INTRODUCTION TO GEOMETRY

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Abstract. This paper is an introduction to Riemannian and semi-Riemannian manifolds of constant sectional curvature. We will introduce the concepts of moving frames, curvature, geodesics and homogeneity on six model spaces: $\mathbb{R}^n, \mathbb{S}^n$, and $\mathbb{H}^n$ in Riemannian geometry, and $\mathbb{R}^{n,1}, dS^n$ and AdS^n in semi-Riemannian geometry. We then claim that up to isometries, these are the only examples in both settings, and state a classification theorem to support that. Some terminology from general relativity will be introduced where relevant.

Contents

1. Introduction

Our goal is to understand Riemannian and semi-Riemannian spaces of constant sectional curvature through six model spaces: $\mathbb{R}^n, \mathbb{S}^n$, and $\mathbb{H}^n$ in Riemannian geometry, and $\mathbb{R}^{n,1}, dS^n$ and AdS^n in semi-Riemannian geometry. To do this, we'll first have to develop an understanding of what Riemannian and semi-Riemannian manifolds are. Specifically, we'll look at the metrics of these spaces. From here, our next step will be to define curvature and understand it through moving frames and connection forms. Then, we will look at geodesics and homogeneity on these spaces. We will state two classification theorems that say that, up to isometries, these are in fact the only examples of Riemannian and semi-Riemannian manifolds of constant sectional curvature. A basic knowledge of differential geometry, algebra and analysis is assumed.

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2. Riemmanian and semi-Riemmanian metrics

2.1. **Riemannian metrics and model spaces.**

The three model spaces of Riemannian manifolds are $\mathbb{R}^n$, $\mathbb{S}^n$, and $\mathbb{H}^n$. We will use these spaces as examples to understand Riemannian metrics.

Definition 2.1. A Riemannian manfiold (M, g) is a smooth manifold M together with a metric g, such that for each $p \in M$, g is an inner product on T_pM, and g varies smoothly between points p on M. In other words, given two vectors $v, w \in T_pM$, $\langle v, w\rangle_p \geq 0$ and $\langle v, w\rangle_p = 0$ iff $v = w$.

Example 2.2. The standard Euclidean metric on $\mathbb{R}^n$ is

$$g = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & & & \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

otherwise known as the n dimensional identity matrix I.

Example 2.3. Consider the n-sphere $\mathbb{S}^n$ which can be described as the set

$$\mathbb{S}^n = \{(x_1, x_2 \dots, x_n) \text{ such that } x_1^2 + x_2^2 + \dots + x_n^2 = +1\}$$

When $n = 2$, we can put coordinates

$$k(\theta, \varphi) = (\sin(\theta)\cos(\varphi), \sin(\theta)\sin(\varphi), \cos(\theta))$$

The metric in coordinates, s, is defined as the pullback of the standard euclidean metric, k^*g. More specifically,

$$s_{i,j} = g_{a,b}(k(\theta, \varphi))\partial_i k^a \partial_j k^b$$

Computing the metric in coordinates gives:

$$s = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(\theta) \end{pmatrix}$$

Generally, a metric on a manifold is achieved by pulling back the metric of the ambient space onto the manifold. Since the n-sphere sits inside $\mathbb{R}^n$, the metric on $\mathbb{S}^n$ is a pullback of the euclidean metric defined in Example 2.2. Next, we'll look at hyperbolic space. The ambient space of hyperbolic space is Minkowski space. Its metric is a pull-back of the Minkowski metric, which we will define below.

Definition 2.4. $I_{p,q}$ is the block matrix

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where A is a p-dimensional identity matrix and B is a q-dimensional negative identity matrix. We say that this matrix has signature q.

Example 2.5.

$$I_{2,1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Definition 2.6. The Minkowski metric, defined on $\mathbb{R}^{n,1}$ is

$$m = I_{n,1}$$

We say that it has signature one. A Minkowski metric is used to describe non-euclidean geometry. Note that it is not required to be positive definite.

Definition 2.7. Minkowski space is $\mathbb{R}^{n,1}$ together with the Minkowski metric.

Since the Minkowski metric is not required to be positive definite, Minkowski space is not a Riemannian manifold, but rather a semi-Riemannian manifold. We will describe semi-Riemannian manifolds in depth in the next section. For now, note that hyperbolic space is a *Riemannian* manifold because the metric on $\mathbb{H}^n$ is positive definite, despite the fact that $\mathbb{H}^n$ sits inside of a semi-Riemannian manifold.

Hyperbolic space has 3 models. We will introduce all of the models, and compute the metric on the hyperboloid model. The relationship between these three models, as well as the proof that they are isometric, can be found in Lee [2]. For the rest of this paper, we will take as a given that these models are isometric.

Definition 2.8. In the Poincaré half-plane model, hyperbolic space can be described as

$$\mathbb{H}^n = \{\text{p} = (x_1,\, x_2,\, \ldots,\, x_{n-1},\, y) \text{ such that } y \geq 0\}.$$

For $u, v \in T_pM$,

$$g_{hyp}(u,v) = \tfrac{1}{y^2} u \cdot v.$$

Definition 2.9. The Poincaré ball model, also known as the disk model of hyperbolic space, is the unit disk in $\mathbb{R}$ with coordinate metric

$$h_{ball} = 4\frac{(dx^1)^2 + \ldots + (dx^n)^2}{(1 - |u|^2)^2}$$

Definition 2.10. In the hyperboloid model, hyperbolic space is treated as the 'upper sheet' of the two-sheeted hyperboloid in $\mathbb{R}^{n,1}$. Here, hyperbolic space is described as

$$\mathbb{H}^n = \{(x_1,\, x_2,\, \ldots,\, x_n) \text{ such that } x_1^2 + x_2^2 + \ldots + x_{n-1}^2 \text{ - } x_n^2 = \text{-}1\}.$$

Example 2.11. Consider $\mathbb{H}^2$ with the hyperboloid model. We may use coordinates

$$k(t,x) = (\sinh(t)\cos(x), \sinh(t)\sin(x), \cosh(t))$$

The metric in coordinates is the pullback of the Minkowski metric

$$h = k^* m$$

Proposition 2.12. *Given the above coordinates, the metric for $\mathbb{H}^2$ is*

$$h = \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2(t) \end{pmatrix}$$

Proof. Generally, since the metric h is defined as the pullback of the Minkowski metric, m, we may write

$$h_{i,j} = m_{a,b}(k(t,x))\partial_i k^a \partial_j k^b$$

where a, b refer to the row and column of the metric m and i, j are in the set {1,2} and ∂_1 refers to the derivative with respect to t and ∂_2 refers to the derivative with respect to x. Computing, we get the following:

$$\begin{aligned}
h_{11} &= m_{a,b}(k(t,x))\partial_1 k^a \partial_1 k^b \\
&= +1(\cosh(t)\cos(x))(\cosh(t)\cos(x)) + 1(\cosh(t)\sin(x))(\cosh(t)\sin(x)) - 1(\sinh(t))(\sinh(t)) \\
&= \cosh^2(t)\cos^2(x) + \cosh^2(t)\sin^2(x) - \sinh^2(t) \\
&= \cosh^2(t)(\cos^2(x) + \sin^2(x)) - \sinh^2(t) \\
&= \cosh^2(t) - \sinh^2(t) \\
&= 1
\end{aligned}$$

$$\begin{aligned}
h_{12} &= m_{a,b}(k(t,x))\partial_1 k^a \partial_1 k^b \\
&= +1(-\cosh(t)\cos(x))(\sinh(t)\sin(x)) + 1(\cosh(t)\sin(x))(\sinh(t)\cos(x)) - 1(\sinh(t)(0)) \\
&= 0 = h_{21}
\end{aligned}$$

$$\begin{aligned}
h_{22} &= m_{a,b}(k(t,x))\partial_1 k^a \partial_1 k^b \\
&= +1(\sinh(t)(-\sin(x)))(\sinh(t)(-\sin(x))) + 1(\sinh(t)\cos(x))(\sinh(t)\cos(x)) - 1(0)(0) \\
&= \sinh^2(t)\sin^2(x) + \sinh^2(t)\cos^2(x) - 0 \\
&= \sinh^2(t)(\cos^2(x) + \sin^2(x)) \\
&= \sinh^2(t)
\end{aligned}$$

Thus the metric for $\mathbb{H}^2$ is

$$h = \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2(t) \end{pmatrix}$$

□

2.2. **Semi-Riemannian metrics and model spaces.**
Semi-Riemannian metrics and manifolds are a generalization of the Riemannian case. Here, the metric is still bilinear, symmetric, and nondegenerate, but need not be positive-definite. When the metric is not positive-definite, we have a Lorentz manifold. The principal use of Lorentz manifolds is for understanding general relativity and spacetime. A connected, time-oriented four dimensional Lorentz manifold is called a *spacetime*. A spacetime is oriented such that the positive time axis is the future, and the negative time axis is the past. Using a Lorentz manifold to describe spacetime allows us to take into account for the fact that the speed of light is constant in all inertial frames of reference. The model Lorentz manifolds are Minkowski space ($\mathbb{R}^{n,1}$), De Sitter space (dS^n), and Anti De Sitter space (AdS^n). A Minkowski spacetime M is isometric to Minkowski 4-space.

Definition 2.13. Minkowski space is $\mathbb{R}^{n,1}$ together with the Minkowski metric. It is the semi-Riemannian analogue of $\mathbb{R}^n$.

Example 2.14. De Sitter space has quadratic form

$$x_1^2 + x_2^2 + \ldots + x_n^2 - x_{n+1}^2 = +1.$$

In other words, as a set,

$$dS^n = \{(x_1,\, x_2,\, \ldots,\, x_{n+1}) \text{ such that } x_1^2 + x_2^2 + \ldots + x_n^2 \text{ - } x_{n+1}^2 = +1\}.$$

The metric is obtained by pulling back the Minkowski metric onto this set. You may notice that this is defined similarly to the hyperboloid, but this is actually the semi-Riemannian analogue of $\mathbb{S}^n$.

For the 2-dimensional case, we can define coordinates

$$k(t, x) = (\sqrt{x^2 + 1}\cos(t), \sqrt{x^2 + 1}\sin(t), x)$$

Note that these coordinates fit the quadratic form. The metric in coordinates, d, is the pullback of the Minkowski metric, k^*m, where

$$d_{i,j} = m_{a,b}(k(t, x))\partial_i k^a \partial_j k^b$$

Computing the metric gives:

$$d = \begin{pmatrix} x^2 + 1 & 0 \\ 0 & \frac{-1}{x^2+1} \end{pmatrix}$$

The proof of this follows a similar calculation as for the hyperbolic metric.

Example 2.15. Anti de Sitter space AdS^n is the set sitting inside $\mathbb{R}^{n-1,2}$

$$x_1^2 + x_2^2 + \ldots - x_n^2 - x_{n+1}^2 = -1$$

where the metric is a pullback of the metric

$$m = I_{n-1,2}.$$

For the 2-dimensional case, we can define coordinates

$$k(t, x) = (x, \sqrt{1 + x^2}\cos(t), \sqrt{1 + x^2}\sin(t))$$

The metric in coordinates, a, is

$$a = \begin{pmatrix} -1 - x^2 & 0 \\ 0 & \frac{1}{1+x^2} \end{pmatrix}$$

3. Moving frames and connection forms

In geometry, we may want to know if there is a way to transport a vector along a smooth curve such that its geometrical information is preserved. In fact, there is a means to do this, known as parallel transport. This idea relies heavily on the notion of a connection form. Parallel transport is a means of transporting vectors in our manifold along curves so that the vectors stay parallel with respect to a connection. We'll develop the notion of a connection in this section, following Do Carmo [1].

Definition 3.1. Let U be an open set in $\mathbb{R}^n$ and let $e_1, e_2, \ldots, e_n$ be n differentiable vector fields such that for each point p in U, $\langle e_i, e_j\rangle = \delta_{i,j}$. This set of vector fields is called an orthonormal moving frame.

In other words, a moving frame is a smoothly varying basis of T_pM, the tangent space on M at p. As the word 'moving' implies, this frame of reference allows us to talk about properties at every point on our manifold, not just one.

Definition 3.2. The coframe associated to $\{e_i\}$ is the set of 1-forms $\{\omega_i\}$ such that $\omega_i(e_j) = \delta_{i,j}$. This forms a basis on the cotangent bundle.

Definition 3.3. $(de_i)_p(v) = \sum_j \omega_{ij} e_j$

where ω_{ij} is a differential 1-form called a *connection form.*

Proposition 3.4. *Let $\{e_i\}$ be a moving frame in an open set U in $\mathbb{R}^n$. Let $\{\omega_i\}$ be the associated coframe and ω_{ij} the connection forms of U in $\{e_i\}$. Then*

$$d\omega_i = \sum_k \omega_k \wedge \omega_{ki}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj}$$

for i, j, k = 1 . . . n

For the proof of the proposition, we refer the reader to Do Carmo [1]. For our purposes, we will focus on the relations in the 2-dimensional case. They are listed below.

$$d\omega_1 = \omega_2 \wedge \omega_{21} \tag{3.5}$$

$$d\omega_2 = \omega_1 \wedge \omega_{12} \tag{3.6}$$

and $\exists \alpha, \beta$ such that

$$\omega_{12} = \alpha\omega_1 + \beta\omega_2 = -\omega_{21} \tag{3.7}$$

Definition 3.8. The Christoffel-Levi Civita connection form is the unique torsion-free connection form that preserves the given metric.

4. Curvature

The basic idea behind curvature is–how far does our manifold deviate from being flat? From calculus, we know the curvature κ of a curve $\gamma(t)$ parametrized by arc length is the norm of the acceleration vector $\ddot{\gamma}(t)$. For a 2-dimensional manifold, we can take a point p and the normal vector at that point N and consider all planes passing through p containing N. The intersection of each plane with the manifold creates a curve along the manifold. The curvature of these curves can be calculated in the manner described above.

The trouble with this, however, is that this gives us extrinsic curvature, but does not give us intrinsic curvature–in other words, it doesn't tell us anything about the manifold as a whole, but rather only about the curvature along a specific curve. Gauss came up with an interpretation of curvature that is actually intrinsic to the manifold because it depends only on the metric and derivatives. The Gaussian curvature K is defined as

$$K = \kappa_{min}\kappa_{max}$$

where κ_{min} and κ_{max} are the minimum and maximum curvature of all curves on the manifold, respectively.

This is the notion of curvature we will use. In this section, we will show the curvature for each of our model spaces in both Riemannian and Semi-Riemannian geometry. We will give equations for curvature and compute it for some examples in 2-dimensions.

4.1. **Two definitions of curvature for 2-dimensional manifolds.**

Proposition 4.1. *The curvature K of a Riemannian or semi-Riemannian manifold is such that*

$$d\omega_{12} = -K\omega_1 \wedge \omega_2 \tag{4.2}$$

See Do Carmo [1] for details.

For the next two lemmas and proposition, consider the following. Let M^2 be a two-dimensional Riemannian manifold. Let $f{:}U \subset \mathbb{R}^2 \to V \subset M$ be a parametrization of $V \subset M^2$ such that $f_u = df(\frac{\partial}{\partial u})$ and $f_v = df(\frac{\partial}{\partial v})$ where u, v are orthogonal vectors in U. Set $E = \langle f_u, f_u \rangle$ and $G = \langle f_v, f_v \rangle$. Choose an orthonormal frame $e_1 = f_u/\sqrt{G}$, $e_2 = f_v/\sqrt{E}$. We will calculate the coframe and connection forms for this moving frame, and use these to develop a new equation for the curvature.

Lemma 4.3. *The associated coframe is given by*

$$\omega_1 = \sqrt{E}du, \omega_2 = \sqrt{G}dv$$

Proof. By defnition 3.2, we know that $\omega_1(e_1) = 1$ and $\omega_1(e_2) = 0$.

$$\begin{aligned}
&\omega_1(f_u/\sqrt{G}) = 1\\
&\omega_1((f_u/\sqrt{G})(\sqrt{G}/f_u) = 1(\sqrt{G}/f_u)\\
&\omega_1 = \sqrt{G}du
\end{aligned}$$

And $\omega_1(f_v/\sqrt{E})$ is 0 because the derivative of f_v with respect to u is 0. The proof that $\omega_2 = \sqrt{G}dv$ follows symmetrically. □

Lemma 4.4. *The connection form is*

$$\omega_{12} = -\frac{\sqrt{E}_v}{\sqrt{G}}du + \frac{\sqrt{G}_u}{\sqrt{E}}dv \tag{4.5}$$

Proof. By (3.5), we know that

$$d\omega_1 = \omega_2 \wedge \omega_{21}$$

Calculating $d\omega_1$ gives

$$\begin{aligned}
d\omega_1 &= (\omega_1) \wedge du + (\omega_1) \wedge dv\\
d\omega_1 &= (\sqrt{E}du) \wedge du + (\sqrt{E}du) \wedge dv\\
d\omega_1 &= 0 + (\sqrt{E})_v dv \wedge du
\end{aligned}$$

Substituting $(\sqrt{E})_v du \wedge dv$ for $d\omega_1$ and $\sqrt{G}dv$ for ω_2 gives

$$\begin{aligned}
&-(\sqrt{E})_v du \wedge dv = \sqrt{G}dv \wedge \omega_{21}\\
&-((\sqrt{E})_v du \wedge dv)/\sqrt{G} = (\sqrt{G}dv \wedge \omega_{21})/\sqrt{G}\\
&-\frac{(\sqrt{E})_v}{\sqrt{G}}du \wedge dv = -\omega_{21} \wedge dv\\
&-\frac{(\sqrt{E})_v}{\sqrt{G}}du \wedge dv = \omega_{12} \wedge dv
\end{aligned}$$

Now by (3.6), we know that

$$d\omega_2 = \omega_1 \wedge \omega_{12}$$

Calculating $d\omega_2$ gives

$$d\omega_2 = (\sqrt{G})_u du \wedge dv$$

Substituting $d\omega_2$ and ω_1 into equation (3.6) gives

$$\begin{aligned} &(\sqrt{G})_u du \wedge dv = \sqrt{E} du \wedge \omega_{12} \\ &((\sqrt{G})_u du \wedge dv)/\sqrt{E} = (\sqrt{E} du \wedge \omega_{12})/\sqrt{E} \\ &\frac{(\sqrt{G})_u}{\sqrt{E}} du \wedge dv = du \wedge \omega_{12} \end{aligned}$$

From here, it's easy to confirm that $\omega_{12} = -\frac{\sqrt{E}_v}{\sqrt{G}} du + \frac{\sqrt{G}_u}{\sqrt{E}} dv$ □

Proposition 4.6. *The Gaussian curvature K of M^2 is*

$$K = -\frac{1}{\sqrt{EG}} \left(\left(\frac{(\sqrt{E})_v}{\sqrt{G}} \right)_v + \left(\frac{(\sqrt{G})_u}{\sqrt{E}} \right)_u \right) \tag{4.7}$$

Proof. Calculating $d\omega_{12}$ gives

$$d\omega_{12} = \left(\left(\frac{(\sqrt{E})_v}{\sqrt{G}} \right)_v + \left(\frac{(\sqrt{G})_u}{\sqrt{E}} \right)_u \right)$$

Substituting $d\omega_{12}$, ω_1 and ω_2 into equation (4.2) gives

$$\begin{aligned} \left(\left(\frac{(\sqrt{E})_v}{\sqrt{G}} \right)_v + \left(\frac{(\sqrt{G})_u}{\sqrt{E}} \right)_u \right) &= -K(\sqrt{E}\sqrt{G}) du \wedge dv \\ K = -\frac{1}{\sqrt{EG}} \left(\left(\frac{(\sqrt{E})_v}{\sqrt{G}} \right)_v + \left(\frac{(\sqrt{G})_u}{\sqrt{E}} \right)_u \right) \end{aligned}$$

□

While we proved this proposition for the Riemannian case, the equation for curvature for a semi-Riemannian manifold is quite similar. Recall that for a semi-Riemannian manifold, the metric need not be positive definite. We have to take this into account in our definition of curvature. We can rewrite our equation as:

$$K = -\frac{1}{eg} \left(\varepsilon_2 (\frac{(e)_v}{g})_v + \varepsilon_1 (\frac{(g)_u}{e})_u \right) \tag{4.8}$$

where $e = \sqrt{|E|}$, $g = \sqrt{|G|}$, and $\varepsilon_1 = \operatorname{sgn}(E)$, $\varepsilon_2 = \operatorname{sgn}(G)$

4.2. **Curvature of Riemannian model spaces.**

Example 4.9. $\mathbb{S}^2$

Recall that the coordinates we put on the 2-sphere were

$$k(t, x) = (\sin(t)\cos(x), \sin(t)\sin(x), \cos(t))$$

And the metric in coordinates was

$$h = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(t) \end{pmatrix}$$

Proposition 4.10. *$\mathbb{S}^2$ has constant Gaussian curvature +1.*

Proof. Define a moving frame on the 2-sphere as

$$e_1 = \frac{\partial}{\partial t}, e_2 = \frac{1}{\sin(t)} \frac{\partial}{\partial x}$$

The coframe is then

$$\omega_1 = dt, \omega_2 = \sin(t) dx$$

We now use (3.5), (3.6), and (3.7) to calculate ω_{12}. First, we calculate $d\omega_1$ and $d\omega_2$.

$$d\omega_1 = dt \wedge dx, d\omega_2 = \cos(t) dt \wedge dx$$

Substituting $d\omega_1$, ω_2 and (3.7) into (3.5) gives

$$\begin{aligned} dt \wedge dx &= -\sin(t) dx \wedge (\alpha dt + \beta \sin(t) dx) \\ &= -\sin(t) dx \wedge (\alpha dt) \\ &= -\cos(t) dt \wedge dx \end{aligned}$$

Therefore, $\alpha = \frac{-1}{\cos(t)}$

Now substituting $d\omega_2$, ω_1, and (3.7) into (3.6) to solve for β gives

$$\begin{aligned} \cos(t) dt \wedge dx &= dt \wedge (\alpha dt + \beta \sin(t) dx) \\ &= \beta dt \wedge (\sin(t) dx) \\ &= \beta \cos(t) dt \wedge dx \end{aligned}$$

Therefore, $\beta = 1$.

Plugging α and β into our expression for ω_{12} gives

$$\omega_{12} = -\frac{dt}{\cos(t)} + \sin(t) dx$$

The derivative of ω_{12} is then

$$d\omega_{12} = dt \wedge \cos(t) dx$$

Finally, using expression (4.2) for our expression for curvature, we get

$$dt \wedge \cos(t) dx = K dt \wedge \cos(t) dx$$

And we conclude that $K = +1$. □

The curvature of $\mathbb{S}^n$ is more difficult to calculate, but it is also +1. $\mathbb{S}^n$ is the model space of constant curvature +1.

The curvature of 2-dimensional hyperbolic space can be calculated using the same general outline as the proof above. Hyperbolic space is a model of spaces with curvature -1. $\mathbb{R}^n$ has curvature 0.

4.3. **Curvature of Semi-Riemannian model spaces.** To compute the curvature of our Semi-Riemannian model spaces, we will use expression (4.8).

$$K = -\frac{1}{eg}\{\varepsilon_2(\frac{(e)_v}{g})_v + \varepsilon_1(\frac{(g)_u}{e})_u\}$$

To use this expression, recall that we must have the following conditions. M^2 must be a two-dimensional (semi)-Riemannian manifold. We let $f{:}U \subset \mathbb{R}^2 \to M$ be a parametrization of M^2 such that $f_u = df(\frac{\partial}{\partial u})$ and $f_v = df(\frac{\partial}{\partial v})$ where u, v are orthogonal vectors in U. Set $E = \langle f_u, f_u\rangle$ and $G = \langle f_v, f_v\rangle$. We choose an orthonormal frame $e_1 = f_u/\sqrt{G}$, $e_2 = f_v/\sqrt{E}$.

Example 4.11. For the following computation, we will consider the 2-dimensional De-Sitter space.

Recall that for dS^2 we used coordinates

$$k(t,x) = (\sqrt{x^2+1}\cos(t), \sqrt{x^2+1}\sin(t), x)$$

and computed the metric

$$d = \begin{pmatrix} x^2+1 & 0 \\ 0 & \frac{-1}{x^2+1} \end{pmatrix}$$

Notice that, by our method of computing the metric on De-Sitter space in section 2, d_{11} is constructed in the same way as E defined above, and similarly, d_{22} is constructed in the same way as G, above. Hence we write:

$$E = x^2+1 \qquad G = \frac{-1}{x^2+1}$$
$$e = \sqrt{x^2+1} \qquad g = \sqrt{\frac{1}{x^2+1}}$$
$$\varepsilon_1 = +1 \qquad \varepsilon_2 = -1$$

Now we have all of the tools we need to evaluate (4.8). Substituting, we get

$$\begin{aligned} K &= -\frac{1}{(\sqrt{x^2+1})(\sqrt{\frac{1}{x^2+1}})}\{-1(\frac{(\sqrt{x^2+1})_x}{\sqrt{\frac{1}{x^2+1}}})_x + 1(\frac{(\sqrt{\frac{1}{x^2+1}})_t}{\sqrt{x^2+1}})_t\} \\ &= -\frac{1}{(\sqrt{x^2+1})(\sqrt{\frac{1}{x^2+1}})}\{-1(\frac{(\frac{x}{x^2+1}}{\sqrt{\frac{1}{x^2+1}}})_x + 0\} \\ &= -1(-1)(x)_x \end{aligned}$$

Therefore, $K = +1$.

De Sitter space, in semi-Riemannian geometry, is the model space for constant curvature +1. It is analogous to the sphere in Riemannian geometry. The curvature of Anti De Sitter space and Minkoski space can be calculated in a similar fashion. The curvature of Anti De Sitter space is -1, and the curvature of 'flat' Minkowski space is 0. Thus these are analogous to hyperbolic space and $\mathbb{R}^n$, respectively.

All of our discussion and calculations have been for the 2-dimensional case. Conceptually, the n-dimensional case is similar. We take a point, p, and look at the tangent space at that point, T_pM, and consider all of the geodesics that pass through p and have their initial tangent vectors in T_pM. These geodesics sweep out a 2-dimensional manifold. From here, we can compute the *sectional curvature* using the mechanics we used above. A manifold has *constant curvature* if the sectional curvature is the same everywhere.

5. Geodesics

In loose terms, geodesics are acceleration free motion on our manifold. They are locally length minimizing–in other words, they are our notion of the shortest path between two points. The most familiar geodesic to us is the straight line on $\mathbb{R}^n$–this is always the shortest path between two points. We give a more precise definition below.

Definition 5.1. Let $\gamma : I \mapsto M$ be a curve parametrized by arc length. Let $\{e_1, e_2\}$ be a local frame with $\dot{\gamma} = e_2|_\gamma$, and let ω_{12} be the associated connection form. Then γ is a geodesic if and only if $\omega_{12}(\dot{\gamma}) = 0$.

Geodesics can also be understood as solutions to a second order ODE. In local coordinates (x^i), a geodesic γ is a solution to the equation $\ddot{\gamma}^i + \Gamma^i_{jk}(\gamma)\dot{\gamma}^j\dot{\gamma}^k = 0$. Importantly, by the Uniqueness Theorem of second order ODE's, a geodesic is uniquely determined by a point it passes through and the initial tangent vector. In this section, we'll use hyperbolic space as our primary example and find the geodesics on the half plane and hyperboloid model.

Example 5.2. $\mathbb{H}^2$ in the half-plane model.

Proposition 5.3. *Vertical lines in* $\mathbb{H}^2$ *are geodesics.*

Proof. Let γ be a vertical line in $\mathbb{H}^2$. Take $\{e_1, e_2\}$ as the standard

$$e_1(x, y) = y\partial_x$$
$$e_2(x, y) = y\partial_y$$

The associated coframe is thus

$$\omega_1 = \frac{1}{y}\partial_x$$
$$\omega_2 = \frac{1}{y}\partial_y$$

Computing the connection form ω_{12} gives

$$\begin{aligned} d\omega_1 &= -\frac{1}{y^2}dy \wedge dx \\ &= \omega_2 \wedge \omega_{21} \\ &= \frac{1}{y}dy \wedge \omega_{21} \end{aligned}$$

Thus we may write

$$\begin{aligned} -\frac{1}{y^2}dy \wedge dx &= \frac{1}{y}dy \wedge \omega_{21} \\ -\frac{1}{y}dx &= \omega_{21} \\ \omega_1 &= \omega_{12} \end{aligned}$$

Then

$$\begin{aligned} \omega_{12}(\dot{\gamma}) &= \omega_{12}(e_2) \\ &= \omega_1(e_2) \\ &= 0. \end{aligned}$$

Thus by definition 5.1, vertical lines (parametrized by arc length) are geodesics. □

Definition 5.4. Let A be a matrix in $SL_2(\mathbb{R})$. Define f_A: $\mathbb{H}^2 \to \mathbb{H}^2$ as

$$\begin{aligned} f_A : \mathbb{H}^2 \to \mathbb{H}^2 \\ z \mapsto \frac{az+b}{cz+d} \end{aligned}$$

This map is called a fractional linear transformation.

Proposition 5.5. *The geodesics in the half plane model are straight vertical lines and semi-circles that intersect the line $y = 0$ at right angles.*

Proof. Let w be in the tangent bundle $T\mathbb{H}^2$. Suppose $w = f_A(v_0)$, where v_0 is some fixed vector along ie^t. $f_A(\gamma_0(t)) = \gamma_w(t)$. Since f_A is in $SL_2(\mathbb{R})$, it preserves distance. Thus $\gamma_w(t)$ is a geodesic.

As a fractional linear transformation, f_A preserves circles and angles. □

Example 5.6. $\mathbb{H}^2$ in the hyperboloid model

Definition 5.7. An isometry is a diffeomorphism $F : M \mapsto M$ that preserves the metric, i.e., $F * g_M = g_M$.

Proposition 5.8. *The $x_1 - x_3$ plane $\cap\mathbb{H}^2$ is a geodesic on $\mathbb{H}^2$.*

Proof. Consider the plane $x_1 - x_3$ plane, call it P. Let γ_0 be equal to the intersection of P with $\mathbb{H}^2$. Consider a point p along γ_0. Imagine that there was a geodesic γ that passed through p that was not γ_0, and let v be it's initial tangent vector. Consider the map $\varphi : \mathbb{R}^3 \mapsto \mathbb{R}^3$ that sends x_2 to $-x_2$ and leaving all other coordinates fixed. This is an isometry fixing $p = \gamma(0)$ and $v = \dot{\gamma}(0)$. Then there would also exist γ' passing through p with initial tangent vector v that was a reflection of γ across γ_0. But by the Uniqueness Theorem of second order ODE's, there is a *unique* geodesic that passes through p with initial tangent vector v. Thus there cannot exist γ and γ_0 must be the unique geodesic through p. □

The geodesics on the sphere are also formed by intersecting a 2-plane that passes through the origin with the sphere. In other words, the geodesics on the sphere are the great circles.

In fact, the geodesics for de Sitter space and Anti de Sitter space are also formed by intersecting planes passing through the origin with the space. There are three types of geodesics in semi-Riemannian manifolds, time-like, light-like (or null), and space-like.

Definition 5.9. A time-like geodesic is such that $\langle \dot{\gamma}, \dot{\gamma} \rangle \lneq 0$. These are paremetrizations of one branch of a hyperbola.

Definition 5.10. A light-like (or null) geodesic is such that $\langle \dot{\gamma}, \dot{\gamma} \rangle = 0$. These are straight lines, or in other words, geodesics of $\mathbb{R}^{n,1}$.

Definition 5.11. A space-like geodesic is such that $\langle \dot{\gamma}, \dot{\gamma} \rangle \gneq 0$. These are periodic parametrizations of an ellipse within the space.

In the next section, we will describe all of our Riemannian and Semi-Riemannian models as homogeneous spaces. This will allow us to find all of the other geodesics on these manifolds.

6. Homogeneous spaces

When a manifold can be described as a homogeneous space, it means that any geometrical properties that hold at one point of M also hold at every other point. Riemannian homogenous spaces are also complete, meaning the geodesics exist for all time. In this section, we give a precise definition of homogeneity and show that all of our model spaces are homogeneous. But first, we review a few definitions from algebra.

Definition 6.1. $O(p, q)$ is the group of linear maps from $\mathbb{R}^{p+q}$ to itself such that for any A in $O(p, q)$, $A^T I_{p,q} A = I_{p,q}$. The group $O(n, 1)$ is called the Lorentz group, and it preserves the Minkowski metric.

Definition 6.2. A group G acts transitively on a space X if given a point q, for any point $p \neq q$ in X, there exists an element g in G such that $g(q) = p$.

Definition 6.3. Given a group action G on a space X and any point $p \in X$, the stabilizer group of P is the set of all $g \in G$ such that $gp = p$. This is also called the isotropy group of p.

Definition 6.4. A Riemannian or Semi-Riemannian manifold M is homogeneous if there is a one-to-one correspondence with a quotient space G/H, where G is a group of isometries that acts transitively on M and H is the stabilizer of M.

Example 6.5. Hyperboloid model of $\mathbb{H}^2$

Using the definitions above, we will prove that $\mathbb{H}^2$ is a homogeneous space.

Lemma 6.6. *$O(2, 1)$ acts transitively on $\mathbb{H}^2$ in the hyperboloid model*

Proof. Let $\langle , \rangle_g$ denote the quadratic form on the hyperboloid. By differentiating

$$\langle x, x \rangle_g = -1$$

we have

$$T_x \mathbb{H}^2 \simeq \{v \in \mathbb{R}^3 | \langle x, v \rangle_g = 0\}$$

Take p = (0,0,1). Then $T_{p^T} \mathbb{H}^2 = \text{span}\{(0, 1, 0)^T, (1, 0, 0)^T\}$.

Consider some other point $\tilde{p}$ in $\mathbb{H}^2$. Let vectors $\{e_1, e_2\}$ in $T_{\tilde{p}}\mathbb{H}^2$ be an orthonormal basis with respect to $\langle , \rangle_g$. Then

$$\langle e_i, e_j \rangle = \delta_{i,j}$$

and hence $\langle x, x \rangle = -1 \Rightarrow$

$$AI_{2,1}A^T = I_{2,1}$$

where $A = ([e_1][e_2][\tilde{p}]) \in O(2,1)$.

Note that

$$\begin{aligned} A(p) &= \tilde{p} \\ A((1,0,0)) &= e_1 \\ A((0,1,0)) &= e_2 \end{aligned}$$

Since A takes p to any other point, $\tilde{p}$, and A takes {(1,0,0), (0,1,0)} to $\{\tilde{E}_i\}$ as a change of basis map, we conclude that $O(2,1)$ acts transitively on $\mathbb{H}^2$ □

In the above proof, we showed that $O(2,1)$ acts transitively on $\mathbb{H}^2$, but we actually showed something stronger. We also showed that $O(2,1)$ acts transitively on orthonormal bases of $\mathbb{H}^2$. This will be important when we get to the semi-Riemannian case. Now we move on to calculate the stabilizer of a point in $\mathbb{H}^2$. For simplicity, we choose the point (0,0,1).

Lemma 6.7. *The stabilizer of (0,0,1) on* $\mathbb{H}^2$ *is* $O(2)$.

Proof. Any matrix that takes (0,0,1) to itself must have (0,0,1) as the values of its last column. Furthermore, by the definition of stabilizer, a matrix A that stabilizes (0,0,1) must belong to $O(2,1)$. This means that $A^T I_{2,1} A = I_{2,1}$

Let $A =$

$$\begin{pmatrix} c_1 & c_2 & 0 \\ c_3 & c_4 & 0 \\ a & b & 1 \end{pmatrix}$$

The criteria described above give us the following system of equations

$$\begin{aligned} &c_1^2 + c_3^2 - a^2 = 1 \\ &c_1c_2 + c_3c_4 - ab = 0 \\ &- a = 0 \\ &c_2^2 + c_4^2 - b^2 = 1 \\ &- b = 0 \end{aligned}$$

This gives

$$A = \begin{pmatrix} c_1 & c_2 & 0 \\ c_3 & c_4 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix}$$

where $C^T IC = I$. Thus C is in $O(2)$. The reverse inclusion is clear. □

Proposition 6.8. $\mathbb{H}^2$ *is a homogeneous space.*

Proof. Define

$$\varphi : O(2,1)/O(2) \to \mathbb{H}^2$$
$$A \mapsto A(0,0,1)$$

By lemma 6.6, lemma 6.7, this map is a well-defined bijection. □

This proof can easily be generalized to the n-dimensional case. The proofs that $\mathbb{R}^n$ and $\mathbb{S}^n$ are homogenous are very similar to the proof that the hyperboloid is homogenous. The homogeneous coset space of $\mathbb{S}^n$ is $O(n+1)/O(n)$, and the homogeneous coset space of $\mathbb{R}^n$ is $(\mathbb{R}^n \rtimes O(n,\mathbb{R}))/O(n)$. The proofs that $\mathbb{R}^{n,1}, dS^n$ and AdS^n are homogeneous are also very similar. We won't go through the entire calculation again for these spaces, but it's worth noting that the homogeneous space of $\mathbb{R}^{n,1}$ is $(\mathbb{R}^{n,1} \rtimes O(n,1))/O(n,1)$, the homogeneous space of dS^n is $SO(n,1)/SO(n-1,1)$ and the homogeneous space of AdS^n is $SO(n,2)/SO(n,1)$. For comparison, we include the proof that $O(2,1)$ acts transitively on dS^2.

Proposition 6.9. *$O(2,1)$ acts transitively on dS^2.*

Proof. Let $\langle , \rangle_d$ denote the quadratic form on dS^2. Take $p = (1,0,0)$ with orthonormal basis $\{E_i\}$ (with respect to $\langle , \rangle_d$) of the tangent space $T_p dS^2$ such that

$$E_1 = (0,1,0)$$
$$E_2 = (0,0,1)$$

Consider some other point $\tilde{p}$ and an orthonormal basis $\{\tilde{E_i}\}$ on $T_{\tilde{p}} dS^2$. Define α as the matrix

$$\alpha = ([\tilde{p}][\tilde{E_1}][\tilde{E_2}])$$

$\langle \tilde{p}, \tilde{p} \rangle_d = 1$ since p is in dS^2 and thus α satisfies $\alpha^T I_{2,1} \alpha = I_{2,1}$. α is in $O(2,1)$. Note that

$$\begin{aligned} \alpha(p) &= \tilde{p} \\ \alpha(E_1) &= \tilde{E_1} \\ \alpha(E_2) &= \tilde{E_2} \end{aligned}$$

Thus α takes $\{E_i\}$ to $\{\tilde{E_i}\}$ as a change of basis map and $\alpha^*\{E_i\} = \{\tilde{E_i}\}$, and O(2,1) acts transitively on dS^2. □

As we said for the hyperboloid, in Proposition 6.9 we have actually proven something stronger than just transitivity. In fact, for our semi-Riemannian model spaces, we have proven something known as *the perfect cosmological principle*. A semi-Riemannian manifold is said to satisfy the perfect cosmological principle in the case that each isometry between tangent spaces T_{p_1}, T_{p_2} can be realized as the differential of a differentiable isometry between neighborhoods of p_1 and p_2. In lay terms, the perfect cosmological principle states that the basic geometry of the world is the same in the vicinity of any two events in spacetime.

The fact that $\mathbb{R}^n, \mathbb{H}^n, \mathbb{S}^n, \mathbb{R}^{n,1}, dS^n$ and AdS^n are homogenous also allows us to find all of the geodesics of these spaces. Isometries take geodesics to geodesics as well. Recall that the geodesics of these spaces are formed by the planes that cut through the space and contain the origin. Again using $\mathbb{H}^2$ as our example, we will show that knowing one geodesic means we can find all other geodesics using the group that acts transitively on $\mathbb{H}^2$. We'll do this by showing that the group that acts transitively on $\mathbb{H}^2$ takes a plane through the origin to any other plane through the origin. The proof for the other spaces is nearly identical.

Proposition 6.10. *$O(2,1)$ takes the x_1x_3 plane to any other 2-plane through the origin*

Proof. The x_1x_3 plane, P, through the origin is equal to the span of (1,0,0) and (0,0,1) with normal vector defined at (0,1,0). Let $\varepsilon_1, \varepsilon_2$ be basis vectors for another plane P' through the origin. Let q be a point in $P' \cap \mathbb{H}^2$. Let $\beta \in O(2,1)$ be the matrix

$$\beta = ([\varepsilon_1][\varepsilon_2][q])$$

Then β takes (0,1,0) to q and is also the push-forward of (1,0,0) and (0,0,1) to $\varepsilon_1, \varepsilon_2$. Thus it takes P to P'. □

Perhaps even more interestingly than just taking geodesics to geodesics, the fact that these spaces have a group that acts transitively on orthonormal frames means that this group also acts transitively on 2-planes in the tangent bundle. Recall from section 3 that sectional curvature is defined on the tangent bundle. It then follows immediately that these spaces have constant sectional curvature.

All of the above proofs can easily be generalized to the n-dimensional case.

7. The importance of the model spaces

Up until this point, we have developed a deep understanding of certain model spaces, namely, $\mathbb{R}^n, \mathbb{S}^n$, and $\mathbb{H}^n$ in Riemannian geometry and $\mathbb{R}^{n,1}, dS^n$, and AdS^n in semi-Riemannian geometry. What can these spaces tell us about other manifolds of constant curvature, if anything at all? This is a question that, historically, troubled many. But, in fact, understanding these model spaces tells us a lot. The following theorems tell us that all manifolds of constant curvature can be classified with the model spaces we've explored.

Theorem 7.1. *For $n \geq 2$, a complete, simply connected, n-dimensional Riemannian manifold of constant curvature K is isometric to*
the sphere $\mathbb{S}^n(r)$ if $K = \frac{1}{r^2}$
Euclidean space $\mathbb{R}^n$ if $K = 0$
hyperbolic space $\mathbb{H}^n(r)$ if $K = -\frac{1}{r^2}$ [3]

Theorem 7.2. *For $n \geq 3$, a complete, simply connected, n-dimensional semi-Riemannian manifold of constant curvature K is isometric to*
de Sitter space $dS^n(r)$ if $K = \frac{1}{r^2}$
Minkowski space space $\mathbb{R}^{n,1}$ if $K = 0$
Anti de Sitter space $AdS^n(r)$ if $K = -\frac{1}{r^2}$ [3]

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