# HYPERBOLIC GEOMETRY: ISOMETRY GROUPS OF HYPERBOLIC SPACE

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ABSTRACT. The goal of this paper is twofold. First, it consists of an introduction to the basic features of hyperbolic geometry, and the geometry of an important class of functions of the hyperbolic plane, isometries. Second, it identifies a group structure in the set of isometries, specifically those that preserve orientation, and deals with the topological properties of their discrete subgroups. In the final section, we show that if one of these Fuchsian groups satisfies certain conditions, then its quotient space is a 2-manifold for which the hyperbolic plane is a covering space.

CONTENTS



## 1. HYPERBOLIC GEOMETRY AND PSL(2,$\mathbb{R}$)

There are several models of hyperbolic space, but for the purposes of this paper we will restrict our view to the Lobachevski upper half plane.

**Definition 1.1.** The upper half plane is the subset $\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$ of $\mathbb{C}$ with the metric

$$ds = \frac{\sqrt{dx^2 + dy^2}}{y}$$

where $z = x + iy$.

We can use this metric to find the length of a differentiable path $\gamma$ through hyperbolic space by integrating over its domain. The hyperbolic length, $h(\gamma)$ of the differentiable path $\gamma : [0,1] \to \mathbb{H}$ where $\gamma(t) = (x(t), y(t))$ is

$$h(\gamma) = \int_0^1 \frac{\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}}{y(t)} dt = \int_0^1 \frac{|\frac{dz}{dt}|}{y(t)} dt$$

since $\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}$ is the Euclidean norm of the vector $\frac{dz}{dt}$.

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**Definition 1.2.** The hyperbolic distance $\rho(z, w)$ between $z, w \in \mathbb{H}$ is given by

$$\rho(z, w) = \inf h(\gamma)$$

where $\gamma$ is an element of the set of all differentiable paths connecting $z$ and $w$.

**Definition 1.3.** An isometry of $\mathbb{H}$ is a function $g : \mathbb{H} \to \mathbb{H}$ such that for any $z, w \in \mathbb{H}$, $\rho(z, w) = \rho(g(z), g(w))$.

This paper will focus on the properties of special *groups* of isometries of $\mathbb{H}$.

**Definition 1.4.** A *group* is a set $G$ with an operation $*$ which satisfies the axioms G1 through G3

(G1) $*$ is associative.
(G2) There is an element $e$ in $G$ such that $a * e = a$ and $e * a = a$ for every element $a$ in $G$.
(G3) For every element $a$ in $G$, there is an element $a^{-1}$ in $G$ such that $a * a^{-1} = e$ and $a^{-1} * a = e$.

One group of isometries (of $\mathbb{R}^2$) particularly important to our aims is the special linear group SL(2,$\mathbb{R}$). SL(2,$\mathbb{R}$) is the set of 2 by 2 matrices with determinant 1 under the group operation of matrix multiplication. That SL(2,$\mathbb{R}$) is a group is easy to show using the multiplicity of determinants, the existence of an identity matrix, and the inversion formula for 2 by 2 matrices.

**Definition 1.5.** The elements of the *projective special linear group* PSL(2,$\mathbb{R}$) are the Möbius transformations, rational functions from $\mathbb{C}$ to $\mathbb{C}$ of the form $z \mapsto \frac{az+b}{cz+d}$ with $ad - bc = 1$.

We now want to show that PSL(2,$\mathbb{R}$) is a group. We see immediately that the set PSL(2,$\mathbb{R}$) is isomorphic to SL(2,$\mathbb{R}$) modulo $\{\pm Id\}$, because for any element $\frac{az+b}{cz+d}$ a factor of $-1$ in the numerator will cancel with a factor of $-1$ in the denominator. Hence, we set up a surjective identification

$$g(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \frac{az + b}{cz + d}$$

from SL(2,$\mathbb{R}$) to PSL(2,$\mathbb{R}$).

**Definition 1.6.** If $G$ and $H$ are groups, a *homomorphism* from $G$ to $H$ is a function $f : G \to H$ such that for any two elements $a$ and $b$ in $G$

$$f(ab) = f(a)f(b).$$

Let the group operation on PSL(2,$\mathbb{R}$) be function composition. A simple calculation confirms that composition of two Möbius transformations corresponds to the multiplication of the corresponding 2 by 2 matrices in SL(2,$\mathbb{R}$). Furthermore, the inverse matrices and identity matrices of SL(2,$\mathbb{R}$) on the subset SL(2,$\mathbb{R}$) $/\{\pm Id\}$ correspond to the inverse and identity elements of PSL(2,$\mathbb{R}$). We conclude that PSL(2,$\mathbb{R}$) is a group and that our identification $g$ is a surjective group homomorphism. Hence, we can say that PSL(2,$\mathbb{R}$) is a *homomorphic image* of SL(2,$\mathbb{R}$).

PSL(2,$\mathbb{R}$) plays an important part in our overall goal to classify the isometries of $\mathbb{H}$. We must first prove that the elements of PSL(2,$\mathbb{R}$) are in fact isometries, and then we will examine what properties distinguish PSL(2,$\mathbb{R}$) from the rest of the isometries of $\mathbb{H}$.

**Lemma 1.7.** *Every Möbius transformation $\phi$ in PSL(2,$\mathbb{R}$) is a homeomorphism of $\mathbb{H}$.*

*Proof.* First show that $\phi$ maps $\mathbb{H}$ into $\mathbb{H}$. Let $\phi(z) = \frac{az+b}{cz+d}$ where $ad - bc = 1$ and $z \in \mathbb{H}$.

$$\phi(z) = \frac{(az+b)(c\overline{z}+d)}{(cz+d)(c\overline{z}+d)} = \frac{ac|z|^2 + adz + bc\overline{z} + bd}{|cz+d|^2}$$

$$Im(\phi(z)) = \frac{\phi(z) - \overline{\phi(z)}}{2i} = \frac{adz + bc\overline{z} - ad\overline{z} - bcz}{2i|cz+d|^2}$$

$$= \frac{(ad-bc)(z-\overline{z})}{2i|cz+d|^2} = \frac{Im(z)}{|cd+z|^2}$$

The imaginary component of $z$ is greater than 0, so clearly $Im(\phi(z)) > 0$, which implies that $\phi(z)$ is in $\mathbb{H}$.

Because $\phi$ is a rational function with a nonzero denominator, continuity is clear. The existence of an inverse follows from the fact that $\phi^{-1}$ is also an element of PSL(2,$\mathbb{R}$). By the argument above, it is continuous and maps into $\mathbb{H}$. Therefore, $\phi$ is a homeomorphism of $\mathbb{H}$. □

**Theorem 1.8.** *PSL(2,$\mathbb{R}$) $\subset$ Isom($\mathbb{H}$).*

*Proof.* Let $\gamma : [0,1] \to \mathbb{H}$ be a piecewise differentiable path in $\mathbb{H}$. Let $\gamma$ be given by $z(t) = (x(t), y(t))$ and $w(t) = T(z(t)) = u(t) + iv(t)$ By the quotient rule,

$$\frac{dw}{dt} = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{acz - caz + ad - bc}{(cz+d)^2} = \frac{1}{(cz+d)^2}$$

Since $Im(T(z)) = \frac{Im(z)}{|cz+d|^2} = v$ and $y = Im(T(z))$ by Lemma 1.7, $|\frac{dw}{dt}| = \frac{v}{y}$. From the definition of hyperbolic length, we obtain the following equation.

$$h(T(\gamma)) = \int_0^1 \frac{|\frac{dw}{dt}|}{v(t)} dt = \int_0^1 \frac{|\frac{dw}{dz}\frac{dz}{dt}|}{v(t)} dt = \int_0^1 \frac{|\frac{v(t)}{y(t)}\frac{dz}{dt}|}{v(t)} dt = \int_0^1 \frac{|\frac{dz}{dt}|}{y(t)} dt = h(\gamma).$$

The hyperbolic distance $\rho(j,k)$ is the infimum of the differentiable paths $\gamma$ between j and k, so since each $\gamma$ is invariant under T, $\rho(j,k)$ is invariant under T. Thus, $T \in$ PSL(2,$\mathbb{R}$) is an isometry. [1] □

There is one further property that classifies PSL(2,$\mathbb{R}$) as a subset of Isom($\mathbb{H}$), which is that PSL(2,$\mathbb{R}$) is the set of all *orientation preserving* isometries of $\mathbb{H}$. A linear operator on a vector space is orientation preserving if its determinant is positive. The equivalent notion for Möbius transformations is the following:

**Definition 1.9.** Let $\phi$ be a map $\mathbb{H} \to \mathbb{H}$. Then $\phi$ is orientation preserving if $\det(D\phi_z) > 0$ for all $z \in \mathbb{H}$.

**Definition 1.10.** The *tangent space* at $z$ in $\mathbb{H}$, written $T_z\mathbb{H}$, is the vector space generated by the collection of vectors based at $z$ tangent to a curve through $z$.

The derivative of $\phi$ at $z$ $D\phi_z$ is a linear map from $T_z\mathbb{H}$ to $T_{\phi(z)}\mathbb{H}$. To show that PSL(2,$\mathbb{R}$) is orientation preserving, we will establish an inner product on $T_z\mathbb{H}$. Observe that each point in $\mathbb{H}$ has two linearly independent tangent vectors associated with it and that $\mathbb{H}$ is a two dimensional subset of $\mathbb{R}^2$, which imply that $T_z\mathbb{H} \cong \mathbb{R}^2$. This observation motivates the following definition.

**Definition 1.11.** The inner product of $v, w \in T_z\mathbb{H}$ is given by

$$< v, w >_z = \frac{1}{(Im(z))^2} < v, w >_{\mathbb{R}^2}$$

where $< v, w >_{\mathbb{R}^2}$ is the standard dot product for $\mathbb{R}^2$.

**Definition 1.12.** A function $\phi : \mathbb{H} \to \mathbb{H}$ is an isometry if for all $z \in \mathbb{H}$ and $v, w \in T_z\mathbb{H}$

$$< v, w >_z = < D\phi_z(v), D\phi_z(w) >_{\phi(z)} .$$

*Remark* 1.13. It follows from a simple calculation that this definition is equivalent to Definition 1.3.

**Lemma 1.14.** *If $\sigma$ and $\phi$ preserve orientation, then $\phi \circ \sigma$ preserves orientation.*

*Proof.* This follows immediately from the multiplicity of determinants. $D(\phi \circ \sigma)_z = D\phi_{\sigma(z)} \circ D\sigma_z$ by the chain rule. Then, $\det D(\phi \circ \sigma)_z = \det(D\phi_{\sigma(z)} \circ D\sigma_z) = \det D\phi_{\sigma(z)} \cdot \det D\sigma_z > 0$, since $\det D\phi_{\sigma(z)} > 0$ and $\det D\sigma_z > 0$ since $\phi$ and $\sigma$ are orientation preserving. □

**Lemma 1.15.** *Any isometry of $\mathbb{H}$ of the form $z \mapsto yz + x$ where $y, x \in \mathbb{R}, y \neq 0$ and $z \in \mathbb{H}$ preserves orientation.*

*Proof.* Let $z = r + is$, or equivalently $z = (r, s) \in \mathbb{C}$ where $r, s \in \mathbb{R}$. Let $f(r, s) = (yr + x, ys)$.

$$Df_{(r,s)} = \begin{pmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial s} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial s} \end{pmatrix} = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}$$

$\det(Df_{(r,s)}) = y^2 > 0$, so $f$ preserves orientation. □

**Theorem 1.16.** *PSL(2,$\mathbb{R}$) is the set of all orientation preserving isometries.*

*Proof.* Let $\phi$ be an orientation preserving isometry, and $\phi(i) = a + bi$. Our goal is to show that $\phi$ is in PSL(2,$\mathbb{R}$). We want to find $x, y \in \mathbb{R}$ such that $x \cdot \phi(i) + y = i$.

$$i = x \cdot (a + bi) + y = xa + xbi + y$$

Choose $x = \frac{1}{b}$ and $y = \frac{-a}{b}$. Let $\sigma = \frac{1}{b} \cdot z + \frac{-a}{b}$ and $(\sigma \circ \phi)(i) = i$. Since $\sigma$ is of the form $z \mapsto yz + x$, $\sigma$ preserves orientation by Lemma 1.15. By assumption, $\phi$ preserves orientation, so by Lemma 1.14 $\sigma \circ \phi$ preserves orientation.

Every Möbius transformation $z \mapsto xz + y$ is an element of PSL(2,$\mathbb{R}$), since dividing the numerator and denominator by $\frac{1}{\sqrt{x}}$ will give an equivalent matrix of determinant 1.

The composition of two isometries is an isometry, so $\sigma \circ \phi$ is an orientation preserving isometry that fixes $i$. The inner product on the tangent space at $i$ $T_i\mathbb{H}$ is the usual inner product on $\mathbb{R}^2$ since $Im(i) = 1$. Since $\sigma \circ \phi$ is an isometry, for any $v, w \in T_i\mathbb{H}$

$$< v, w >_i = < D(\sigma \circ \phi)_i(v), D(\sigma \circ \phi)_i(w) >_{\sigma\circ\phi(i)} = < D(\sigma \circ \phi)_i(v), D(\sigma \circ \phi)_i(w) >_i$$

This equation shows us that $D(\sigma \circ \phi)_i$ is a linear transformation in $\mathbb{R}^2$ that is represented by an orthogonal matrix. The composition $\sigma \circ \phi$ preserves orientation, so $\det(D(\sigma \circ \phi)_i) > 0$. This in turn implies that $\det D(\sigma \circ \phi)_i = 1$ because the determinant of an orthogonal matrix is $\pm 1$. Basic linear algebra tells us that an orthogonal matrix with determinant 1 is a rotation of the plane, so there exists a $\theta$ such that $D(\sigma \circ \phi)_i$ can be written in the form

$$D(\sigma \circ \phi)_i = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

To complete the proof that $\phi$ is in PSL(2,$\mathbb{R}$), we need to find a Möbius transformation $\frac{az+b}{cz+d}$ that fixes $i$, has determinant greater than 0 and has the same differential as $\sigma \circ \phi$ at $i$. To fix $i$, we need $i = \frac{ai+b}{ci+d}$, so $ai + b = di - c$, and $a = d, b = -c$. This calculation gives us $\chi = \frac{az+b}{-bz+a}$, with $\det \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = a^2 + b^2 > 0$ and $\chi(i) = i$. All that remains is to check the derivative. A lengthy elementary calculation will reveal the following partial derivatives for $D\chi_{(0,1)}$ where $z = (x, y)$:

$$\left.\frac{\partial \chi_1}{\partial x}\right|_{(0,1)} = \frac{a^2 - b^2}{a^2 + b^2} \quad \left.\frac{\partial \chi_1}{\partial y}\right|_{(0,1)} = \frac{-2ab}{a^2 + b^2}$$

$$\left.\frac{\partial \chi_2}{\partial x}\right|_{(0,1)} = \frac{2ab}{a^2 + b^2} \quad \left.\frac{\partial \chi_2}{\partial y}\right|_{(0,1)} = \frac{a^2 - b^2}{a^2 + b^2}$$

It is clear that the numerators follow the double angle formulas for sine and cosine. If we take $\theta$ from $D(\sigma \circ \phi)_i$ and let $a = \cos(\frac{\theta}{2})$ and $b = \sin(\frac{\theta}{2})$, then $a^2 + b^2 = 1$ and $a^2 - b^2 = \cos^2(\frac{\theta}{2}) - \sin^2(\frac{\theta}{2}) = \cos\theta$ and $2ab = 2\cos(\frac{\theta}{2})\sin(\frac{\theta}{2}) = \sin\theta$. This substitution gives us the following Jacobian.

$$D\chi_{(0,1)} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = D(\sigma \circ \phi)_i.$$

Our transformation $\chi = \sigma \circ \phi$ and $\chi \in$ PSL(2,$\mathbb{R}$), so $\sigma \circ \phi \in$ PSL(2,$\mathbb{R}$). Since $\sigma \in$ PSL(2,$\mathbb{R}$), $\sigma^{-1}$ exists and is in PSL(2,$\mathbb{R}$). Therefore,

$$\phi = \sigma^{-1} \circ \chi$$

Since $\sigma^{-1}, \chi \in$ PSL(2,$\mathbb{R}$) and PSL(2,$\mathbb{R}$) is a group and is thus closed under composition, every orientation preserving isometry $\phi$ can be written as a product of two elements of PSL(2,$\mathbb{R}$) and is thus itself an element of PSL(2,$\mathbb{R}$). We conclude that PSL(2,$\mathbb{R}$) = Isom$^+(\mathbb{H})$, the set of orientation preserving isometries of $\mathbb{H}$. □

## 2. Geodesics

A *geodesic* is a curve passing through two distinct, arbitrary points in $\mathbb{H}$ along which distance is locally minimized. In this section, we classify the geodesics of $\mathbb{H}$, prove the equivalence of hyperbolic length and length along the geodesic, and prove the preservation of geodesics under PSL(2,$\mathbb{R}$).

**Definition 2.1.** A straight line or semicircle in $\mathbb{H}$ that is orthogonal to $\mathbb{R} \cup \{\pm\infty\}$ is given by an equation of the form

$$\alpha z\bar{z} + \beta z + \beta\bar{z} + \gamma = 0 \tag{2.2}$$

where $\alpha$, $\beta$, and $\gamma \in \mathbb{R}$.

**Lemma 2.3.** *If $L$ is a semicircle or straight line in $\mathbb{H}$ orthogonal to the real axis and $\phi \in$ PSL(2,$\mathbb{R}$), then $\phi(L)$ is a semicircle or straight line orthogonal to the real axis.*

*Proof.* Let $\phi(z) = \frac{az+b}{cz+d}$ be an arbitrary element of PSL(2,$\mathbb{R}$). Then, $\phi^{-1}(w) = \frac{dw-b}{-cw+a}$. Let $w = \phi(z)$ for an arbitrary $z \in \mathbb{H}$. Substitute $z = \phi^{-1}(w)$ into Equation 2.2 to implicitly find the image of the $z$ that lie along some given curve.

$$\alpha\left(\frac{dw-b}{-cw+a}\right)\left(\frac{d\overline{w}-b}{-c\overline{w}+a}\right)+\beta\left(\frac{dw-b}{-cw+a}\right)+\beta\left(\frac{d\overline{w}-b}{-c\overline{w}+a}\right)+\gamma=0$$

Clearing the denominators by multiplying both sides by a factor of $(-cw+a)(-c\overline{w}+a)$ gives us

$$\begin{aligned}\alpha(dw-b)(d\overline{w}-b)+\beta(dw-b)(-c\overline{w}+a)+\beta(d\overline{w}-b)(-cw+a)\\ +\gamma(-cw+a)(-c\overline{w}+a)=0\\ (\alpha d^2-2\beta cd+\gamma c^2)w\overline{w}+(-\alpha bd+\beta ad+\beta bc-\gamma ac)w\\ +(-\alpha bd+\beta ad+\beta bc-\gamma ac)\overline{w}+(\alpha b^2-2\beta ab+\gamma a^2)=0\end{aligned}$$

This follows equation 2.2, and thus $L$ is mapped to a semicircle or straight line orthogonal to the real axis. [2] □

**Proposition 2.4.** *If L is a Euclidean circle or straight line orthogonal to the real axis, and meets the real axis at some finite point $\alpha$, there exists an element of PSL(2,$\mathbb{R}$) that maps L to the imaginary axis.*

*Proof.* If $L$ is the straight vertical line $Re(z) = \alpha$, then $z \mapsto z - a$ will map $L$ to the imaginary axis. The determinant of the corresponding matrix is 1, so it is an element of PSL(2,$\mathbb{R}$).
If $L$ is a Euclidean circle orthogonal to the real line, then it must intersect the real axis at some second point $\beta$. Without loss of generality, say $\alpha > \beta$. Then the map $g$ that sends z to $\frac{z-\alpha}{z-\beta}$ is an element of PSL(2,$\mathbb{R}$) because $-\beta + \alpha > 0$. If we take the image of the endpoints, then $g$ maps $\alpha$ to 0 indicating a straight line or circle that intersects the origin, and the limit of $g(z)$ as $z$ approaches $\beta$ is infinite.

$$\lim_{z\to\beta}\frac{z-\alpha}{z-\beta}=\infty$$

Since a Euclidean circle must have a finite radius, it will intersect the real axis at some point. Therefore, the image must be a straight vertical line with $Re(z) = 0$, that is, the imaginary axis. [2] □

**Definition 2.5.** A *geodesic* is a curve $L \in \mathbb{H}$ such that for two points $z, w \in \mathbb{H}$, the shortest path between them is $L$.

**Theorem 2.6.** *The geodesics in $\mathbb{H}$ are semicircles and straight lines orthogonal to the real axis $\mathbb{R} \cup \{\pm\infty\}$.*

*Proof.* Let $ia, ib \in \mathbb{H}$ be two distinct points on the imaginary axis, and $\gamma : [0, 1] \to \mathbb{H}$ be a differentiable path joining them. With $\gamma(t) = (x(t), y(t))$,

$$h(\gamma)=\int_0^1\frac{\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}}{y(t)}dt\geq\int_0^1\frac{|\frac{dy}{dt}|}{y(t)}dt\geq\int_0^1\frac{\frac{dy}{dt}}{y(t)}dt=\int_a^b\frac{dy}{y}=\ln\frac{a}{b}$$

We can see that the first inequality becomes an equality if there is no change in $x(t)$ and $\frac{dx}{dt} = 0$. That is, $Re(z)$ does not vary with $t$ and the path moves only vertically. The second inequality becomes an equality if $\frac{dy}{dt} > 0$, which means that the path moves positively in the y direction as $t$ varies from 0 to 1. This means that along the path, $y$ varies hyperbolically (by a factor of $\frac{1}{y}$) from $a$ to $b$ as $t$ increases, and $\gamma$ is thus a vertical line segment from $ia$ to $ib$ with length $\ln \frac{a}{b}$. Since the length $h(\lambda)$ of any arbitrary path $\lambda$ between these points is greater than or equal to $\ln \frac{a}{b}$, the shortest possible path must be the straight vertical line.

Take arbitrary $z, w \in \mathbb{H}$, and let $L$ be a unique Euclidean circle or straight line orthogonal to $\mathbb{R}$ passing through them. There is a unique element $\phi \in$ PSL(2,$\mathbb{R}$) that maps $L$ to the imaginary axis, and the segment of $L$ joining $z$ and $w$ is mapped to the segment of the imaginary axis joining $Tz$ and $Tw$. By the definition of an isometry though, for all $\gamma$ joining $z$ and $w$, and their images $T\gamma$ joining $Tz$ and $Tw$, $h(\gamma) = h(T\gamma)$. Thus, the shortest path joining $Tz$ and $Tw$ is the straight line, which is mapped to the shortest path joining $z$ and $w$, the curve $L$. We conclude that the geodesics of $\mathbb{H}$ are the straight vertical lines and Euclidean circles orthogonal to the real axis. □

**Corollary 2.7.** *The hyperbolic distance $\rho$ between two points $z, w \in \mathbb{H}$ is equal to the hyperbolic length of the unique geodesic segment $L$ connecting them.*

*Proof.* We are assuming the circle $L$ centered on the real axis connecting z and w is unique by the principles of Euclidean geometry. We know $h(\gamma) \geq h(L)$ for any differentiable path $\gamma$ connecting $z$ and $w$. Thus, $\rho(z, w) = \inf h(\gamma) = h(L)$. □

**Corollary 2.8.** *For distinct $z, w \in \mathbb{H}$,*

$$\rho(z, w) \leq \rho(z, v) + \rho(v, w)$$

*with equality holding if and only if $v \in L$, where $L$ is the geodesic connecting $v$ and $w$.*

**Theorem 2.9.** *Any element of PSL(2,$\mathbb{R}$) maps geodesics onto geodesics in $\mathbb{H}$.*

*Proof.* Let $\phi$ be a Möbius transformation from PSL(2,$\mathbb{R}$) and $z, w$ be distinct points in $\mathbb{H}$ with geodesic $L$ connecting them. Let $v$ be an arbitrary point on $L$. By corollary 2.8, we know $\rho(z, w) = \rho(z, v) + \rho(v, w)$. Since $\phi$ is an isometry, $\rho(\phi(z), \phi(w)) = \rho(\phi(z), \phi(v)) + \rho(\phi(v), \phi(w))$ and thus by the same corollary, $\phi(v)$ must be an element of the geodesic $L_\phi$ that connects $\phi(z)$ and $\phi(w)$. Thus, every element of the geodesic $L$ gets mapped to an element of the geodesic $L_\phi$, and vice versa for $\phi^{-1}$. Thus any transformation in PSL(2,$\mathbb{R}$) maps geodesics to geodesics in $\mathbb{H}$. □

Given this fact, we can see some of the intuition behind Theorem 1.16. If we take the image of the point $i$ under $\phi \in$ PSL(2,$\mathbb{R}$) and the image of any other arbitrary point $z$, there is a unique geodesic connecting them. The unique preimage of this geodesic is another geodesic in $\mathbb{H}$ connecting $i$ and $z$. Thus, $\phi \in$ PSL(2,$\mathbb{R}$) can be classified by the action of $\phi$ on $i$ together with a rotation of the unit vector based at $i$ tangent to the geodesic containing $[i, z]$. More specifically, the unit tangent vector at $i$ is transformed into the unit vector based at $\phi(i)$ tangent to the geodesic containing $[\phi(i), \phi(z)]$. This corresponds to the two transformations that constitute $\phi$ in the proof of Theorem 1.13: $\chi$, a rotation of the tangent space at $i$, and $\sigma^{-1}$ a translation that moves $i$ to $\phi(i)$.

## 3. Discrete Isometry Groups and Proper Discontinuity

Recall that in section 1 we established that PSL(2,$\mathbb{R}$) was a group under the operation of function composition. In the following section we will discuss a special class of subgroups of PSL(2,$\mathbb{R}$) called *Fuchsian groups*.

**Definition 3.1.** A discrete subgroup of Isom($\mathbb{H}$) is called a Fuchsian group if it consists of orientation-preserving transformations. Equivalently, a Fuchsian group is a discrete subgroup of PSL(2,$\mathbb{R}$).

To make sense of Definition 3.1, we must establish a notion of discreteness for a subgroup of PSL(2,$\mathbb{R}$). Our strategy will be to introduce a definition of discreteness for general topological spaces and to topologize PSL(2,$\mathbb{R}$) in order to apply this definition to it.

**Definition 3.2.** Let $S$ be a set in a topological space $X$. $S$ is discrete if for every $t$ in $S$, there exists an open set $U \subset X$ such that $U \cap S = \{t\}$.

To apply this definition to PSL(2,$\mathbb{R}$), we must induce a topology on PSL(2,$\mathbb{R}$). Let $S = \{(a,b,c,d) \in \mathbb{R}^4 \mid ad - bc = 1\}$ and identify the elements of SL(2,$\mathbb{R}$), 2 by 2 matrices with entries $a, b, c, d$, with the elements of the subset $S$.

**Definition 3.3.** Given a topological space $(X, \tau)$ and a subset $B$ of $X$, the *subspace topology* on $B$ is defined as

$$\tau_B = \{B \cap U \mid U \in \tau\}.$$

Let us induce the subspace topology $\tau_S$ of $S$ from the natural topology of $\mathbb{R}^4$ using Definition 3.3. Now we can topologize SL(2,$\mathbb{R}$) with the subspace topology of $S$.

However, we cannot equip PSL(2,$\mathbb{R}$) with the topology $\tau_S$ because the elements $(a,b,c,d)$ and $(-a,-b,-c,-d)$ are distinct in SL(2,$\mathbb{R}$) but not in PSL(2,$\mathbb{R}$). Let us define an equivalence relation $\sim$ on $S$ such that $(a,b,c,d) \sim (-a,-b,-c,-d)$.

**Definition 3.4.** Let $(X, \tau_X)$ be a topological space and $\sim$ be an equivalence relation on $X$. The *quotient space* $Y = X/\sim$ is defined as the set of equivalence classes of elements of X

$$Y = \{\{v \in X \mid v \sim x\} \mid x \in X\}$$

equipped with the topology where the open sets $\tau_Y$ of $Y$ are defined to be the sets of equivalence classes whose unions are open sets in $X$

$$\tau_Y = \{U \subset Y \mid \bigcup U \in \tau_X\}.$$

We now topologize PSL(2,$\mathbb{R}$) by identifying it with the quotient space $S/\sim$. Since we have shown that PSL(2,$\mathbb{R}$) is a topological space, it is possible to define discreteness for a subgroup of PSL(2,$\mathbb{R}$) in the sense given by Definition 3.2.

**Definition 3.5.** A subgroup $\Gamma$ of PSL(2,$\mathbb{R}$) is called discrete if $\Gamma$ is a discrete set in the topological space PSL(2,$\mathbb{R}$).

To further our understanding of discreteness, our next result will show the implications of a discrete topology on the convergence of sequences. Before we prove this result, we must know first what it means for a sequence of elements to converge in a topological space.

**Definition 3.6.** Let $(X, \tau)$ be a topological space and $x_n \in X$ a sequence. We say that the sequence $x_n$ converges to $x_0 \in X$ if for every open set $U \subset X$ which contains $x_0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the points $x_n$ lie in $U$.

**Proposition 3.7.** *A subgroup $\Gamma$ of PSL(2,$\mathbb{R}$) is discrete if and only if the convergence of a sequence of elements of $\Gamma$, $(T_n)_{n=1}^{\infty}$, to the identity element $Id$ implies that $T_n$ equals $Id$ for sufficiently large $n$.*

*Proof.* Let us prove the forward implication. Suppose $\Gamma$ contains the identity and is a discrete subset of PSL(2,$\mathbb{R}$). Then, there exists an open set $U$ in PSL(2,$\mathbb{R}$) such that $U \cap \Gamma = Id$. By Definition 3.6, there is an $n_0 \in \mathbb{N}$ such that if $n > n_0$, then the points $T_n$ lie in $U$. Since $T_n$ is a subset of $\Gamma$, $T_n = Id$ for $n > n_0$.

We now prove the contrapositive of the reverse implication. Suppose $\Gamma$ is not a discrete subset of PSL(2,$\mathbb{R}$). Then there is a $T$ in $\Gamma$ such that there is no open set $U \subset$ PSL(2,$\mathbb{R}$) for which $U \cap \Gamma = \{T\}$. If the homeomorphism $g : \Gamma \to \Gamma$ is defined as right multiplication by $T^{-1}$, we see that the previous statement is also true for $g(T)$, where $g(T) = Id$. This implies that every open set $U$ will contain a non-identity element of $\Gamma$. Hence, we can construct a sequence that satisfies Definition 3.6, but for which $\{T_n\} \cap U \neq \{Id\}$ for any $U$. □

Fuchsian groups can be considered analogously with lattices in $\mathbb{R}^n$ which are discrete groups of isometries in n-dimensional Euclidean space. These lattices act *discontinuously* on $\mathbb{R}^n$ in the sense that for any point $x$ in $\mathbb{R}^n$, there is a neighborhood $U$ of $x$ such that $g(U) \cap U = \emptyset$ for all non-identity elements $g$ in the lattice.

Since it is possible for elements of a Fuchsian group to fix points in $\mathbb{H}$, there cannot be an analogous neighborhood for these fixed points. We can, however, establish *proper discontinuity.* Proper discontinuity is a weaker notion of discontinuity that ensures that for every point in $\mathbb{H}$, there is a similar neighborhood that is carried outside of itself by all but finitely many elements of the Fuchsian group.

**Definition 3.8.** Let $X$ be a metric space, and let $G$ be a group of homeomorphisms of $X$. $G$ acts *properly discontinuously* on $X$ if for any compact set $K$ in $X$, $g(K) \cap K \neq \emptyset$ for only finitely many $g \in G$.

*Remark* 3.9. For the remainder of this section, $X$ is a metric space, and $G$ is a group of homeomorphisms of $X$ unless noted otherwise.

**Definition 3.10.** The set $Gx = \{g(x) \mid g \in G\}$ is the *$G$-orbit of the point $x$.* Each point of $Gx$ is contained with multiplicity equal to the cardinality of the set $G_x = \{g \in G \mid g(x) = x\}$, called the *stabilizer of $x$ in $G$.*

**Theorem 3.11.** *$G$ acts properly discontinuously on $X$ if and only if the order of the stabilizer $G_x$ of every point $x$ in $X$ is finite and the $G$-orbit of each point $x$ in $X$ is discrete.*

*Proof.* Let us prove the forward implication by proving the contrapositive. Suppose the order of the stabilizer $G_x$ is infinite. By the subsequence definition of compactness, we know that the singleton $\{x\}$ is a compact set. The intersection $g(\{x\}) \cap \{x\}$ equals $\{x\}$ for infinitely many $g$ in $G$, which implies that the action of $G$ on $X$ is not properly discontinuous.

Furthermore, suppose that the $G$-orbit of some $x$ in $X$ is not discrete. This implies that there is a point $g(x)$ in the $G$-orbit of some arbitrary $x$ in $X$ that is not isolated. That is, there exists a sequence of elements in $G$ $(g_n)_{n=1}^{\infty}$ such that $g_1(x), g_2(x), ..., g_n(x)...$ converges to $g(x)$ in $X$. Let $S$ equal $\{g_n(x)\} \subset X$, the set of terms in this sequence. It is apparent that any infinite sequence of elements in $S$ with infinitely many distinct terms has a subsequence converging to $g(x)$. Hence, $S$ is compact.

For any $g_n \in G$ from $(g_n)_{n=1}^{\infty}$, the intersection $g_n(S) \cap S$ contains $g_n(x)$ and is non-empty. Since $S$ is compact and there are an infinite number of $g_n$'s, $G$ does not act properly discontinuously on $X$.

To summarize, we have shown that if the order of the stabilizer $G_x$ is not finite or if the $G$-orbit of some $x$ in $X$ is not discrete, then $G$ does not act properly discontinuously on $X$. Therefore, if $G$ acts properly discontinuously on $X$, then the order of the stabilizer $G_x$ of every point $x$ in $X$ is finite and the $G$-orbit of each point $x$ in $X$ is discrete.

We now prove the converse by contradiction. Suppose the order of $G_x$ is finite and the $G$-orbit of every point is discrete, but there is a compact set $K$ in $X$ with an infinite sequence $(g_n)_{n=1}^{\infty}$ of $g_n \in G$ such that the intersection $g_n(K) \cap K$ is non-empty. Let us form a sequence of points in $X$ by taking an arbitrary element $a_n$ from the corresponding intersection $g_n(K) \cap K$.

Since $(a_n)_{n=1}^{\infty}$ is in a compact space $K$, we can take a subsequence $(b_j)_{j=1}^{\infty}$ of $(a_n)_{n=1}^{\infty}$ that converges to some point $b$ in $K$. For each $b_j$ in $(b_j)_{j=1}^{\infty}$, there is a $g_n$ in $G$ that takes $b_j$ into $K$. Let us re-name these $g_n$'s as $h_j$'s corresponding to $b_j$'s. We can create yet another sequence $(h_j(b_j))_{j=1}^{\infty}$ and a convergent subsequence $(h_{j_k}(b_{j_k}))_{k=1}^{\infty}$ that converges to $y$.

To recap, $(b_j)_{j=1}^{\infty}$ converges to $b$ and $(h_{j_k}(b_{j_k}))_{k=1}^{\infty}$ converges to $y$. We will use the convergence of these sequences to prove that the $G$-orbit of $b$ has an accumulation point.

$$d(h_{j_k}(b), y) \leq d(h_{j_k}(b), h_{j_k}(b_{j_k})) + d(h_{j_k}(b_{j_k}), y) \tag{3.12}$$

Equation 3.12 obviously follows from the triangle inequality. We know that $(b_{j_k})_{k=1}^{\infty}$ is a subsequence of the convergent sequence $(b_j)_{j=1}^{\infty}$. Hence, $(b_{j_k})_{k=1}^{\infty}$ converges to $b$, the limit of $(b_j)_{j=1}^{\infty}$, as $k$ approaches infinity. The function $h_{j_k}$ is a homeomorphism, so continuity tells us that for any $\epsilon$ greater than 0, for large enough $k$ $d(h_{j_k}(b_{j_k}), h_{j_k}(b))$ is less than $\epsilon$.

We also know that $(h_{j_k}(b_{j_k}))_{k=1}^{\infty}$ converges to $y$. We conclude that for large enough $k$, $d(h_{j_k}(b_{j_k}), y)$ is less than the $\epsilon$ given above. We conclude

$$d(h_{j_k}(b), y) \leq d(h_{j_k}(b), h_{j_k}(b_{j_k})) + d(h_{j_k}(b_{j_k}), y) < 2\epsilon$$

for large enough $k$ and arbitrary $\epsilon > 0$. Therefore, $(h_{j_k}(b))_{k=1}^{\infty}$ converges to $y$ in K. Since this sequence is convergent in a metric space, it is Cauchy, so for infinitely many integers $p, q$ greater than $k$, $d(h_{j_p}(b), h_{j_q}(b)) < 2\epsilon$. Hence, the $G$-orbit of $b$ in $X$ is not discrete, which contradicts our assumption. □

The definition we have been using for proper discontinuity is a more formal, and in my opinion illustrative, definition, but this is not to say it is the only definition. There are many equivalent definitions, and in order to prove some results about the properly discontinuous action of Fuchsian groups on $\mathbb{H}$, we will introduce one of these alternatives. First, we will establish a concept of *local finiteness* for a set, or family of sets.

**Definition 3.13.** A family $\{M_\alpha \mid \alpha \in A\}$ of subsets of $X$ indexed by elements of a set $A$ is called *locally finite* if for any compact subset $K \subset X$, the intersection $M_\alpha \cap K$ is non-empty for only finitely many $\alpha$ in $A$.

To demonstrate local finiteness, we will often take advantage of the fact that a discrete set intersected with a compact set in a metric space is finite. This is the main strategy in the proof of the following statement.

**Theorem 3.14.** *A group of homeomorphisms $G$ acts properly discontinuously on a metric space $X$ if and only if the $G$-orbit of any point $x \in X$ is locally finite.*

*Proof.* We first prove the forward implication. We know from Theorem 3.11 that if $G$ acts properly discontinuously on $X$, then the $G$-orbit $Gx$ of each point $x$ is discrete. For every compact subset $K \subset X$, the intersection $K \cap Gx$ is finite because $Gx$ is discrete. Thus, the $G$-orbits of all $x$ in $X$ are locally finite.

We now prove the reverse direction by proving the contrapositive. By the same reasoning as in the proof of Theorem 3.11, if $Gx$ the $G$-orbit of some $x$ in $X$ is not discrete and has a limit point $y$, there is an infinite compact subset of $Gx$. Thus, $Gx$ is not locally finite. Furthermore, if the stabilizer of $x$ is infinite, $\{x\}$ is a compact set that is not locally finite and $Gx$ is not locally finite. Therefore, if the $G$-orbit of any point $x \in X$ is locally finite, then $G$ acts properly discontinuously on $X$. □

To prove properly discontinuous action of Fuchsian groups on $\mathbb{H}$, we will first need to prove a Lemma.

**Lemma 3.15.** *Let $\Gamma$ be a subgroup of PSL(2,$\mathbb{R}$)acting properly discontinuously on $\mathbb{H}$, and $p \in \mathbb{H}$ be fixed by some element of $\Gamma$. Then there is a neighborhood $W$ of $p$ such that no other point of $W$ is fixed by an element of $\Gamma$ other than the identity.*

*Proof.* Suppose a point $p$ in $\mathbb{H}$ is fixed by some non-identity element $T$ in $\Gamma$ such that $T(p) = p$, and there are fixed points of transformations in $\Gamma$ in all neighborhoods of $p$. This implies that there is a sequence of points $(p_n)_{n=1}^{\infty}$ in $\mathbb{H}$ that converges to $p$ and $T_n(p_n) = p_n$.

Let $\overline{B_{3\epsilon}(p)}$ with $\epsilon > 0$ be the closed hyperbolic $\epsilon$-neighborhood of $p$. $\overline{B_{3\epsilon}(p)}$ is closed and bounded, and thus compact, as the hyperbolic plane $\mathbb{H}$ inherits the Euclidean topology of $\mathbb{R}^2$.

The group of isomorphisms $\Gamma$ acts properly discontinuously on $\mathbb{H}$, so the $\Gamma$-orbit of $p$ contained in $\overline{B_{3\epsilon}(p)}$ is finite. This implies that the set $\{T \in \Gamma \mid T(p) \in \overline{B_{3\epsilon}(p)}\}$ is finite.

The sequence of isomorphisms $T_n$ mentioned above are distinct and neccesarily do not fix $p$. Because there are only finitely many $T$ such that $T(p)$ is within $3\epsilon$ of p, we conclude that for $N \in \mathbb{N}$ sufficiently large, $n > N$ implies that $\rho(T_n(p), p) > 3\epsilon$. Additionally, by basic convergence $\rho(p_n, p)$ is less than epsilon $\epsilon$ for large enough n. By the triangle inequality and the invariance of the hyperbolic metric under isomorphism, we are left with the following inequality.

$$\rho(T_n(p), p) \leq \rho(T_n(p), T_n(p_n)) + \rho(T_n(p_n), p) = 2\rho(p_n, p) < 2\epsilon.$$

This is a contradiction. [1] □

For the proof of the next theorem, we will make use of the following fact.

**Lemma 3.16.** *Let $z_0 \in \mathbb{H}$ be given and let $K$ be a compact subset of $\mathbb{H}$. Then the set*

$$E = \{T \in PSL(2,\mathbb{R}) \mid T(z_0) \in K\}$$

*is compact.*

*Proof.* We will not prove this formally, but instead give a sketch of a proof. Because PSL(2,$\mathbb{R}$)'s open set topology is inherited from $\mathbb{R}^4$, there is a continuous projective map from $\mathbb{R}^4$ to PSL(2,$\mathbb{R}$). If we prove that the set of pre-images of $E$ is compact in $\mathbb{R}^4$, then compactness in PSL(2,$\mathbb{R}$) naturally follows. From the Heine-Borel theorem for Euclidean space, all that remains to prove is that the set of pre-images of $E$ is closed and bounded in $\mathbb{R}^4$, which can be done easily. □

**Theorem 3.17.** *Let $\Gamma$ be a subgroup of PSL(2,$\mathbb{R}$). Then $\Gamma$ is a Fuchsian group if and only if $\Gamma$ acts properly discontinuously on $\mathbb{H}$.*

*Proof.* We begin by proving the forward implication. Let $z$ be an arbitrary point in $\mathbb{H}$ and $K$ be a compact subset of $\mathbb{H}$.

$$\{T \in \Gamma \mid T(z) \in K\} = \{T \in PSL(2,\mathbb{R} \mid T(z) \in K\} \cap \Gamma \tag{3.18}$$

By Lemma 3.16 and the definition of a Fuchsian group, the set given in (3.18) is finite, since it is the intersection of a compact set and a discrete set. This implies that the $\Gamma$-orbits of any point $z \in \mathbb{H}$ are locally finite. Thus, $\Gamma$ acts properly discontinuously on $\mathbb{H}$.

We now prove the reverse implication. Suppose $\Gamma$ acts properly discontinuously on $\mathbb{H}$, but it is not a discrete subgroup of PSL(2,$\mathbb{R}$). By Lemma 3.15, there is a point $p$ in $H$ not fixed by any non-identity element of $\Gamma$. If $\Gamma$ is not discrete, then by Proposition 3.7 there is a sequence $(T_n)_{n=1}^{\infty}$ of distinct elements of $\Gamma$ that converges to the identity element $Id$. The existence of such a sequence implies that the sequence $(T_n(s))_{n=1}^{\infty}$ converges to $s$. Since $s$ is not a fixed point, $T_k(s) \neq s$ for all $T_k \neq Id$. Thus, the $\Gamma$-orbit of $s$ clusters at $s$ and therefore $\Gamma$ does not act properly discontinuously. □

**Corollary 3.19.** *Let $\Gamma$ be a subgroup of PSL(2,$\mathbb{R}$). Then $\Gamma$ acts properly discontinuously on $\mathbb{H}$ if and only if for all $z \in \mathbb{H}$, $\Gamma z$, the $\Gamma$-orbit of $z$, is a discrete subset of $\mathbb{H}$.*

*Proof.* This is immediate from Theorem 3.11. □

## 4. Topological Properties of Fuchsian Groups

In section 3, we used proper discontinuity to link the discrete topology of Fuchsian groups to its action on hyperbolic space. More specifically, we showed that a Fuchsian group's $\Gamma$-orbits must be discrete and its stabilizers finite. A Riemannian surface that naturally arises from the action of these discrete isometry groups is the quotient space that is comprised of the equivalence classes of points in the same $\Gamma$-orbit.

Our aim in this section is to show that in certain cases, the discreteness of the Fuchsian group's action allows the quotient space to behave "nicely" locally, without cusps, edges or tears. We demonstrate this in two similar ways. First, we show that if a Fuchsian group $\Gamma$ does not fix any points, then the entire hyperbolic plane acts as a universal cover (a simply connected covering space) for the quotient space $\mathbb{H}/\Gamma$. Our next result shows that this quotient space is a topological 2-manifold. Both of these results essentially show that the quotient space locally resembles more familiar spaces like $\mathbb{H}$ and $\mathbb{R}^2$.

As in Section 3, Let $X$ be a metric space and $G$ be a group of homeomorphisms acting properly discontinuously on $X$.

**Definition 4.1.** A closed region $F \subset X$ is defined to be a *fundamental region* for $G$ if

(i)
$$\bigcup_{T \in G} T(F) = X$$

(ii)
$$int(F) \cap T(int(F)) = \emptyset \qquad \forall T \in G - \{Id\}$$

where $int(F)$ is the union of all open subsets of $F$ and $\partial F$ is the boundary of $F$.

**Definition 4.2.** The *Dirichlet region for* $\Gamma$ *centered at* $p$ is the set

$$D_p(\Gamma) = \{z \in \mathbb{H} \mid \rho(z,p) \leq \rho(z, T(p)) \text{ for all } T \in \Gamma\}$$

Equivalently, because of the invariance of $\rho$ under PSL(2,$\mathbb{R}$)

$$D_p(\Gamma) = \{z \in \mathbb{H} \mid \rho(z,p) \leq \rho(T(z), p) \text{ for all } T \in \Gamma\}$$

**Theorem 4.3.** *If $p$ is not fixed by any element of a Fuchsian group $\Gamma$ besides the identity, then $D_p(\Gamma)$ is a fundamental region for $\Gamma$.*

*Proof.* Let $z \in \mathbb{H}$ and $\Gamma z$ be its $\Gamma$-orbit. $\Gamma z$ is a discrete set, so there exists $z_0 \in \Gamma z$ with the smallest distance from $p$. Then $\rho(z_0, p) \leq \rho(T(z_0), p)$ for all $T \in \Gamma$. Hence, $z_0$ is an element of the Dirichlet region $D_p(\Gamma)$ by Definition 4.2. This shows that $D_p(\Gamma)$ contains at least one point from every $\Gamma$-orbit. Thus, condition (i) of Definition 4.1 is satisfied.

Now we show that no two points of the same $\Gamma$-orbit can lie in the interior of $D_p(\Gamma)$. Suppose two points $z, Tz \in \Gamma z$ are in $D_p(\Gamma)$. By the definition of the Dirichlet region, $\rho(z,p) = \rho(Tz, p)$. By the invariance of $\rho$ under isometry, $\rho(Tz, p)$ equals $\rho(z, T^{-1}p)$.

Let $w$ be an element of the geodesic segment $[z, T^{-1}p]$ such that $\rho(p,w) \geq \rho(p,z)$ and $\rho(z,w) = \epsilon$ where $\epsilon > 0$. This gives us the following inequality.

$$\rho(w,p) \geq \rho(p,z) = \rho(z,w) + \rho(w, T^{-1}p) = \epsilon + \rho(w, T^{-1}p) > \rho(w, T^{-1}p).$$

This inequality implies that $w$ is not an element of $D_p(\Gamma)$ despite being in the open $2\epsilon$-neighborhood of $z$. Hence, $z$ is not in the interior of $D_p(\Gamma)$. We conclude that the interior of the dirichlet region $D_p(\Gamma)$ contains at most one point in each $\Gamma$-orbit. Observe that condition (ii) of Definition 4.1 is satisfied and $D_p(\Gamma)$ is a fundamental region. [1] □

*Remark* 4.4. The reason that $p$ must not be fixed is that in the second part of the proof, it is possible that if $p = T^{-1}p$, then both $z$ and $Tz$, for which $\rho(z,p) = \rho(Tz,p)$, could be in the interior, violating condition (ii) of Definition 4.1.

*Remark* 4.5. The fundamental region defined by the Dirichlet region for $\Gamma$ centered at a non-fixed point $p$ is convex and connected.

We will not formally justify this assertion, but it follows from the fact that the Dirichlet region can be generated by intersecting closed hyperbolic half-planes. This is done by taking a point $p$ not fixed by any $T \in \Gamma$, and finding all the points closer or equally close to $p$ than $T_j(p)$ by taking the half plane below the geodesic perpendicular bisector of $[p, T_j(p)]$. The intersection of all the resulting closed half planes is also closed and hyperbolically convex, implying path-connectedness which in turn implies connectedness.

Let $\Gamma$ be a Fuchsian group. Furthermore, let us establish an equivalence relation $\sim_\Gamma$ on $\mathbb{H}$ where $z \sim_\Gamma w$ if $z$ and $w$ are in the same $\Gamma$-orbit. For convenience, we will denote the quotient space generated by this relation as $\mathbb{H}/\Gamma$. Clearly, the elements of $\mathbb{H}/\Gamma$ are the equivalence classes of $\sim_\Gamma$, the $\Gamma$-orbits.

*Remark* 4.6. We define the *natural projection* $\pi : \mathbb{H} \to \mathbb{H}/\Gamma$ as the function that takes points in $\mathbb{H}$ to their $\Gamma$-orbit in the quotient space. By Definition 3.4, a set $U \subset \mathbb{H}/\Gamma$ is open if and only if the preimage of $U$ under $\pi$ is open in $\mathbb{H}$. Equivalently, we say that the open set topology on $\mathbb{H}/\Gamma$ is induced by the natural projection $\pi$.

**Definition 4.7.** A *covering space* of a topological space $X$ is a space $\tilde{X}$ together with a surjective map $p : \tilde{X} \to X$ such that: There exists an open cover $\{U_\alpha\}$ of $X$ such that for each $\alpha$, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped homeomorphically onto $U_\alpha$ by $p$.

**Theorem 4.8.** *If* $\Gamma$ *acts on* $\mathbb{H}$ *without fixed points,* $\mathbb{H}$ *is a covering space of* $\mathbb{H}/\Gamma$*.*

*Proof.* Let $F$ be a fundamental region equal to the Dirichlet region of some non fixed point $p \in \mathbb{H}$ by Theorem 4.4. By Definition 4.6, we know that for any open neighborhood $U \subset int(F)$ of some $z \in int(F)$, $\pi(U)$ is an open subset of $\mathbb{H}/\Gamma$ since its preimage is $U$, an open set. Every point in $U$ is associated with a unique point in $\Gamma \setminus \mathbb{H}$ since for all $z$ in $U \subset int(F)$, $z$ is the only element of the $\Gamma$-orbit of $z$ in $int(F)$ by the second condition in the definition of a fundamental region (Definition 4.1). Thus, $\pi : U \to \pi(U)$ is bijective, where $\pi(U)$ is an open subset of $\mathbb{H}/\Gamma$.

The open set condition for continuity states that a function $f : M \to N$ is continuous if the preimage of each open set in $N$ is open in $M$. Thus, $\pi$ and $\pi^{-1}$ are continuous as a direct consequence of Remark 4.6. Thus, $\pi : U \to \pi(U)$ is a homeomorphism.

By assumption, any point $z$ in the boundary $\partial F$ is not fixed by any $T \in \Gamma$ except the identity. Thus, we can form a Dirichlet region $D_z(\Gamma)$ around it. It is a fundamental region by Theorem 4.3. By Corollary 3.19, $\Gamma z$ is a discrete set, so there is some open neighborhood $U$ of $z$ contained in $D_z(\Gamma)$. This neighborhood is by definition in the interior of a fundamental region, so $\pi$ maps $U$ homeomorphically to $\pi(U) \subset \mathbb{H}/\Gamma$.

For a certain point $z \in F$, let such a neighborhood $U$ for $z$ as described above be denoted $U_z$. Take each of these neighborhoods $U_z$ and apply the projection $\pi : \mathbb{H} \to \mathbb{H}/\Gamma$ to obtain the family of open sets $\{\pi(U_z)\}$. This is an open cover of $\mathbb{H}/\Gamma$ since $F$ contains at least one point of each $\Gamma$-orbit.

Let $T \in \Gamma$, $z \in F$. That $T(U_z) \cap U_z = \emptyset$ has been established above, and if $y$ is in $U_z$, then $T(y)$ is in $\Gamma y$, so $\pi(T(y)) \in \pi(U_z)$. Thus $\pi$ maps $T(U_z)$ homeomorphically to $\pi(T(U_z)) \subset \mathbb{H}/\Gamma$. The preimage of every point $\pi(y) \in \pi(U_z)$ is $T(y)$ where $T$ is some element of $\Gamma$. Therefore, the preimage of $\pi(U_z)$ is the disjoint union

$$\bigsqcup_{T \in \Gamma} T(U_z), \quad T(U_z) \subset \mathbb{H}$$

Therefore, $\mathbb{H}$ is a covering space of $\mathbb{H}/\Gamma$. □

**Definition 4.9.** A topological manifold $M$ of dimension $n$ is a topological space that is locally homeomorphic to $\mathbb{R}^n$. Equivalently, for each point $m$ in $M$, there is a neighborhood $U$ of $m$ and a one-to-one, continuous map $\phi$ of $U$ into $\mathbb{R}^n$ onto some open set $\phi(U)$ in $\mathbb{R}^n$ such that the inverse map $\phi^{-1} : \phi(U) \to U$ is also continuous.

**Theorem 4.10.** *The quotient space of* $\Gamma$, $\mathbb{H}/\Gamma$, *is a topological manifold of dimension 2.*

*Proof.* To prove that $\mathbb{H}/\Gamma$ is a topological manifold of dimension 2, we need to show that $\mathbb{H}/\Gamma$ is locally homeomorphic to $\mathbb{R}^2$. We will approach this in two steps, first showing that $\mathbb{H}/\Gamma$ is locally homeomorphic to $\mathbb{H}$, and then showing that $\mathbb{H}$ is locally homeomorphic to $\mathbb{R}^2$.

Let $w$ be an arbitrary point in $\mathbb{H}/\Gamma$. The point $w$ represents the $\Gamma$-orbit of some point $z$ in $\mathbb{H}$. As in the proof of Theorem 4.9, generate an open region $U_z$ around $z$ such that it does not contain more than one point from each $\Gamma$-orbit. Then, the inverse of the natural projection $\pi : U_z \to \pi(U_z)$ is a local homeomorphism from a neighborhood of $w$ onto an open set in $\mathbb{H}$.

Now we show that $\mathbb{H}$ is locally homeomorphic to $\mathbb{R}^2$. Let $\mu : \mathbb{H} \to \mathbb{R}^2$ be the projection $(x, y) \mapsto (x, y)$ for $(x, y) \in \mathbb{H}$. The set $\{(x, y) \mid x, y \in \mathbb{R}, y \leq 0\}$ is not contained in $\mathbb{H}$. Hence, $\mu$ is locally bijective for regions $R \subset \mathbb{R} \times \mathbb{R}^+ \setminus \{0\}$.

Let us take an arbitrary point $z \in \mathbb{H}$, and the open $\epsilon$ neighborhood $B_\epsilon(z)$ where $\epsilon > 0$. Then, the image under the projection $\mu(B_\epsilon(z))$ is an open set in $\mathbb{R}^2$. Clearly, the pre-images of the open subsets of $\mu(B_\epsilon(z))$ under $\mu$ are open in $\mathbb{H}$, so $\mu : B_\epsilon(z) \to \mu(B_\epsilon(z))$ is continuous. The projection $\mu$ is bijective and the same is true of its inverse, and we conclude that $\mu : B_\epsilon(z) \to \mu(B_\epsilon(z))$ is a local homeomorphism.

Let $U_z$ be a neighborhood of $z$ as defined in the second paragraph. Then, $\mu : U_z \to \mu(U_z)$ is a homeomorphism. With the projection $\pi$ defined as above, the function $\mu \circ \pi^{-1} : \pi(U_z) \to U_z \to \mu(U_z)$ is a homeomorphism. We conclude that $\mathbb{H}/\Gamma$ is locally homeomorphic to $\mathbb{R}^2$ and that $\mathbb{H}/\Gamma$ is a topological manifold of dimension 2. □

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