = t(k \times b)$. Let $P_X a$ be the equalizer of π_1 and t, as in the diagram

$$\begin{array}{ccccc} B & \xrightarrow{k\times b} & PA\times X & \underset{t}{\overset{\pi_1}{\rightrightarrows}} & PA \\ \downarrow & \nearrow{\scriptstyle e} & \downarrow{\scriptstyle 1\times\{\cdot\}_X} & & \uparrow{\scriptstyle\wedge} \\ P_X a & & PA\times PX & \xrightarrow[1\times Pa]{} & PA\times PA \end{array} \quad .$$

Our succession of bijections shows that $\mathcal{E}/X(\langle a,b\rangle,\Omega_0)\cong\{k\colon B\to PA\,|\,p(k\times b)=t(k\times b)\}$, and the universality of the equalizer $P_X a$ gives a further bijection into $\mathcal{E}/X(b,P_X a)$. Thus, the category $\mathcal{E}/X$ also has power objects, and so $\mathcal{E}/X$ is a topos, for any object X. □

To say that the Heyting algebra structure is natural in X intuitively means that given some $f\colon X\to Y$ in $\mathcal{E}$, it does not matter whether we first perform operations on subobjects of Y and then convert them to subobjects of X, or vice versa. Essentially it amounts to the property that our Heyting algebras $\mathrm{Sub}_{\mathcal{E}}(X)$ are not biased by our choice of X, and that the Heyting algebra structures respect, to some extent, the structure of $\mathcal{E}$. As any element of $\mathrm{Sub}_{\mathcal{E}}(X)$ is an object of $\mathcal{E}/X$, it turns out that proving the following for any pullback functor $f^*\colon\mathcal{E}/Y\to\mathcal{E}/X$ will suffice.

Theorem 3.10. *For any morphism $f\colon X\to Y$ in the topos $\mathcal{E}$, the functor $f^*\colon\mathcal{E}/Y\to\mathcal{E}/X$, defined by pullback along f, has both a left and a right adjoint and preserves exponentials in the sense that $f^*(m^n)\cong f^*m^{f^*n}$, for all objects m,n of $\mathcal{E}/Y$.*

Proof. We first prove the result for $Y=1$, that is, pullback along morphisms $f\colon X\to 1$, so that $f^*\colon\mathcal{E}/1\to\mathcal{E}/X$. Notice that the functor f^* here is simply $-\times X$, as the product $B\times X$ is the pullback of $!_B$ along f, where $!_B$ is the unique map $B\to 1$. By the definition of the exponential, we thus have a right adjoint $-^X$ to $-\times X=f^*$.

Define the functor $\overline{f}\colon\mathcal{E}/X\to\mathcal{E}/1$ to send an object $l\colon L\to X$ to the unique map $!_L\colon L\to 1$. For any objects $l\colon L\to X$ and $!_M\colon M\to 1$ of $\mathcal{E}/X$ and $\mathcal{E}/1$, respectively, we have a bijection $\mathcal{E}/1(\,!_L,!_M)\cong\mathcal{E}/X(\,l\,,f^*\,!_M)$. This is because arrows $h\colon l\to f^*!_M$ in $\mathcal{E}/X$ are morphisms $h\colon L\to M\times X$ in $\mathcal{E}$ for which $\pi_2 h=l$, and so they uniquely determine a morphism $\pi_1 h\colon L\to M$ in $\mathcal{E}$ and, as $!_M\pi_1 h\colon L\to 1$, we have $!_M\pi_1 h=!_L$, so that $\pi_1 h\colon !_L\to !_M$ in $\mathcal{E}/1$. Conversely, each arrow $j\colon !_L\to !_M$ is a morphism $j\colon L\to M$ in $\mathcal{E}$, and it uniquely determines a map $j\times l\colon L\to M\times X$, for which $\pi_2(j\times l)=l$ and so $j\times l\colon l\to f^*m$. That is, $\overline{f}\dashv f^*$.

To see that f^* preserves exponentials, note first that $f^*(m^n)\cong f^*m^{f^*n}$ if and only if the diagram

$$\begin{array}{ccc} \mathcal{E}/1 & \xrightarrow{(-)^n} & \mathcal{E}/1 \\ {\scriptstyle f^*}\downarrow & & {\scriptstyle f^*}\downarrow \\ \mathcal{E}/X & \xrightarrow{(-)^{f^*n}} & \mathcal{E}/X \end{array}$$

commutes, for all objects m,n in $\mathcal{E}/1$. However, recall from the proof of Theorem 3.9 that the pullback in the topos of objects from the slice topos is precisely the product in the slice topos, and, by the definition of the exponential, we therefore have left adjoints $-\times n \dashv (-)^n$ and $-\times f^*n \dashv (-)^{f^*n}$. From the above, we also have the left adjoint $\overline{f} \dashv f^*$. Notice that we have an isomorphism between the compositions of these left adjoints, as indicated by the commutative diagram

$$\begin{array}{ccccc} P & \longrightarrow & f^*N & \longrightarrow & N \\ \downarrow & & \downarrow{\scriptstyle f^*n} & & \downarrow{\scriptstyle n} \\ K & \xrightarrow{k} & X & \xrightarrow{f} & 1 \end{array} ,$$

in which both squares are pullbacks, so that the entire rectangle is a pullback. In particular, the diagram shows that $\overline{f}k \times n = fk \times n \cong \overline{f}(k \times f^*n)$, for any objects k and n of $\mathcal{E}/X$ and $\mathcal{E}/1$, respectively.

As in the square at the bottom of the previous page, we have that the composite $(-)^{f^*n} \circ f^*$ is right adjoint to $\overline{f} \circ (- \times f^*n)$, and so, by the isomorphism just shown, it is also right adjoint to $(- \times n) \circ \overline{f}$, which in turn is left adjoint to $f^* \circ (-)^n$. By the uniqueness of left adjoints, we therefore have that $f^* \circ (-)^n \cong (-)^{f^*n} \circ f^*$, so that f^* preserves exponentials.

For any function $f\colon X \to Y$ — not necessarily with $Y = 1$ — notice that f is an object in $\mathcal{E}/Y$, and that an object in the slice topos $(\mathcal{E}/Y)/(f)$ is preicsely an arrow $w\colon M \to X$, for which $fw = m$. Every object $w\colon M \to X$ of $\mathcal{E}/X$ uniquely determines the arrow fw of $(\mathcal{E}/Y)/(f)$. We thus have a canonical isomorphism $(\mathcal{E}/Y)/(f) \cong \mathcal{E}/X$ and a pullback functor $\hat{f}\colon (\mathcal{E}/Y)/(1_Y) \to (\mathcal{E}/Y)/(f)$ which is naturally isomorphic to f^*. By the above, this functor $\hat{f}$ has both adjoints and preserves exponentials, and therefore so does the isomorphic f^*. □

4 External and internal Heyting algebras

Definition 4.1. A *homomorphism of Heyting algebras* H and H' is a morphism $m\colon H \to H'$ which preserves $\wedge, \vee$, and $\Rightarrow$, as well as both top and bottom elements. This means that both diagrams

$$\begin{array}{ccc} H \times H & \xrightarrow{\wedge_H} & H \\ {\scriptstyle m\times m}\downarrow & & \downarrow{\scriptstyle m} \\ H' \times H' & \xrightarrow{\wedge_{H'}} & H' \end{array} \quad , \quad \begin{array}{ccc} 1 & \xrightarrow{\top_H} & H \\ \| & & \downarrow{\scriptstyle m} \\ 1 & \xrightarrow{\top_{H'}} & H' \end{array}$$

commute, as do the analogous diagrams with $\vee$ and $\Rightarrow$ or $\bot$ in place of $\wedge$ or $\top$.

Our tedious work above now allows us to draw some surprising connections between topoi and IPC.

Theorem 4.2. *For any object X in a topos $\mathcal{E}$, the set $Sub_{\mathcal{E}}(X)$ is a Heyting algebra. This structure is natural in X, in the sense that for any arrow $f\colon X \to Y$ in the topos $\mathcal{E}$, the functor $f^{-1}\colon Sub_{\mathcal{E}}(Y) \to Sub_{\mathcal{E}}(X)$, defined by pullback along f, is a Heyting algebra homomorphism.*

Proof. We have already shown in Lemma 3.5 that $\mathrm{Sub}_{\mathcal{E}}(X)$ is a lattice. We will first concern ourselves with the lattice of subobjects of the terminal object, $\mathrm{Sub}_{\mathcal{E}}(1)$. Note that for any subobjects $u\colon U \to 1, v\colon V \to 1$, the objects U and V are in the topos $\mathcal{E}$ and consequently there is an exponential U^V, and a unique arrow into the terminal object, $u^v\colon U^V \to 1$. Let f and g be morphisms from some object S to U^V. Under the (exponential) bijection $\mathcal{E}(S, U^V) \cong \mathcal{E}(S \times V, U)$, we have for f and g corresponding morphisms $\overline{f}, \overline{g}\colon S \times V \to U$. If $u^v f = u^v g$, then $u\overline{f} = u\overline{g}$, as in

$$S \times V \underset{\overline{g}}{\overset{\overline{f}}{\rightrightarrows}} U \xrightarrow{u} 1 \,.$$

However, u is monic, so we have that $\overline{f} = \overline{g}$, and so $f = g$ under the bijection. Thus, u^v is monic and therefore is in $\mathrm{Sub}_{\mathcal{E}}(1)$, which is consequently a Heyting algebra.

If 1 is the terminal object of the topos $\mathcal{E}$, and X is some object of the topos, then we have a bijection $\mathrm{Sub}_{\mathcal{E}}(X) \cong \mathrm{Sub}_{\mathcal{E}/X}(1_X)$. By Theorem 3.9, $\mathcal{E}/X$ is a topos, and so by the above $\mathrm{Sub}_{\mathcal{E}/X}(1_X)$ is a Heyting algebra. Thus, under the isomorphism $\mathrm{Sub}_{\mathcal{E}}(X) \cong \mathrm{Sub}_{\mathcal{E}/X}(1_X)$, we have that $\mathrm{Sub}_{\mathcal{E}}(X)$ is a Heyting algebra, for any object X of the topos $\mathcal{E}$.

As pullbacks preserve monics, we have that the pullback of any subobject $m\colon M \to Y$ of Y along $f\colon X \to Y$ produces a subobject of X. Let $i_Y\colon \mathrm{Sub}_{\mathcal{E}}(Y) \to \mathcal{E}/Y$ and $i_X\colon \mathrm{Sub}_{\mathcal{E}}(X) \to \mathcal{E}/X$ be the canonical inclusions, i.e. $i_Y(m\colon M \to Y) = m \in \mathrm{Ob}(\mathcal{E}/Y)$. By the commutative diagram

$$\begin{array}{ccc} \mathrm{Sub}_{\mathcal{E}}(Y) & \xrightarrow{f^{-1}} & \mathrm{Sub}_{\mathcal{E}}(X) \\ {\scriptstyle i_Y}\downarrow & & \downarrow{\scriptstyle i_X} \\ \mathcal{E}/Y & \xrightarrow{f^*} & \mathcal{E}/X \end{array}$$

and Theorem 3.10, we have that f^{-1} preserves limits, and therefore pullbacks, and therefore the operation $\cup$. Notice from our proof of Lemma 3.5 that the operation $\cap$ is the image of the coproduct, which is, from our proof of Proposition 3.4, an equalizer of a pushout. As f^* has a left and right adjoint, it preserves all finite limits and colimits, so we have that f^{-1} must preserve $\cap$. As we defined $x \Rightarrow y$ to be the exponential y^x, and as f^* preserves exponentials by Theorem 3.10, we have that f^{-1} too preserves exponentials. As the pullback of the bottom element $!_Y\colon 0 \to Y$, where 0 is the initial object, is the bottom element $!_X\colon 0 \to X$ of $\mathrm{Sub}_{\mathcal{E}}(X)$, and as the pullback of the top element $1_Y\colon Y \to Y$ is the top element $1_X\colon X \to X$ of $\mathrm{Sub}_{\mathcal{E}}(X)$, we consequently have that f^{-1} is a Heyting algebra homomorphism. □

The result is significant because it demonstrates that every topos provides us with a model of IPC; namely, $\mathrm{Sub}_{\mathcal{E}}(1)$. Many logicians and mathematicians are concerned with not just the law of excluded middles but various axioms of set theory, and it is by this token desireable to have an "internal" or **Set**-free analog of a Heyting algebra. The hope is that, by working with internal objects, we might cleanse ourselves of any **Set**-specific logical pathologies. By an *external* Heyting algebra, we mean an object in **Set** which satisfies Definition 2.5.

Definition 4.3. An *internal lattice* in a topos $\mathcal{E}$ is an object L equipped with two arrows, meet $\wedge\colon L \times L \to L$ and join $\vee\colon L \times L \to L$, such that the diagrams expressing the associative, commutative, and idempotent laws for both $\vee$ and $\wedge$ as well as the absorption law $X \wedge (Y \vee X) = X = (X \wedge Y) \vee X$ are commutative; see the below definition for an example of such a diagrammatic expression. An internal lattice L has top $\top$ and bottom $\bot$ elements if there are arrows $\top\colon 1 \to L$ and $\bot\colon 1 \to L$ from the terminal object of $\mathcal{E}$ such that $\wedge(1 \times \bot) = \vee(1 \times \top) = 1_L$.

Definition 4.4. An *internal Heyting algebra* in a topos $\mathcal{E}$ is an internal lattice H with top and bottom elements with an additional binary operation $\Rightarrow\colon H \times H \to H$ rendering commutative the diagrams expressing the laws

$$X \Rightarrow X = 1, X \wedge (X \Rightarrow Y) = X \wedge Y, Y \wedge (X \Rightarrow Y)$$

$$\text{and}\, X \Rightarrow (Y \wedge Z) = (X \Rightarrow Y) \wedge (X \Rightarrow Z).$$

For example, the last equation is diagrammatically expressed as

$$\begin{array}{ccc}
H \times H \times H & \xrightarrow{1\times\wedge} & H \times H\ , \\
\downarrow{\scriptstyle \delta\times 1\times 1} & & \\
H \times H \times H \times H & & \\
\downarrow{\scriptstyle 1\times\tau\times 1} & & \downarrow{\scriptstyle \Rightarrow} \\
H \times H \times H \times H & & \\
\downarrow{\scriptstyle \Rightarrow\times\Rightarrow} & & \\
H \times H & \xrightarrow{\wedge} & H
\end{array}$$

in which $\tau\colon H \times H \to H \times H$ is the twist map, for which $\pi_1 = \pi_2\tau$ and $\pi_2 = \pi_1\tau$; $\delta\colon H \to H \times H$ is the diagonal map; and by $\Rightarrow(X, Y)$ and $\wedge(X, Y)$ we mean $X \Rightarrow Y$ and $X \wedge Y$, respectively.

We also want internal Heyting algebras because each one gives rise to a whole family of external Heyting algebras, through the bijection $\mathrm{Sub}_{\mathcal{E}}(X \times Y) \cong \mathcal{E}(Y, PX)$. We want naturality internally so that the map $Pf\colon PY \to PX$ preserves the structure of PY, i.e., so that $\wedge \circ (Pf \times Pf) = Pf \circ \wedge$.

Theorem 4.5. *For any object X in a topos $\mathcal{E}$, the power object PX is an internal Heyting algebra. For any morphism $f\colon X \to Y$ in $\mathcal{E}$, the corresponding map $Pf\colon PY \to PA$ is a homomorphism of internal Heyting algebras. For each object Y in $\mathcal{E}$, the set $\mathcal{E}(Y, PX)$ is an (external) Heyting algebra.*

Proof. From the previous theorem we have that $\mathrm{Sub}(X \times B)$ is an external Heyting algebra, for any objects X and B, and hence there is some meet operation $\cap\colon \mathrm{Sub}(X \times B) \times \mathrm{Sub}(X \times B) \to \mathrm{Sub}(X \times B)$. Theorem 4.2 shows that $\cap\colon \mathrm{Sub}(X \times B) \times \mathrm{Sub}(X \times B) \to \mathrm{Sub}(X \times B)$ is natural in B, and so we have that the map $\wedge_B$ in the diagram

$$\begin{array}{ccc}
\mathrm{Sub}(X \times B) \times \mathrm{Sub}(X \times B) & \xrightarrow{\cap} & \mathrm{Sub}(X \times B) \\
\big\wr\!\!\downarrow & & \big\wr\!\!\downarrow \\
\mathcal{E}(B, PX) \times \mathcal{E}(B, PX) & \xrightarrow{\wedge_B} & \mathcal{E}(B, PX)
\end{array}$$

is also natural in B. We therefore have a natural transformation $\wedge_{-}$ from $\mathcal{E}(-, PX) \times \mathcal{E}(-, PX)$ to $\mathcal{E}(-, PX)$. By the Yoneda Lemma, this natural transformation uniquely corresponds to a morphism $\wedge\colon PX \times PX \to PX$; this $\wedge$ is the meet operation for PX. We analogously obtain the operations $\vee$ and $\Rightarrow\colon PX \times PX \to PX$ from the Yoneda Lemma and under the isomorphism $\mathrm{Sub}_{\mathcal{E}}(X \times B) \cong \mathcal{E}(B, PX)$, as Theorem 4.2 ensures that $\vee$ and $\Rightarrow$ are also natural in B.

Theorem 4.2 also shows that both $\top, \bot\colon 1 \to \mathrm{Sub}_{\mathcal{E}}(X \times B)$ are natural in B, so we have a natural transformation $\top_{-}\colon * \to \mathcal{E}(-, PX)$, where the functor $*\colon \mathcal{E} \to \mathbf{Set}$ sends every object to the singleton set, $\{*\}$. As $* \cong \mathcal{E}(-, 1)$, the Yoneda Lemma uniquely determines a corresponding arrow $\top'\colon 1 \to PX$. The bottom element is likewise obtained.

That these operations satisfy the requirements of an internal Heyting algebra is an immediate consequence of Theorem 4.2 and Proposition 1.6. To see this, consider the diagram

$$\begin{array}{ccccc}
PX & & \xleftarrow{\quad\wedge\quad} & & PX \times PX \\
\uparrow{\scriptstyle \pi_1} & & & & \uparrow{\scriptstyle 1\times\vee} \\
PX \times PX & \xrightarrow{\delta\times 1} & PX \times PX \times PX & \xrightarrow{\tau\times 1} & PX \times PX \times PX \\
\downarrow{\scriptstyle \pi_1} & & & & \downarrow{\scriptstyle \wedge\times 1} \\
PX & & \xleftarrow{\quad\vee\quad} & & PX \times PX
\end{array}$$

and the isomorphisms

$$\mathcal{E}(PX \times PX, PX) \cong \mathrm{Nat}(\mathcal{E}(-, PX \times PX), \mathcal{E}(-, PX))$$

$$\cong \mathrm{Nat}(\mathcal{E}(-, PX) \times \mathcal{E}(-, PX), \mathcal{E}(-, PX)) \cong \mathrm{Nat}(\mathcal{E}(X \times -, \Omega) \times \mathcal{E}(X \times -, \Omega), \mathcal{E}(X \times -, \Omega))$$

$$\cong \mathrm{Nat}(\mathrm{Sub}_{\mathcal{E}}(X \times -) \times \mathrm{Sub}_{\mathcal{E}}(X \times -), \mathrm{Sub}_{\mathcal{E}}(X \times -)).$$

The operation $\wedge$, for instance, thus corresponds to the natural transformation

$$\cap\colon \mathrm{Sub}_{\mathcal{E}}(X \times -) \times \mathrm{Sub}_{\mathcal{E}}(X \times -) \to \mathrm{Sub}_{\mathcal{E}}(X \times -).$$

Proposition 1.6 tells us that the above diagram commutes for these morphisms on PX if and only if it commutes for the corresponding natural transformations on $\mathrm{Sub}_{\mathcal{E}}(X \times -)$. Commutativity of the corresponding natural transformations is precisely the commutativity of the components, for every object. But for every object Y, we have by Theorem

4.2 that $\mathrm{Sub}_{\mathcal{E}}(Y \times X)$ is a Heyting algebra, in which it is true that $x \wedge (y \vee x) = x = (x \wedge y) \vee x$. Hence, each necessary diagram commutes, including the above, and PX is an internal Heyting algebra.

To see the naturality of this internal structure, consider the diagrams

$$\begin{array}{ccc} PY & \xleftarrow{\wedge} & PY \times PY \\ \Big\downarrow{\scriptstyle Pf} & & \Big\downarrow{\scriptstyle Pf \times Pf} \\ PX & \xleftarrow[\wedge]{} & PX \times PX \end{array} \quad , \quad \begin{array}{ccc} \mathrm{Sub}_{\mathcal{E}}(Y \times -) & \xleftarrow{\cap} & \mathrm{Sub}_{\mathcal{E}}(Y \times -) \times \mathrm{Sub}_{\mathcal{E}}(Y \times -) \\ \Big\downarrow{\scriptstyle f \times 1^{-1}} & & \Big\downarrow{\scriptstyle (f\times 1)^{-1} \times (f \times 1)^{-1}} \\ \mathrm{Sub}_{\mathcal{E}}(X \times -) & \xleftarrow{\cap} & \mathrm{Sub}_{\mathcal{E}}(X \times -) \times \mathrm{Sub}_{\mathcal{E}}(X \times -) \end{array}$$

and the sequence of isomorphisms connecting them:

$$\mathcal{E}(PY, PX) \cong \mathrm{Nat}(\mathcal{E}(-, PY), \mathcal{E}(-, PX))$$

$$\cong \mathrm{Nat}(\mathcal{E}(Y \times -, \Omega), \mathcal{E}(X \times -, \Omega)) \cong \mathrm{Nat}(\mathrm{Sub}_{\mathcal{E}}(Y \times -), \mathrm{Sub}_{\mathcal{E}}(X \times -)).$$

The left square commutes if and only if the right square does, which in turn commutes if and only if, for every object Z in the topos $\mathcal{E}$, the diagram

$$\begin{array}{ccc} \mathrm{Sub}_{\mathcal{E}}(Y \times Z) & \xleftarrow{\cap} & \mathrm{Sub}_{\mathcal{E}}(Y \times Z) \times \mathrm{Sub}_{\mathcal{E}}(Y \times Z) \\ \Big\downarrow{\scriptstyle (f \times 1)^{-1}} & & \Big\downarrow{\scriptstyle (f\times 1)^{-1} \times (f \times 1)^{-1}} \\ \mathrm{Sub}_{\mathcal{E}}(X \times Z) & \xleftarrow{\cap} & \mathrm{Sub}_{\mathcal{E}}(X \times Z) \times \mathrm{Sub}_{\mathcal{E}}(X \times Z) \end{array}$$

does. However, by Theorem 4.2, we have that $(f \times 1)^{-1}$ is a Heyting algebra homomorphism, and so each diagram displayed commutes, as do the analogs for $\vee, \Rightarrow, \top$, and $\bot$. Therefore, for any $f\colon X \to Y$, we have that $Pf\colon PY \to PX$ is an internal Heyting algebra homomorphism, so that the structure PX is natural in X. For any object Y of $\mathcal{E}$, we have that $\mathcal{E}(Y, PX)$ is an external Heyting algebra by the canonical isomorphism $\mathcal{E}(Y, PX) \cong \mathrm{Sub}_{\mathcal{E}}(X \times Y)$. □

5 Conclusion and acknowledgements

In effect, we have shown that there is an intimate relationship between topoi and Heyting algebras, which we have seen are models for IPC. Anything that can be proven without the law of excluded in middles — in IPC — must get sent to 1 under every valuation into *every* Heyting algebra that we can obtain from a topos.

I can not possibly express how grateful I am for my mentor Michael Smith and all of his time that I wasted. He informed me early on that topos theory was rather difficult, but he was incredibly patient, without fail, throughout all of my struggles. Without his genuine concern for my studies and his remarkable ability to explain category theory, I probably wouldn't have made it past page 6.

References

[1] Saunders Mac Lane, *Categories for the Working Mathematician*. 2nd edition, 1998.

[2] Saunders Mac Lane and Ieke Moerdijk, *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*. 1992.

[3] Erik Palmgren, *Semantics of Intuitionistic Propositional Logic*. 2009.

[4] Peter T. Johnstone, *Sketches of an Elephant: A Topos Theory Compendium*. 2002.