PREISSMAN'S THEOREM

ALEXANDER GRANT

ABSTRACT. This paper presents a proof of Preissman's Theorem, a theorem from the study of Riemannian Geometry that imposes restrictions on the fundamental group of compact, negatively curved spaces. This particular proof presented hinges upon the notion that, in a negatively curved space, the interior angles of triangles add up to less than π.

CONTENTS

1. INTRODUCTION

In essence, Preissman's theorem takes a local restriction, the curvature of the space, combined with a global restriction, compactness, and concludes something about the nature of the fundamental group $\pi_1(M)$, a global and purely topological object. Here is a statement of the theorem:

Theorem 1.1 (Preissman)**.** *Suppose M is a compact Riemannian manifold with negative sectional curvature. Then any nontrivial abelian subgroup of $\pi_1(M)$ is isomorphic to $\mathbb{Z}$.*

However, first we need to have a firm grasp on what the sectional curvature of a manifold is. To this end, we will define the sectional curvature in the second section. It will then be used to talk about two important concepts from Riemannian geometry, namely Jacobi fields and the Rauch Comparison Theorem. Since the proof of this theorem is lengthy on its own, the statement of the theorem will be more instructive than the proof.

To those unfamiliar with the fundamental group, this will be defined at the outset of section 3. This will be a particularly brief introduction, covering just enough to ensure that we know what this theorem really means. This section will continue by connecting the covering transformations of algebraic topology to isometries. We will then conclude with Cartan's Theorem, which concerns the existence of closed geodesics, a crucial requirement in showing that these isometries are translations along geodesics of the space.

Finally, in the last section, we prove Preissman's Theorem. To provide some motivation, we start with the Cartan-Hadamard Theorem, which assures us that all of the topological information in a space of negative curvature is contained in its fundamental group, as all higher homotopy groups are trivial. We begin the proof in earnest with the well-known result from elementary geometry: a triangle in negatively curved spaces has angles whose sum is less than π. This result can be related to how covering transformations commute, and Preissman's Theorem then follows naturally from a combination of this and Cartan's Theorem. We will then prove a final theorem that is related to Preissman's Theorem, although not a direct corollary, and we will conclude with some examples.

The definitions and proofs in this paper are primarily based on those presented by do Carmo in [1], unless otherwise noted. The proof of Preissman's Theorem itself comes from the ninth chapter.

2. Basics of Riemannian Geometry

2.1. **Basic concepts.** Assumed for this paper is the understanding of the covariant derivative of a tensor field on a manifold. We will denote such a covariant derivative by ∇_u, with $u \in T_pM$ being the vector along which we are taking this derivative. For the ease of notation, we have the following:

Notation. For a vector field v defined at points along a curve $\gamma : [0, a] \to M$, we say that $v : [0, a] \to TM$ and define $v' : [0, a] \to TM$ as the vector field $v'(t) = \nabla_{\gamma'(t)} v(t)$.

Continuing, the covariant derivative also gives us the *Riemann tensor*, which is defined by

$$R(u, v) = \nabla_u \nabla_v - \nabla_v \nabla_u + \nabla_{[u,v]},$$

where $[u, v]$ is the commutator of two vector fields, considered as functions from $C^\infty(M)$ to itself.

Given a knowledge of the Riemann tensor at a given point, we can define the sectional curvature as follows.

Definition. Given $u, v \in T_pM$ linearly independent,

$$K(u, v) = \frac{\langle R(u, v)v, u\rangle}{|u|^2|v|^2 - \langle u, v\rangle^2}.$$

This could be stated in terms of the point p and any two-dimensional subspace σ of T_pM, for it does not depend upon a choice of basis for such a subspace. Therefore, we will also denote the value of the sectional curvature $K_p(\sigma)$. For a two-dimensional manifold, because the tangent space has only two dimensions, the only choice of σ is T_pM. In this case, it also turns out that $K_p(T_pM)$ is equal to the *Gaussian* curvature at that point.

Furthermore, there is the concept of a geodesic, or a curve that minimizes the length between two points. More formally, a geodesic is defined as follows, in terms of the covariant derivative.

Definition. A differentiable curve $\gamma : [0, a] \to M$ is called *geodesic at a point* $t_0 \in [0, a]$ if $\nabla_{\gamma'(t_0)}\gamma'(t_0) = 0$ (also written $\gamma''(t_0) = 0$ to emphasize that these are curves with zero acceleration). We say that it is a *geodesic* if it is geodesic at all points $t \in [0, a]$. It is a *closed geodesic* if it is geodesic at all points $t \in [0, a]$, in addition to having $\gamma(0) = \gamma(a)$ and $\gamma'(0) = \gamma'(a)$.

When referring to geodesics, what really matters is the image of the geodesic. After all, the curve can always be reparameterized to yield "different" curves. Fortunately, most of the properties of these curves are invariant under reparameterization, so we can make our notation more concise by *referring to the images of geodesics by the same symbol as the geodesics themselves.*

From the Christoffel symbols, which give us the "coordinates" of the covariant derivative, we get the following differential equation for geodesics:

$$\frac{d^2x_k}{dt^2} + \sum_{i,j=1}^{n} \Gamma_{ij}^{k} \frac{dx_i}{dt}\frac{dx_j}{dt} = 0.$$

By some very basic applications of the theory of the existence and uniqueness of differential equations, we get that for each point $p \in M$ and for each vector $v \in T_pM$, there is a unique geodesic passing through that point with a velocity vector $\frac{d\gamma}{dt} = v$ at that point. Let us denote this geodesic $\gamma_{p,v}$.

Definition. The map $\exp_p : T_pM \to M$ such that $\exp_p(v) = \gamma_{p,v}(1)$ is called the *exponential map.*

The first and most important thing to notice about the exponential map is that $(\mathrm{d}\exp_p)_0(v) = v$, as we have defined vectors in T_pM as derivatives of curves in M.

2.2. **Jacobi Fields and the Rauch Comparison Theorem.** In order to prove an important result on geodesic triangles in spaces with negative curvature, we need to discuss a result from differential geometry known as the Rauch Comparison Theorem. This theorem is presented in the language of vector fields known as Jacobi fields, so we first present their definition:

Definition. Given a geodesic $\gamma : [0, a] \to M$, a vector field $J : [0, a] \to TM$ along γ is said to be a *Jacobi field* if it satisfies the following differential equation (*Jacobi's equation*).

$$J''(t) + R(\gamma'(t), J(t))\gamma'(t) = 0.$$

The idea behind Jacobi fields is that, given a one-parameter family of geodesics γ_s, with $\gamma_0 = \gamma$, the vector field

$$J(t) = \left.\frac{\partial \gamma_s}{\partial s}\right|_{s=0}$$

is a Jacobi field that describes how geodesics behave in a neighborhood around γ. The reason why the Riemann tensor appears in Jacobi's equation is that the curvature is related to how geodesics spread out around a point.

The next definition is required for the statement of the Rauch Comparison Theorem, though on its own it is not the most important concept.

Definition. Consider a geodesic $\gamma : [0, a] \to M$. For $t_0 \in (0, a]$, the point $\gamma(t_0)$ is said to be *conjugate* to $\gamma(0)$ along γ if there exists a non-zero Jacobi field along γ that vanishes at $\gamma(0)$ and $\gamma(t_0)$. Since $\gamma(0)$ is clearly also conjugate to $\gamma(t_0)$ (consider the geodesic $\bar{\gamma}(t) = \gamma(t_0 - t)$), we just call these points *conjugate points*.

An example of conjugate points includes the antipodal points of the sphere (a geodesic being a great circle between the two), but more important for many considerations is the fact that a conjugate point is also a critical point of the exponential map. This, however, is beyond the scope of this paper.

Now we can turn to the Rauch Comparison Theorem, which is unfortunately too long to prove in its own right. The statement made is as follows.

Theorem 2.1 (Rauch Comparison)**.** *Let M and $\bar{M}$ be two Riemannian manifolds such that $\dim M \leq \dim \bar{M}$, and $\gamma : [0, a] \to M$ and $\bar{\gamma} : [0, a] \to \bar{M}$ be two geodesics such that $|\gamma'(t)| = |\bar{\gamma}'(t)|$ for all $t \in [0, a]$. Further, let J and $\bar{J}$ be Jacobi fields along these two geodesics, respectively, such that*

$$J(0) = \bar{J}(0) = 0,\ \langle J'(0),\ \gamma'(0)\rangle = \langle \bar{J}'(0), \bar{\gamma}'(0)\rangle, \text{ and}$$
$$|J'(0)| = |\bar{J}'(0)|.$$

Suppose that $\bar{\gamma}$ has no conjugate points on $(0, a]$, and that, for all $t \in [0, a]$ and $x \in T_{\gamma(t)}(M), \bar{x} \in T_{\bar{\gamma}(t)}(\bar{M})$,

$$K(x, \gamma'(t)) \leq K(\bar{x}, \bar{\gamma}'(t)).$$

Then $|J| \geq |\bar{J}|$. Furthermore, if for some $t_0 \in (0, a]$, we have that $|J(t_0)| = |\bar{J}(t_0)|$, then for all $t \in [0, t_0]$, $K(J(t), \gamma'(t)) = K(\bar{J}(t), \bar{\gamma}'(t))$.

Intuitively, the more curvature a space has, the shorter lengths are in the space. We use this theorem to prove the following proposition.

Proposition 2.2. *Let M and $\bar{M}$ be Riemannian manifolds of dimension n such that, for all $p \in M$, $\bar{p} \in \bar{M}$, $\sigma \subseteq T_pM$, and $\bar{\sigma} \subseteq T_{\bar{p}}\bar{M}$, we have that $K_p(\sigma) \leq K_{\bar{p}}(\bar{\sigma})$. Fix $p \in M$, $\bar{p} \in \bar{M}$, and a linear isometry $i : T_pM \to T_{\bar{p}}\bar{M}$. Let $r > 0$ be small enough that the restriction $\exp_p|_{B_r(0)}$ is a diffeomorphism and the restriction $\exp_{\bar{p}}|_{B_r(0)}$ is non-singular. If $c : [0, a] \to \exp_p(B_r(0))$ is a differentiable curve, define $\bar{c} : [0, a] \to \exp_{\bar{p}}(B_r(0))$ by*

$$\bar{c}(s) = (\exp_{\bar{p}} \circ\, i \circ \exp_p^{-1})(c(s)).$$

Then $\ell(c) \geq \ell(\bar{c})$. If we instead have the condition $K_p(\sigma) < K_{\bar{p}}(\bar{\sigma})$, we have that $\ell(c) > \ell(\bar{c})$.

Proof. We first turn this curve into a curve in T_pM, defining $C(s) = \exp_p^{-1}(c(s))$. This gives us a family of geodesics $\gamma_s(t) = \exp_p(tC(s))$ in M, which define a parameterized surface $f(t, s) = \gamma_s(t)$, with $f(0, s) = p$ and $f(1, s) = c(s)$.

We therefore have a Jacobi field $J_s(t) = \frac{\partial}{\partial s} f(t, s)$, where $J_s(0) = 0$ and $J_s(1) = c'(s)$. Likewise, consider the Jacobi field $\bar{J}_s(t) = \frac{\partial}{\partial s} \bar{f}(t, s)$, where $\bar{f}(t, s) = \bar{\gamma}_s(t) = \exp_{\bar{p}}(tiC(s))$: we also have that $\bar{J}_s(0) = 0$ and $\bar{J}_s(1) = \bar{c}'(s)$.

What remains is to consider $J_s'(0)$ and $\bar{J}_s'(0)$. We have that

$$J_s(t) = \frac{\partial}{\partial s} \exp_p(tC(s)) = (\mathrm{d}\exp_p)_{tC(s)}(tC'(s)) = t(\mathrm{d}\exp_p)_{tC(s)}(C'(s))$$

from the chain rule, so

$$J_s'(0) = (\mathrm{d}\exp_p)_0(C'(s)) = C'(s).$$

By a similar argument, we have $\bar{J}_s'(0) = iC'(s)$, and, since i is an isometry, we get that all of the conditions for the Rauch Comparison Theorem are met for each s:

$$|J_s(0)| = |\bar{J}_s(0)| = 0,\ |J_s'(0)| = |\bar{J}_s'(0)|, \text{ and}$$

$$\begin{aligned}\langle J_s'(0), \gamma_s'(0)\rangle &= \langle C(s), \gamma_s'(0)\rangle \\ &= \langle iC(s), i\gamma_s'(0)\rangle \\ &= \langle \bar{J}_s'(0), \bar{\gamma}_s'(0)\rangle,\end{aligned}$$

so we have that $|c'(s)| = |J_s(1)| \geq |\bar{J}_s(1)| = |\bar{c}'(s)|$, which yields $\ell(c) \geq \ell(\bar{c})$ by integration. Furthermore, if we have the strict inequality in the sectional curvatures, we must have a strict inequality in comparing $|J_s(1)|$ and $|\bar{J}_s(1)|$, which follows from the second conclusion of the Rauch Comparison Theorem. Thus, in that case, by integration we have that $\ell(c) > \ell(\bar{c})$. □

3. The Fundamental Group and Riemannian Geometry

3.1. **Basics of Algebraic Topology.** We begin with some definitions:

Definition. Consider a topological space X with a fixed point (the "base point") $x_0 \in X$. The *nth homotopy group* of X (denoted $\pi_n(X, x_0)$) is the set of all maps $f : [0,1]^n \to X$ such that $f(\partial[0,1]^n) = x_0$, under the equivalence relation of homotopies that preserve $\partial[0,1]^n$. This is given group structure by defining

$$(fg)(t_1, \ldots, t_n) = \begin{cases} f(2t_1, \ldots, t_n) & \text{for } 0 \leq t_1 \leq 1/2 \\ g(2t_1 - 1, \ldots, t_n) & \text{for } 1/2 \leq t_1 \leq 1. \end{cases}$$

This operation is called *concatenation*. The constant map serves as the identity. We call the first homotopy group the *fundamental group*, and all others are the *higher homotopy groups*.

There are a few things to note with this definition: first, that the binary operation on $\pi_n(X, x_0)$ is well-defined, and second, that it is associative and each map has an inverse with respect to this operation. In the case where $n \geq 2$, we also have that the group is abelian. This follows from the fact that, with the addition of a second dimension, we can consider a concatenation with the identity along another axis, and rearrange the two by homotopy.

Definition. A space is *simply connected* if its fundamental group $\pi_1(X, x_0)$ is trivial. In this case, our choice of base point does not matter.

Furthermore, note that, given a path $\alpha : [0,1] \to X$ that goes from x_0 to x_1, we can create an isomorphism between the two fundamental groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ by going along α, going around a loop, and then going back along α the other way. This means that for path connected spaces we do not have to worry about our choice of base point, and throughout the rest of this paper we will not do so, denoting *the* fundamental group by $\pi_1(X)$.

We now consider the *universal cover*, which we introduce with the following series of definitions.

Definition. A point $x \in X$ is *evenly covered* by a map $\pi : \tilde{X} \to X$ if there is a neighborhood U_x of x and a collection K_x of disjoint open sets in $\tilde{X}$ such that each one is mapped homeomorphically to U via π. If each point of X is evenly covered by a map $\pi : \tilde{X} \to X$, we say that π is a *covering map*, and $\tilde{X}$ is a *covering space*. If $\tilde{X}$ is simply connected, we call $\tilde{X}$ the *universal cover*.

Next, consider paths in $\tilde{X}$, a covering space of X. Composing these with π, the covering map, clearly yields paths in X. What about turning paths in X into paths in $\tilde{X}$? This is an important property of covering maps, but the next two propositions, which cover this case, will not be proven in this paper. They can be found in texts on basic algebraic topology such as [2].

Proposition 3.1. *Let $\pi : \tilde{X} \to X$ be a covering map and $\tilde{x} \in \tilde{X}$ be such that $\pi(\tilde{x}) = x$. Then, given a path $f : [0,1] \to X$ such that $f(0) = x$, there exists a unique path $\tilde{f} : [0,1] \to \tilde{X}$ with $\tilde{f}(0) = \tilde{x}$ and $\pi \circ \tilde{f} = f$.*

Proposition 3.2. *Let $\pi : \tilde{X} \to X$ be a covering map and $f, g : [0,1] \to \tilde{X}$ be paths such that $f(0) = g(0)$. If $\pi \circ f$ is homotopic to $\pi \circ g$, then f is homotopic to g, and $f(1) = g(1)$.*

These are the *path lifting properties of covering spaces.* Another proposition from [2] concludes this section:

Proposition 3.3. *If Y is a connected space, $\pi : \tilde{X} \to X$ is a covering map, and $f, g : Y \to \tilde{X}$ are continuous maps such that $\pi \circ f = \pi \circ g$, then $\{y \in Y | f(y) = g(y)\} = Y$ or $\varnothing$.*

In the case where $Y = \tilde{X}$ and $\pi \circ f = \pi \circ g = \pi$, such maps are called *covering transformations.* The above proposition therefore implies that all covering transformations are determined by their action on a given element.

In the following sections, unless otherwise specified, we will use $\tilde{X}$ to denote the universal cover of X. This is because the universal cover of X is universal in the sense that it is a covering space of all connected covering spaces of X.

3.2. **Covering Transformations as Isometries.** Before proceeding, we must note that covering transformations, as discussed in the previous subsection, are in fact isometries. Isometries of what? Well, the so-called covering metric:

Definition. Given a Riemannian manifold (M, g), the universal cover $\tilde{M}$ can be given a Riemannian metric $\tilde{g}$ with $\tilde{g} = \pi^* g$, where π^* as usual denotes the pullback. This is called the *covering metric.*

This definition is chosen such that the sectional curvature of a space is preserved when going to the universal cover, a result that will be crucial in the following arguments. Similarly, the length of a path in M will be the same as the lifted path in $\tilde{M}$. Moreover, from this definition, it is also clear that covering transformations are isometries, since the condition for being an isometry (in the language of pullbacks) is that $f^* \tilde{g} = \tilde{g}$.

From the comments at the end of the previous section, we can conclude not only that these are isometries, but that, other than the identity, they are isometries that have no fixed points. A similar result is the following proposition, which connects these isometries with the fundamental group.

Proposition 3.4. *The set of all covering transformations of a manifold M is isomorphic to $\pi_1(M)$.*

Proof. First, we take a loop g in $\pi_1(M,p)$ and construct a covering transformation α_g. Let $\tilde{p}$ be any point in $\tilde{M}$ such that $\pi(\tilde{p}) = p$. Consider another point $q \in M$, with $\pi(\tilde{q}) = q$. Let σ be a path from p to q, and take the unique lifting of $\sigma^{-1}g\sigma$ starting at $\tilde{q}$, and let $\alpha_g(\tilde{q})$ be the endpoint of this lifted path. This is a covering transformation of $\tilde{M}$ because the endpoints of the lifted path have the same image under π. Since α can be specified by its action on $\tilde{p}$ alone, the below paragraphs concern themselves only with $\alpha(\tilde{p})$, although it is defined elsewhere on M.

The above operation turns concatenation of loops in $\pi_1(M,p)$ into composition of covering transformations: consider two loops g_1 and g_2 in $\pi_1(M,p)$ and their concatenation g_1g_2. The above process, when performed on g_1g_2, results in a lift that goes from $\tilde{p}$ to $\alpha_{g_2}(\alpha_{g_1}(\tilde{p}))$. The order of composition is reversed, but because we are merely showing that these groups are isomorphic, it does not matter because a group is naturally isomorphic to its opposite (the group where the binary operator is reversed).

Now, we just have to prove that the operation outlined above is both injective and surjective. That this is surjective follows from the fact that, given a covering transformation α, we can take $\tilde{p} \in \tilde{M}$ and take a path between $\tilde{p}$ and $\alpha(\tilde{p})$, then compose that path with π to form a loop in M based at $p = \pi(\tilde{p})$. Doing the above operation takes this process in reverse, yielding the same covering transformation because the resulting path from lifting the loop will be between the same points.

Finally, to show that it is injective, suppose it took two loops g_1 and g_2 to the same covering transformation α. Then the paths between $\tilde{p}$ and $\alpha(\tilde{p})$ would be homotopic (as $\tilde{M}$ is simply connected), hence their compositions with the covering map π would be homotopic, yielding homotopic loops g_1 and g_2. Thus, we have an isomorphism. □

3.3. **Cartan's Theorem.** Cartan's Theorem is a statement about the existence of closed geodesics in compact manifolds. It is here that the condition for compactness for Preissman's theorem becomes necessary, and is necessary for imposing conditions on the covering transformations of such spaces.

Definition. Consider the set of all closed paths in M. Let $C_1(M)$ be the set of all equivalence classes of such paths, under the equivalence relation of homotopy. Such a class is referred to as a *free homotopy class.*

Note that these homotopies do not necessarily preserve base point, like those in the definition of the homotopy groups.

Theorem 3.5 (Cartan)**.** *If M is a compact manifold and $[\gamma] \in C_1(M)$ is not the constant class, then there exists a closed geodesic of M that is in the class $[\gamma]$.*

Proof. Since it is not the constant class, we have that there is a $d > 0$ that is the infimum of the lengths of piecewise differentiable curves belonging to $[\gamma]$. Therefore, we can let γ_j be a sequence of piecewise differentiable curves such that $\ell(\gamma_j) \to d$. In fact, we can make each γ_j a broken geodesic (defined on $[0,1]$ and parameterized proportionally to arc length) because doing so will only *shorten* the total length of the curve, not impeding convergence. Then we have that

$$d(\gamma_j(a), \gamma_j(b)) \leq \int_a^b |\gamma_j'(t)| \, \mathrm{d}t \leq \sup_{j \in \mathbb{N}} \ell(\gamma_j)(b-a)$$

for all $a \leq b \in [0,1]$. Since the only constraint on these curves is that their length is bounded from below and their lengths converge to d, we can ensure that $\sup_{j\in\mathbb{N}} \ell(\gamma_j)$ is finite. This family of broken geodesics is equicontinuous, as there exists a single Lipshitz constant for all members of the family. Therefore, since M is compact, a subsequence of them converge uniformly to a continuous, closed curve $\gamma_0 : [0,1] \to M$, which is also in $[\gamma]$ by the local simple-connectedness of M, an inevitable consequence of it being a manifold.

Now, this is not exactly what we wanted, but consider a partition of the unit interval $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = 1$, chosen such that γ_0 is smooth on each (t_i, t_{i+1}). Then connect each point $\gamma_0(t_i)$ to $\gamma_0(t_{i+1})$ by a geodesic β_i, and let β be the broken geodesic formed by concatenating all of these geodesics. That this also belongs to $[\gamma]$ follows again from the local simple-connectedness of M. Thus, we have that $\ell(\beta) \geq d$.

To show that $\ell(\beta) = d$, we suppose that $\ell(\beta) > d$, so that we can have $\epsilon = \frac{\ell(\beta)-d}{2n+1} > 0$. Because of convergence of $\ell(\gamma_j)$, we have that there is a positive integer j such that $\ell(\gamma_j) - d < \epsilon$ and $d(\gamma_j(t), \gamma_0(t)) < \epsilon$ for all $t \in [0,1]$. Let us denote $\gamma_j|_{[t_{i-1},t_i]}$ by $\gamma_{j,i}$, so that

$$\sum_{i=1}^{n}(\ell(\gamma_{j,i}) + 2\epsilon) = \ell(\gamma_j) + 2n\epsilon < d + (2n+1)\epsilon = \ell(\beta) = \sum_{i=1}^{n} \ell(\beta_i).$$

Since all of these are positive quantities, this means there must be an integer i between 1 and n such that $\ell(\gamma_{j,i}) + 2\epsilon < \ell(\beta_i)$. However, by the equicontinuity condition above, we have that $d(\gamma_{j,i}(t_{i-1}), \beta_i(t_{i-1}))$ and $d(\gamma_{j,i}(t_i), \beta_i(t_i))$ are both less than ϵ, so we can join these points by geodesics and obtain a curve *shorter* than β_i, but with the same endpoints. This contradiction implies that $\ell(\beta) = d$, so β is the shortest curve in $[\gamma]$.

All that remains is to show that it is actually a closed geodesic, which we do as follows: we know that it is geodesic at all points except possibly at $\beta(t_0), \ldots, \beta(t_n)$. Suppose that it is not geodesic at, say, $p_i = \beta(t_i)$. Take two points $q_1 \in \beta((t_{i-1}, t_i))$ and $q_2 \in \beta((t_i, t_{i+1}))$ (with proper adjustments, since this is a loop, to the cases $i = 0$ and $i = n$) such that they are within a simply-connected neighborhood of p_i. Then a minimizing geodesic between q_1 and q_2 will be homotopic to the segment of β that connects q_1 and q_2, but be shorter, which is a contradiction. Thus, β must be a geodesic at each p_i, so it is a closed geodesic in the homotopy class $[\gamma]$. □

Definition. An isometry $f : M \to M$ without fixed points is called a *translation* if there exists a geodesic γ such that $f(\gamma)$ is (at most) a reparameterization of γ. Such a geodesic is said to be *left invariant* by the translation.

Proposition 3.6. *Any covering transformation $\alpha : \tilde{M} \to \tilde{M}$ is also a translation on $\tilde{M}$.*

Proof. In the case where α is the identity, this follows easily. Otherwise, consider the loop $g \in \pi_1(M,p)$ corresponding to α in the isomorphism outlined in Proposition 3.4, and let $p = \pi(\tilde{p})$, where $\tilde{p} \in \tilde{M}$. Cartan's Theorem implies that there is a closed geodesic γ in M that is homotopic to g. Let q be a point along γ, and σ be a path from p to q. Then γ is homotopic to $\sigma^{-1}g\sigma$, because it was homotopic to g, but now they *have the same start point.* Let $\tilde{q}$ be the endpoint of the lift of σ starting at $\tilde{p}$, and let $\tilde{\gamma}$ be the lift of γ starting from $\tilde{q}$.

Now, let $\alpha_{\tilde{q}}$ be the covering transformation corresponding to γ, considered as a loop in $\pi(M, q)$. Of course, we have that $\alpha_{\tilde{q}}(\tilde{q})$ is just the endpoint of $\tilde{\gamma}$, which is also the endpoint of the lift of $\sigma^{-1}g\sigma$, which also happens to be $\alpha(\tilde{q})$, since this is how we defined the isomorphism in Proposition 3.4. Since covering transformations are determined by their action on one element, that means that $\alpha_{\tilde{q}} = \alpha$.

However, consider a point $\tilde{\gamma}(t)$. We have that $\alpha_{\tilde{q}}(\tilde{\gamma}(t))$ is a point along the lift of γ from $\tilde{q}$, as we can go from $\gamma(t)$ to q along γ, so by the uniqueness of the lift, $\alpha(\tilde{\gamma}(t)) = \alpha_{\tilde{q}}(\tilde{\gamma}(t)) \in \tilde{\gamma}$, so α is a translation along γ. □

4. Preissman's Theorem

4.1. **The Cartan-Hadamard Theorem.** Preissman's Theorem is useful, to be sure, as it gives us information on the fundamental group of a compact manifold of negative curvature. However, it is the Cartan-Hadamard Theorem that makes it crucial: it implies that for such spaces, the higher homotopy groups are trivial. This means that all of the topological information about the space is contained in the fundamental group. Here is a statement of the theorem:

Theorem 4.1 (Cartan-Hadamard). *Let M be a simply connected, complete Riemannian manifold with sectional curvature $K_p(\sigma) \leq 0$ for all $p \in M$ and $\sigma \subseteq T_pM$. Then M is diffeomorphic to $\mathbb{R}^n$, where $n = \dim M$.*

Most notably, this means that, since negative sectional curvature is preserved by going to the universal cover, the universal cover of any space of negative sectional curvature will be diffeomorphic to Euclidean space. It can be shown that this implies that the higher homotopy groups of M are trivial, but this proof is outside the scope of this paper.

4.2. **Geodesic Triangles.** For the first step in Preissman's Theorem, we turn to a familiar result from classical geometry, namely that the angles of a triangle in a negatively curved space fail to add up to π.

Definition. A *geodesic triangle* T is a set containing the images of three geodesic segments (the *sides* of T)

$$\gamma_A : [0,1] \to M,\ \gamma_B : [0,1] \to M,\ \text{and}\ \gamma_C : [0,1] \to M,$$

parameterized such that $\gamma_A(1) = \gamma_B(0)$, $\gamma_B(1) = \gamma_C(0)$, and $\gamma_C(1) = \gamma_A(0)$. These three common endpoints are called the *vertices* of T, and the angles between adjacent sides are called the *interior angles* of their respective vertices, all of which must be nonzero.

Lemma 4.2. *Let T be a geodesic triangle with vertices $a, b, c \in M$, where M is a complete, simply connected Riemannian manifold with sectional curvature $K_p(\sigma) \leq 0$ for all $p \in M$ and $\sigma \subseteq T_pM$. Let the corresponding interior angles be labeled α, β, and γ, side lengths A, B, and C (corresponding oppositely, so the side of length A is opposite the angle α, etc.). Then we have that*

$$A^2 + B^2 - 2AB\cos(\gamma) \leq C^2 \qquad (< C^2,\ \text{if}\ K_p(\sigma) < 0)$$

and

$$\alpha + \beta + \gamma \leq \pi \qquad (< \pi,\ \text{if}\ K_p(\sigma) < 0).$$

Proof. Let us denote the geodesics that make up this triangle by γ_A, γ_B, and γ_C, where the subscripts also denote the length of the geodesic segments. Let $\Gamma_I(t) = \exp_c^{-1}(\gamma_I)$ for $I = A, B, C$. These are curves in T_cM, and since γ_A and γ_B are radial geodesics, we have that $A = \ell(\gamma_A) = \ell(\Gamma_A)$ and $B = \ell(\gamma_B) = \ell(\Gamma_B)$ (this is a property of the exponential map). Furthermore, if we join the two endpoints of Γ_C by a straight line Γ, we have that $\ell(\Gamma) \leq \ell(\Gamma_C) \leq \ell(\gamma_C)$ ($< \ell(\gamma_C)$, if $K_p(\sigma) < 0$), by Proposition 2.2. Thus, we have that

$$\begin{aligned} A^2 + B^2 - 2AB\cos\gamma &= \ell(\Gamma)^2 \\ &\leq \ell(\Gamma_C)^2 \\ &\leq \ell(\gamma_C)^2 \qquad (< C^2, \text{ if } K_p(\sigma) < 0). \end{aligned}$$

To prove the other conclusion, consider a triangle with side lengths A, B, and C in Euclidean space, and denote interior angles α', β', and γ'. Observe that

$$A^2 + B^2 - 2AB\cos\gamma \leq C^2 = A^2 + B^2 - 2AB\cos\gamma',$$

so that $\cos\gamma \geq \cos\gamma'$ ($> \cos\gamma'$, if $K_p(\sigma) < 0$), which, since cosine is a strictly decreasing function on $[0, \pi]$, means that $\gamma \leq \gamma'$ ($< \gamma'$, if $K_p(\sigma) < 0$). Repeating this for the other two angles yields

$$\alpha + \beta + \gamma \leq \pi \qquad (< \pi, \text{ if } K_p(\sigma) < 0).$$

□

In some ways, the miraculous thing about this lemma is that this is the only way in which we will be concerned about the curvature of M: given this statement about geodesic triangles alone, we can prove what is required for Preissman's Theorem.

4.3. **The Commutativity of Isometries.** As a demonstration of how we can use the previous result, here is a little lemma:

Lemma 4.3. *If M is a simply connected manifold of negative curvature and $f : M \to M$ is a translation along a geodesic γ, f different from the identity, then γ is the only geodesic left invariant by f.*

Proof. Suppose f left invariant two geodesics γ_1 and γ_2. If $\gamma_1 \cap \gamma_2$ is nonempty, then there is a point $p \in \gamma_1 \cap \gamma_2$, and we have a distinct point $f(p) \in \gamma_1 \cap \gamma_2$ because f has no fixed points. Take a point $\bar{p} \in \gamma_1$ such that $\gamma_1^{-1}(\bar{p}) \in [\gamma_1^{-1}(p), \gamma_1^{-1}(f(p))]$. Then $\bar{p}$, p, and $f(p)$ forms the vertices of a geodesic triangle, with one side being γ_2, and another two being segments of γ_1. However, the interior angle associated with $\bar{p}$ is π already, so this cannot be the case. Therefore, we must have that $\gamma_1 \cap \gamma_2 = \varnothing$.

In this case, from two points $p_1 \in \gamma_1$ and $p_2 \in \gamma_2$, we can form the geodesic *quadrilateral* by considering the geodesic segment γ_3 that joins p_1 and p_2 and the geodesic $f(\gamma_3)$, which joins $f(p_1)$ and $f(p_2)$ (which is still a geodesic because f is an isometry). Let us label the interior angles at p_1 and p_2 α and β, respectively. Then we have that the interior angles at $f(p_1)$ and $f(p_2)$ are going to be $\pi - \alpha$ and $\pi - \beta$, respectively, as f preserves angles (being an isometry), and angles are still additive. However, this means that the interior angles of this geodesic quadrilateral add up to 2π, which doesn't work: draw the diagonal geodesic γ_4 from p_2 to $f(p_1)$, say, and you get two geodesic triangles, the interior angles of one of which must add up to more than π. This contradiction means that γ must be the only geodesic left invariant by f. □

This may not seem to be a step in the right direction, but consider the following (almost immediate) result.

Lemma 4.4. *If M is a simply connected manifold of negative curvature, $f, g : M \to M$ are isometries without fixed points that commute, and f is a translation along a geodesic γ, then g is also a translation along γ.*

Proof. This follows because $f(g(\gamma)) = g(f(\gamma)) = g(\gamma)$, so by the above lemma $g(\gamma) = \gamma$. □

Our final lemma is the following.

Lemma 4.5. *Consider a nontrivial subgroup H of $\pi_1(M)$, considered as a group of isometries of $\tilde{M}$. If M has negative curvature and all members of H leave invariant a single geodesic, then $H \cong \mathbb{Z}$.*

Proof. Let $\tilde{\gamma}$ be the the geodesic left invariant by all members of H, and let $\tilde{p} \in \tilde{\gamma}$. Consider the map $\theta : H \to \mathbb{R}$ such that $\theta(h) = \pm d(\tilde{p}, h(\tilde{p}))$, where the sign is determined by whether or not $\tilde{\gamma}^{-1}(\tilde{p}) < \tilde{\gamma}^{-1}(h(\tilde{p}))$. That this is well-defined is ensured by the fact that there are no closed geodesics of $\tilde{M}$, since those would surely count as geodesic triangles whose interior angles added up to more than π. Since each h is an isometry, we have that θ is a homomorphism of H into $\mathbb{R}$, considered as an additive group. Moreover, it is injective, since otherwise $h_1(\tilde{p}) = h_2(\tilde{p})$ with $h_1 \neq h_2$ would imply that $\tilde{p}$ would be a fixed point of $h_1 h_2^{-1}$. Since this implies that H is (isomorphic to) an additive subgroup of $\mathbb{R}$, and the only one (up to isomorphism) that is discrete is the integers, we have that $H \cong \mathbb{Z}$. □

In the above proof, two key results are assumed: first, that H must be a discrete group. This follows because π is a covering map, and hence the inverse image under π of a neighborhood of a point is broken up into a collection of *disjoint* neighborhoods. Since a covering transformation can only permute these neighborhoods, this group is discrete. Second, we assume the reasonable fact that additive subgroups of $\mathbb{R}$ are either dense in $\mathbb{R}$ or isomorphic to $\mathbb{Z}$.

4.4. **Preissman's Theorem.** Finally, we have the proof of Preissman's Theorem: if we have a nontrivial abelian subgroup of $\pi_1(M)$, all members of that subgroup will be translations, by Lemma 3.6. They will all be translations along a single geodesic by Lemma 4.4, and by Lemma 4.5, we then have that the subgroup is isomorphic to $\mathbb{Z}$.

The requirement that the fundamental group only have abelian subgroups isomorphic to $\mathbb{Z}$, however, is not the the strictest we can impose. In fact, the following theorem holds.

Theorem 4.6. *Suppose M is a compact Riemannian manifold with negative sectional curvature, such that $\pi_1(M)$ is abelian. Then $\pi_1(M)$ is trivial.*

Proof. Suppose $\pi_1(M)$ is both abelian and non-trivial, so there exists a geodesic $\tilde{\gamma}$ that is left invariant by all covering transformations of $\tilde{M}$. Let $\tilde{p}$ be any point along $\tilde{\gamma}$, and take a geodesic $\tilde{\beta} : [0, 1] \to \tilde{M}$ that is perpendicular to $\tilde{\gamma}$ at p, with $\tilde{\beta}(0) = \tilde{p}$. This exists because the exponential map is defined and (since we can even talk about sectional curvature) $\dim M \geq 2$.

Given these geodesics in $\tilde{M}$, we can turn them into curves in M by means of the covering map: let $\gamma = \pi \circ \tilde{\gamma}$, $\beta = \pi \circ \tilde{\beta}$, and $p = \pi(\tilde{p})$. Let α be the geodesic that joins $\beta(1)$ to $\beta(0) = p$, so that the concatenation of the two paths α and β is, in fact, a loop. We can associate this loop, as before, with an isometry of $\tilde{M}$, which must, of course, be a translation along γ. This means, among other things, that the endpoint of the lifting of α starting at $\tilde{\beta}(1)$ must be along $\tilde{\gamma}$, as this is the image of $\tilde{p}$ under this isometry. Let us call this lifting $\tilde{\alpha}$.

Thus, we have a geodesic triangle in $\tilde{M}$, with side lengths $\ell(\tilde{\alpha})$, $\ell(\tilde{\beta})$, and $\ell(\tilde{\gamma})$, where the angle between $\tilde{\beta}$ and $\tilde{\gamma}$ is $\frac{\pi}{2}$. By Lemma 4.2, we have that $\ell(\tilde{\alpha}) > \ell(\tilde{\beta})$, since

$$\begin{aligned} \ell(\tilde{\beta})^2 &< \ell(\tilde{\alpha})^2 - \ell(\tilde{\gamma})^2 \\ &\leq \ell(\tilde{\alpha})^2. \end{aligned}$$

However, we also have that $\ell(\tilde{\alpha}) = \ell(\alpha) \leq \ell(\beta) = \ell(\tilde{\beta})$, so there is a contradiction. Thus, $\pi_1(M)$ cannot be abelian. □

In conclusion, we turn to some applications of Preissman's Theorem. First, there is the torus: the torus has a fundamental group that is abelian ($\mathbb{Z}^n$, for the n-torus), so it *cannot* be given a metric with negative curvature everywhere. The same thing happens for $\mathbb{RP}^n$, the nth real projective space, which has a fundamental group of $\mathbb{Z}/2\mathbb{Z}$ for $n \geq 2$. This is a rather impressive statement about how the topology of a space can restrict its geometry.

There are also some examples that go the other way: the surfaces of genus $g \geq 2$ can be given a metric with negative sectional curvature everywhere (though the details are hard to go through), and the fundamental group is

$$\langle a_1, b_1, \ldots, a_g, b_g | a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = e \rangle.$$

For $g \geq 2$, this is not abelian, and all of the abelian subgroups are isomorphic to $\mathbb{Z}$.

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