# TOPICS IN PERCOLATION

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Abstract. Percolation is the study of connectedness in a randomly-chosen subset of an infinite graph. Following Grimmett[4], we set up percolation on a square lattice. We establish the existence of a critical edge-density, prove several results about the behavior of percolation systems above and below this critical density, and use these results to find the critical density of percolation on the two-dimensional square lattice.

Contents



## 1. Introduction

In nature, fluids (such as water) percolate through porous substances (such as earth). The key feature of such a system is the porosity of the medium, which means that the fluid can travel along some, "open," paths but not along other, "closed," paths. The exact configuration of the open and closed paths in a given system is of course very complicated, and a specific configuration is not of independent interest. We instead want to study *random* percolation systems, in which each path is declared open or closed with a given probability. This is analogous to a complex physical system in which it is impossible to predict, say, the permeability of a given particle in the earth, but in which we know the general density of permeable particles in the entire section of earth under consideration.

While natural percolation is the inspiration for the subject, the mathematical study of percolation that we describe in this paper does not attempt to model any

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particular physical system. Rather, we think of a percolation system as a mathematical graph and establish results about graphs constructed randomly according to certain parameters. In this paper, we restrict our attention to configurations of the $d$-dimensional square lattice $\mathbb{L}^d$. We generate such configurations by choosing (as "open") a random subset of the *edges* of $\mathbb{L}^d$, which creates a system known as *bond-percolation*. (In the more general *site percolation*, subgraphs are created by choosing a subset of the *vertices*.) Percolation is subject to the *density parameter* $p$, which defines the probability of any given edge being chosen open or closed.

Given this setup, percolation theory seeks to understand the sizes of "open clusters" in configurations. In particular, we want to know whether these open clusters are of finite or infinite size. Curiously, as the density parameter varies, the probability of the existence of an infinite cluster changes sharply at the *critical point* $p_c$, which depends on the density of the lattice. The percolation system behaves in qualitatively different ways in the "subcritical" and "supercritical" phases. When $p < p_c$, open clusters are almost surely finite, and we can ask questions about their size (Section 4.2). When $p > p_c$, there is almost surely an infinite open cluster, and we can ask questions about how many such infinite open clusters there are (Section 4.3). Combining results from both phases, we can actually derive the value of $p_c$ for $d = 2$ (Section 4.4), which for general $d$ is very difficult to compute.

Chapter 2 sets down some basic concepts of measure theory and probability, and Chapter 3 establishes various definitions and theorems that will be useful for working with graphs and lattices. We apply this built-up theory to percolation systems in Chapter 4.

## 2. Measure and probability

We briefly formulate basic probability concepts using measure theory.

**Notation 2.1** (Set-theoretic notations)**.** We write $|A|$ to denote the cardinality of a set $A$ and $\mathscr{P}(A)$ to denote the power set of $A$. Given sets $A$ and $B$, we define the *symmetric difference* $A \triangle B$ as the set of elements that are members of either $A$ or $B$ but not both. In symbols, we have $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

**Definition 2.2.** Given a set $\Omega$, an *algebra* on $\Omega$ is a collection $\mathscr{F} \subseteq \mathscr{P}(\Omega)$ such that

1. $\emptyset, \Omega \in \mathscr{F}$.
2. $A \in \mathscr{F} \implies A^c \in \mathscr{F}$.
3. If $(A_i)_i$ is a finite collection of elements of $\mathscr{F}$, then $\bigcup_i A_i \in \mathscr{F}$.

**Definition 2.3.** An algebra $\mathscr{F}$ on $\Omega$ is a $\sigma$*-algebra* if, given a *countable* collection $(A_i)_i$ of elements of $\mathscr{F}$, we have $\bigcup_i A_i \in \mathscr{F}$.

**Fact 2.4.** *By de Morgan's Laws, an algebra is closed under finite intersections and a $\sigma$-algebra is closed under countable intersections.*

**Proposition 2.5.** *If $\mathscr{F}_1 \subseteq \mathscr{F}_2 \subseteq \mathscr{F}_3 \subseteq \cdots$ is an increasing sequence of $\sigma$-algebras, then $\mathscr{F} = \bigcup_i \mathscr{F}_i$ is an algebra.*

*Proof.* Clearly, $\emptyset, \Omega \in \mathscr{F}$. If $A \in \mathscr{F}$, then $A \in \mathscr{F}_i$ for some $i$, so $A^c \in \mathscr{F}_i \subseteq \mathscr{F}$. If $A_1, \ldots, A_n \in \mathscr{F}$, then $A_1, \ldots, A_n \in \mathscr{F}_i$ for some $i$, so $\bigcup_{j=1}^n A_j \in \mathscr{F}_i \subseteq \mathscr{F}$. □

**Definition 2.6.** Given a set $\Omega$ and a $\sigma$-algebra $\mathscr{F}$ on $\Omega$, a function $\mathbf{P} : \mathscr{F} \to [0, 1]$ is a *probability measure* on $\mathscr{F}$ if the following conditions hold:

1. $\mathbf{P}(\emptyset) = 0$.
2. $\mathbf{P}(\Omega) = 1$.
3. If $(A_i)_i$ is a countable collection of disjoint elements of $\mathscr{F}$, then $\mu(\bigcup_i A_i) = \sum_i \mu(A_i)$.

**Definition 2.7.** A *probability space* is a triple $(\Omega, \mathscr{F}, \mathbf{P})$ of a set $\Omega$ (called the *sample space*), a $\sigma$-algebra $\mathscr{F}$ on $\Omega$ (whose members are called *events*), and a probability measure $\mathbf{P}$ on $\mathscr{F}$. If $\mathscr{F}$ and $\mathbf{P}$ are clear from the context, we will often speak of the probability space $\Omega$ to refer to the probability space $(\Omega, \mathscr{F}, \mathbf{P})$.

**Definition 2.8.** We say that an event $A$ is *almost sure*, or that $A$ happens *almost always*, if $\mathbf{P}(A) = 1$, and that it happens *almost never* if $\mathbf{P}(A) = 0$.

**Facts 2.9.** *The following are basic properties of probability measures:*

1. *(Monotonicity.) If $A \subseteq B$, then $\mu(A) \leq \mu(B)$.*
2. *(Countable subadditivity.) If $(A_i)_i$ is a countable collection of elements of $\mathscr{F}$, not necessarily disjoint, then $\mu(\bigcup_i A_i) \leq \sum_i \mu(A_i)$.*
3. *(Continuity from below.) If $A_1 \subseteq A_2 \subseteq \cdots$, and $A = \bigcup_i A_i$, then $\mu(A) = \lim_{n\to\infty} \mu(A_i)$.*
4. *(Continuity from above.) If $A_1 \supseteq A_2 \supseteq \cdots$, and $A = \bigcap_i A_i$, then $\mu(A) = \lim_{n\to\infty} \mu(A_i)$.*

**Definition 2.10.** Two events $A$ and $B$ are *independent* with respect to a measure $\mathbf{P}$ if $\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B)$.

**Definition 2.11.** Given events $A, B \in \mathscr{F}$ such that $\mathbf{P}(B) > 0$, we define the *conditional probability* $\mathbf{P}(A \mid B) = \mathbf{P}(A \cap B)/\mathbf{P}(B)$.

**Fact 2.12.** *If $A$ and $B$ are independent, then $\mathbf{P}(A \mid B) = \mathbf{P}(A)$.*

**Definition 2.13.**

1. Suppose that $(\Omega, \mathscr{F}, \mathbf{P})$ is a probability space and $X$ is a real-valued function on $\Omega$ such that $X(\Omega)$ is finite or countable. We say that $X$ is *$\mathscr{F}$-measurable*, and that $X$ is a *discrete random variable*, if $\{\omega \mid X(\omega) = x\} \in \mathscr{F}$ for each $x \in X(\Omega)$.
2. If $A \in \mathscr{F}$, we define the *indicator function of $A$* to be the discrete random variable given by
$$\mathbf{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A; \\ 0 & \text{otherwise.} \end{cases}$$

**Notation 2.14.** We will write $\mathbf{P}(X = x)$ to denote $\mathbf{P}\{\omega \in \Omega \mid X(\omega) = x\}$.

**Definition 2.15.** Let $\Omega$ be a sample space.

1. If $\mathscr{A}$ is a collection of subsets of $\Omega$, then the *$\sigma$-algebra generated by $\mathscr{A}$*, denoted $\sigma(\mathscr{A})$, is the intersection of all $\sigma$-algebras $\mathscr{F}$ of $\Omega$ such that $\mathscr{A} \subseteq \mathscr{F}$.
2. If $\mathscr{X}$ is a collection of discrete random variables $X : \Omega \to \mathbf{R}$, then the *$\sigma$-algebra generated by $\mathscr{X}$*, denoted $\sigma(\mathscr{X})$, is the intersection of all $\sigma$-algebras $\mathscr{F}$ of $\Omega$ such that $X$ is $\mathscr{F}$-measurable for all $X \in \mathscr{X}$.

**Fact 2.16.** *If $\mathscr{X}$ is as in Definition 2.15.2, then*
$$\sigma(\mathscr{X}) = \sigma\left\{X^{-1}(x) \,\middle|\, X \in \mathscr{X}, x \in X(\Omega)\right\}.$$

It will be useful to approximate events in a $\sigma$-algebra generated by an algebra by events in the original algebra.

**Theorem 2.17.** *Suppose that* $\mathbf{P}$ *is a probability measure,* $\tilde{\mathscr{F}}$ *is an algebra on* $\Omega$*, and* $\mathscr{F} = \sigma(\tilde{\mathscr{F}})$*. For every* $A \in \mathscr{F}$ *and* $\varepsilon > 0$*, there is an* $\tilde{A} \in \tilde{\mathscr{F}}$ *so that* $\mathbf{P}(A \triangle \tilde{A}) < \varepsilon$*.*

*Proof.* Let $\mathscr{G} = \left\{ A \in \mathscr{F} \,\middle|\, \forall \varepsilon > 0,\ \exists \tilde{A} \in \tilde{\mathscr{F}} \text{ so that } \mathbf{P}(A \triangle \tilde{A}) < \varepsilon \right\}$. It is clear that $\tilde{\mathscr{F}} \subseteq \mathscr{G}$. We claim that $\mathscr{G}$ is a $\sigma$-algebra, which will imply that $\mathscr{F} = \sigma(\tilde{F}) \subseteq \mathscr{G}$ and thus prove the theorem.

We clearly have $\emptyset, \Omega \in \mathscr{G}$. Suppose $A \in \mathscr{G}$ and $\varepsilon > 0$. Then there is an $\tilde{A} \in \tilde{\mathscr{F}}$ so that $\mathbf{P}(A \triangle \tilde{A}) < \varepsilon$. But since $A \triangle \tilde{A} = A^c \triangle (\tilde{A})^c$, and $(\tilde{A})^c \in \tilde{\mathscr{F}}$, this means that $A^c \in \mathscr{G}$.

Suppose that $(A_i)_{i \in \mathbf{N}}$ is a countable collection of elements of $\mathscr{G}$, let $A = \bigcup_{i \in \mathbf{N}} A_i$, and let $B_n = \bigcup_{i=1}^n A_i$. Let $\varepsilon > 0$. By continuity of the measure $\mathbf{P}$, there is an $n \in \mathbf{N}$ so that $|\mathbf{P}(B_n) - \mathbf{P}(A)| < \varepsilon$, which implies that $\mathbf{P}(B_n \triangle A) = \mathbf{P}(A \setminus B_n) < \varepsilon$. For each $i = 1, \ldots, n$, there is an $\hat{A}_i \in \tilde{\mathscr{F}}$ such that $\mathbf{P}(\hat{A}_i \triangle A_i) < \varepsilon/n$, which implies that $\mathbf{P}(\bigcup_{i=1}^n \hat{A}_i \triangle B_n) < \varepsilon$. Therefore, $\mathbf{P}(\bigcup_{i=1}^n \hat{A}_i \triangle A) < 2\varepsilon$, so $A \in \mathscr{G}$ since $\bigcup_{i=1}^n \hat{A}_i \in \tilde{\mathscr{F}}$ by the definition of algebra. □

**Definition 2.18.** Given a countable collection of probability spaces $\{(\Omega_i, \mathscr{F}_i, \mathbf{P}_i)\}$, respectively, we say that $A \subseteq \Omega = \prod_i \Omega_i$ is a *cylinder set* if $A = \prod_i A_i$, where $A_i \in \mathscr{F}_i$ for all $i$ and $A_i = \Omega_i$ for all but finitely many $i$. We can then define the *product* $\sigma$*-algebra* $\mathscr{F}$ on $\Omega$ as the $\sigma$-algebra generated by the set of all cylinder sets. A *product (probability) measure* $\mathbf{P} = \prod_i \mathbf{P}_i$ is a (probability) measure on $\mathscr{F}$ such that $\mathbf{P}\left(\prod_i A_i\right) = \prod_i \mathbf{P}(A_i)$.

The actual construction of product measure, and thus the proof of the following theorem, is omitted.

**Theorem 2.19.** *For a countable product of probability spaces, a unique product measure exists.*

**Notation 2.20.** If $\Omega = \prod_i \Omega_i$, we have the usual projection functions $p_i : \Omega \to \Omega_i$.

**Proposition 2.21.** *If* $\{\Omega_i\}$ *is a countable collection of probability spaces and* $\Omega = \prod_i \Omega_i$ *is endowed with the product* $\sigma$*-algebra* $\mathscr{F}$*, then* $\mathscr{F} = \sigma(\{p_i\})$*.*

**Notation 2.22.** Let $p \in [0,1]$. We can define a probability measure $\mu_p$ on $\mathscr{P}(\{0,1\})$ by $\mu_p(\{1\}) = p$. Given a finite or countable set $E$, we will write $\mathbf{P}_p$ for the measure on $\{0,1\}^E$ given by $\mathbf{P}_p = (\mu_p)^E$.

**Definition 2.23.** Given a probability space $(\Omega, \mathscr{F}, \mathbf{P})$ and a discrete random variable $X$, we define the *expected value* of $X$ by

$$\mathbf{E}[X] = \int_\Omega X\, d\mathbf{P} = \sum_{x \in X(\Omega)} x \mathbf{P}(X = x).$$

**Facts 2.24.** *The following are basic properties of expected value:*

1. *(Linearity.) Given two random variables* $X$ *and* $Y$ *and a real number* $a$*, we have* $\mathbf{E}[aX + Y] = aE[X] + E[Y]$*.*
2. *(Monotonicity.) Given two random variables* $X$ *and* $Y$ *such that* $X(\omega) \leq Y(\omega)$ *for almost all* $\omega \in \Omega$*, we have* $\mathbf{E}[X] \leq E[Y]$*.*
3. *The expected value of an indicator function of an event is the probability of the event. Symbolically,* $\mathbf{E}[\mathbf{1}_A] = \mathbf{P}(A)$*.*

**Notation 2.25.** With setup as in Notation 2.22, we will write $\mathbf{E}_p$ to denote the expected value operator with respect to the product measure $\mathbf{P}_p$.

# 3. Graphs

## 3.1. Definitions and basic theory.

**Definition 3.1.** An *undirected graph* $G$ is a pair $(V, E)$ consisting of a set $V$ and a set $E$ of unordered pairs of elements in $V$. The elements of $V$ are called *vertices* and the elements of $E$ are called *edges*. If $e$ is an edge between vertices $v_1, v_2$, we will write $e = \langle v_1, v_2 \rangle = \langle v_2, v_1 \rangle$. In this paper, the term *graph* will always refer to an undirected graph.

**Definition 3.2.** A *configuration* of a graph $G = (V, E)$ is a function $\omega : E \to \{0, 1\}$, or equivalently an element $w \in \{0, 1\}^E$. An edge $e$ is called *open* in $\omega$ if $\omega(e) = 1$ and *closed* if $\omega(e) = 0$.

We will often wish to speak of configurations derived from other configurations by opening or closing certain subsets of edges, and will use the following notation frequently.

**Notation 3.3.** Let $G = (V, E)$ be a graph and let $\omega$ be a configuration of $G$. Suppose that $F \subseteq E$ is a subset of the edges of $G$. Define the configurations $\omega^F$ and $\omega_F$ by

$$\omega^F(e) = \begin{cases} 1 & \text{if } e \in F; \\ \omega(e) & \text{otherwise;} \end{cases} \qquad \omega_F(e) = \begin{cases} 0 & \text{if } e \in F; \\ \omega(e) & \text{otherwise.} \end{cases}$$

In other words, $\omega^F$ and $\omega_F$ are the configurations obtained from $\omega$ by taking all edges of $F$ to be open and closed, respectively.

**Definition 3.4.** Let $G = (V, E)$ be a graph, $\omega$ be a configuration on $G$, and $x, y \in V$. We say that $x$ and $y$ are *connected* in $\omega$, and write $x \leftrightarrow y$, if there is a finite sequence $x = z_1, z_2, \ldots, z_{n-1}, z_n = y$ such that $(z_i, z_{i+1})$ is an open edge in $\omega$ for all $i = 1, \ldots, n-1$.

**Fact 3.5.** *The connectedness relation $\leftrightarrow$ is an equivalence relation.*

**Definition 3.6.** A *cluster* in a configuration $\omega$ is an equivalence class under $\leftrightarrow$. We will write $C_x(\omega)$ for the cluster containing the node $x$; in particular, $C_0(\omega)$ will denote the cluster containing the origin.

3.2. **Properties of probability measure on graphs.** We review several important theorems for considering probability measure on graphs. While these theorems are not specific to probability on graphs, graphs provide both an application for the theorems and a set of examples for understanding them. In this section, we will work with a discrete set $E$, the sample space $\Omega = \{0, 1\}^E$, the product $\sigma$-algebra $\mathscr{F}$, and the probability measure and expectation operators $\mathbf{P}_p$ and $\mathbf{E}_p$ defined in Notations 2.22 and 2.25, respectively. For example, $E$ might be the set of edges on a graph $G = (V, E)$ and $\Omega$ the set of configurations of $G$; in this case, $\mathbf{P}_p$ would be the probability measure that independently takes each edge to be open with probability $p$.

It is clear that $\Omega$ is partially ordered under the usual partial order for functions: $\omega \leq \omega'$ if $\omega(e) \leq \omega'(e)$ for all $e \in E$. In our example of graphs and edges, we have $\omega \leq \omega'$ if all of the edges that are open in $\omega$ are also open in $\omega'$. Since $\Omega$ is partially ordered, we thus also have a notion of an increasing random variable on $\Omega$.

**Theorem 3.7.** *If $f : \Omega \to \mathbf{R}$ is a nondecreasing function, and $p \leq r$, then $\mathbf{E}_p[f] \leq \mathbf{E}_r[f]$.*

*Proof.* Consider the sample space $\Xi = [0,1]^E$ with probability measure given by the product of uniform measures on each copy of the unit interval. For $p \in [0,1]$ and $\xi \in \Xi$, define a random configuration $\omega_p(\xi)$ of $\Omega$ by

$$(\omega_p(\xi))(e) = \mathbf{1}_{[0,p]}(\xi(e)).$$

Note that $\omega_p$ is distributed according to the product measure $\mathbf{P}_p$. If $p \leq r$, it is clear that $\omega_p \leq \omega_r$. Therefore, we have $f(\omega_p) \leq f(\omega_r)$. But this means, since the expected value operator is order-preserving, that $\mathbf{E}_p[f] = \mathbf{E}(f(\omega_p)) \leq \mathbf{E}(f(\omega_r)) = \mathbf{E}_r[f]$. □

**Definition 3.8.** An event $A \in \mathscr{F}$ is *increasing* if $\omega \in A, \omega' \in \Omega, \omega' \geq \omega$ implies $\omega' \in A$. An event $B \in \mathscr{F}$ is *decreasing* if $B^c$ is increasing.

**Examples 3.9.** Given a graph $G = (V, E)$, we have the following examples of increasing events.

1. Let $E' \subseteq E$. Then the event $A = \{\omega \in \Omega \,|\, \omega(E') = 1\}$ is increasing.
2. Let $v_1, v_2 \in V$. Then the event
$$A = \{\omega \in \Omega \,|\, v_1 \text{ and } v_2 \text{ are connected in } \omega\}$$
is increasing.

We state the following theorem, a special case of what is called the FKG inequality, with the proof[4] omitted.

**Theorem 3.10.** *Suppose that $\Omega$ is as above, with $E$ finite, and suppose that $\mathbf{P}$ is a product measure on $\Omega$. If $A$ and $B$ are increasing events on $\Omega$, then $\mathbf{P}(A \cap B) \geq \mathbf{P}(A)\mathbf{P}(B)$.*

**Corollary 3.11.** *By symmetry, the (unmodified) conclusion of Theorem 3.10 also holds under the hypothesis that $A$ and $B$ are decreasing: if $A$ and $B$ are decreasing events on $\Omega$, then $\mathbf{P}(A \cap B) \geq \mathbf{P}(A)\mathbf{P}(B)$.*

*Remark* 3.12. The intuition behind Theorem 3.10 (in the setup of Examples 3.9) is that an increasing event is one that requires edges to be open and that only becomes more likely when it is given that more edges are open. Heuristically, then, the occurrence of one increasing event increases the likelihood of another increasing event.

**Definition 3.13.** Suppose that $A$ and $B$ are increasing subsets of $\Omega$. Define

$$A \circ B = \left\{\omega \in \Omega \,\middle|\, \text{there is an } F \subseteq E \text{ such that } \omega_{E \setminus F} \in A \text{ and } \omega_F \in B\right\},$$

using the notation from 3.3.

*Remark* 3.14. Intuitively, $\omega \in A \circ B$ if there is an $F \subseteq E$ so that we only need to examine the edges of $F$ to know that $\omega \in A$, and we only need to examine the edges outside of $F$ to know that $\omega \in B$.

**Proposition 3.15.** *If $A$ and $B$ are increasing subsets of $\Omega$, then $A \circ B$ is increasing.*

*Proof.* If $\omega \in A \circ B$, then there is an $F \subseteq E$ so that $\omega_{E \setminus F} \in A$ and $\omega_F \in B$. If $\omega' \geq \omega$, then $\omega'_{E \setminus F} \geq \omega_{E \setminus F}$ and thus $\omega'_{E \setminus F} \in A$ since $A$ is increasing. Similarly, $\omega'_F \in B$. Therefore, $\omega' \in A \circ B$, so $A \circ B$ is increasing. □

**Theorem 3.16** (BK inequality)**.** *Let $\Omega$ be defined as above and endowed with a product measure $\mathbf{P}$. Suppose that the edge-set $E$ is finite, and suppose moreover that $A$ and $B$ are increasing subsets of $\Omega$. Then $\mathbf{P}(A \circ B) \leq \mathbf{P}(A)\mathbf{P}(B)$.*

*Proof.* Let $1, \ldots, N$ be an ordering of $E$. Let $\Omega \times \Omega'$ be the product of two copies of $\Omega$ and let $\hat{\mathbf{P}} = \mathbf{P} \times \mathbf{P}$ be the product measure on $\Omega \times \Omega'$. If $\omega \in \Omega$ and $\omega' \in \Omega'$, let $H_j(\omega, \omega') = (\omega'(1), \ldots, \omega'(j-1), \omega(j), \ldots, \omega(N))$. Clearly, $\omega_1 = \omega$ and $\omega_{N+1} = \omega'$. For each $j = 1, \ldots, N$, define

$$\hat{A}_j = \{(\omega, \omega') \,|\, H_j(\omega, \omega') \in A\} \qquad \hat{B} = \{(\omega, \omega') \,|\, \omega \in B\} = B \times \Omega.$$

It is clear that $\hat{B}$ and all of the $\hat{A}_j$s are increasing since $A$ and $B$ are increasing. Moreover, $\hat{A}_1 = A \times \Omega'$, so

$$\hat{\mathbf{P}}(\hat{A}_1 \circ \hat{B}) = \hat{\mathbf{P}}((A \times \Omega') \circ (B \times \Omega')) = \hat{\mathbf{P}}(A \circ B).$$

Also, $\hat{A}_{N+1} = \Omega \times A$, so

$$\hat{\mathbf{P}}(\hat{A}_{N+1} \circ \hat{B}) = \hat{\mathbf{P}}((\Omega \times A) \circ (B \times \Omega')) = \hat{\mathbf{P}}\{(\omega, \omega') \,|\, \omega \in B \text{ and } \omega' \in A\} = \mathbf{P}(A)\mathbf{P}(B)$$

by the definition of the product measure $\hat{\mathbf{P}}$. The rest of the proof will establish the chain of inequalities

$$\hat{\mathbf{P}}(A \circ B) = \hat{\mathbf{P}}(\hat{A}_1 \circ B) \leq \hat{\mathbf{P}}(\hat{A}_2 \circ B) \leq \cdots \leq \hat{\mathbf{P}}(\hat{A}_{N+1} \circ B) = \mathbf{P}(A)\mathbf{P}(B),$$

which will imply the desired result.

Fix $1 \leq j \leq N$. We want to show that

$$\hat{\mathbf{P}}(\hat{A}_j \circ \hat{B}) \leq \hat{\mathbf{P}}(\hat{A}_{j+1} \circ \hat{B}). \tag{3.17}$$

We do this by conditioning on the value of $\omega(i), \omega'(i)$ for all $i \neq j$. Let $\underline{E} = E \setminus \{j\}$ and let $\underline{G} = (V, \underline{E})$. We will use the notation $\omega|_{\underline{E}}$ to mean the restriction of $\omega$ (as a function $E \to \{0, 1\}$) to $\underline{E}$. Let $\underline{\omega}$ and $\underline{\omega}'$ be configurations of $\underline{G}$. Define the event $C \subseteq \Omega \times \Omega'$ by

$$C = \left\{\omega|_{\underline{E}} = \underline{\omega} \text{ and } \omega'|_{\underline{E}} = \underline{\omega}'\right\}.$$

The goal is then to show that

$$\hat{\mathbf{P}}\left((\omega, \omega') \in \hat{A}_j \circ \hat{B} \,\middle|\, C\right) \leq \hat{\mathbf{P}}\left((\omega, \omega') \in \hat{A}_{j+1} \circ \hat{B} \,\middle|\, C\right); \tag{3.18}$$

if (3.18) holds for all $C$, then (3.17) clearly follows. Let $\underline{\omega}^j, \underline{\omega}'^j, \underline{\omega}_j, \underline{\omega}'_j$ be configurations of $G$ that agree on $E \setminus \{j\}$ with $\underline{\omega}$ and $\underline{\omega}'$, respectively, and have edge $j$ open and closed, respectively. Consider three cases:

1. $(\underline{\omega}^j, \underline{\omega}'^j) \notin \hat{A}_j \circ \hat{B}$. Then
$$\hat{\mathbf{P}}\left((\omega, \omega') \in \hat{A}_j \circ \hat{B} \,\middle|\, C\right) = 0 \leq \hat{\mathbf{P}}\left((\omega, \omega') \in \hat{A}_{j+1} \circ \hat{B} \,\middle|\, C\right),$$
with the inequality following trivially from the definition of measure.
2. $(\underline{\omega}_j, \underline{\omega}'_j) \in \hat{A}_j \circ \hat{B}$, so $(\underline{\omega}_j, \underline{\omega}'_j) \in \hat{A}_{j+1} \circ \hat{B}$. Then
$$\hat{\mathbf{P}}\left((\omega, \omega') \in \hat{A}_{j+1} \circ \hat{B} \,\middle|\, C\right) = 1 \geq \hat{\mathbf{P}}\left((\omega, \omega') \in \hat{A}_j \circ \hat{B} \,\middle|\, C\right).$$
3. Neither of the above, so $(\underline{\omega}^j, \underline{\omega}'^j) \in \hat{A}_j \circ \hat{B}$ and $(\underline{\omega}_j, \underline{\omega}'_j) \notin \hat{A}_j \circ \hat{B}$. Membership of a pair of configurations $(\omega, \omega')$ in $\hat{A}_j \circ \hat{B}$ does not depend on the value of $\omega'(j)$, so this implies
$$(\underline{\omega}^j, \underline{\omega}'_j) \in \hat{A}_j \circ \hat{B} \qquad (\underline{\omega}_j, \underline{\omega}'^j) \notin \hat{A}_j \circ \hat{B}.$$

Since $\hat{A}_j \circ \hat{B}$ is increasing, this means that the conditional probability that $(\omega, \omega')$ is in $\hat{A}_j \circ \hat{B}$ exactly the probability that $\omega(j) = 1$. More formally, we have

$$\hat{\mathbf{P}}\left((\omega, \omega') \in \hat{A}_j \circ \hat{B} \,\middle|\, C\right) = \mathbf{P}(\omega(j) = 1).$$

Since $(\underline{\omega}^j, \underline{\omega}'_j) \in \hat{A}_j \circ \hat{B}$, there are sets $F, F' \subseteq E$ so that we have $((\underline{\omega}^j)_{E \setminus F}, (\underline{\omega}'_j)_{E \setminus F'})) \in \hat{A}_j$ and $((\underline{\omega}^j)_F, (\underline{\omega}'_j)_{F'})) \in \hat{B}$. The latter membership is equivalent to $(\underline{\omega}^j)_F \in B$.

Consider two cases.

(a) $j \in F$, so $\underline{\omega}^j(j)$ "contributes" to $\hat{A}_j$. The value of $w'(j)$ does not affect whether or not $\hat{B}$ occurs, and the value of $w(j)$ does not affect whether or not $\hat{A}_{j+1}$ occurs, so we can certainly remove $j$ from $F$ and add $j$ to $F'$ when considering membership in $\hat{A}_{j+1} \circ \hat{B}$. With this arrangement of $F$ and $F'$ (and conditioning on $C$), we have $(\omega_{E \setminus F'}, \omega'_{E \setminus F'}) \in \hat{A}_{j+1}$ if and only if $\omega'(j) = 1$. There may be other arrangements of $F$ and $F'$ that increase the membership of of $\hat{A}_{j+1} \circ \hat{B}$, so, in general,

$$\begin{aligned} \hat{\mathbf{P}}\left((\omega, \omega') \in \hat{A}_{j+1} \circ \hat{B} \,\middle|\, C\right) &\geq \mathbf{P}(\omega'(j) = 1) \\ &= \mathbf{P}(\omega(j) = 1) \\ &= \hat{\mathbf{P}}\left((\omega, \omega') \in \hat{A}_j \circ \hat{B} \,\middle|\, C\right). \end{aligned}$$

(b) $j \in E \setminus F$. Now $\omega(j)$ "contributes" to $\hat{B}$. There is no effect from $w'(j)$ on membership in $\hat{B}$, so we can add $j$ to $F$ when considering membership of a given configuration in $\hat{A}_{j+1} \circ \hat{B}$. This may make the probability of membership in $\hat{A}_{j+1} \circ \hat{B}$ higher than that in $\hat{A}_j \circ \hat{B}$, but certainly cannot make it lower. Therefore,

$$\hat{\mathbf{P}}\left((\omega, \omega') \in \hat{A}_{j+1} \circ \hat{B} \,\middle|\, C\right) \geq \hat{\mathbf{P}}\left((\omega, \omega') \in \hat{A}_j \circ \hat{B} \,\middle|\, C\right).$$

In all cases, (3.18) holds. This applies for all fixed configurations $\underline{\omega}, \underline{\omega}'$, so (3.17) holds, which implies the desired result in the manner indicated above. □

3.3. **Lattices and translation.** We single out a special class of graphs, called *lattices*, for the study of percolation. We will write $\mathbf{Z}^d$ for the Cartesian product of $d$ copies of the integers.

**Definition 3.19.** The $\ell^1$ *norm* on $\mathbf{Z}^d$ is given by $\|x\|_1 = \sum_{i=1}^d |x_i|$. The $\ell^\infty$ *norm* is given by $\|x\|_\infty = \max_{1 \leq i \leq d} |x_i|$.

**Definition 3.20.** The *d-dimensional square lattice* is the (infinite) graph $\mathbb{L}^d = (\mathbf{Z}^d, \mathbb{E}^d)$, with the edge set $\mathbb{E}^d$ consisting of pairs of (non-diagonally) adjacent nodes. More precisely, we set $\mathbb{E}^d = \left\{\langle x, y\rangle \,\middle|\, x, y \in \mathbf{Z}^d \text{ and } \|x - y\|_1 = 1\right\}$.

For the remainder of the paper, $(\Omega, \mathscr{F}, \mathbf{P}_p)$ will be the probability space of configurations of $\mathbb{L}^d$ with the product measure $\mathbf{P}_p$.

**Definition 3.21.** A *box* $B$ in $\mathbb{L}^d$ is a subgraph of $\mathbb{L}^d$ with vertex set $V = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$ and edge set consisting of all edges in $\mathbb{E}^d$ with both endpoints in $V(B)$. The *boundary* of $B$, denoted $\partial B$, is (as its name suggests) the subgraph of $\mathbb{L}^d$ with vertex set

$$V(\partial B) = \{x = (x_1, \dots . x_n) \in V(\partial B) \,|\, \text{there is an } i \text{ such that } x_i \in \{a_i, b_i\}\}$$

and edge set consisting of all edges in $\mathbb{E}^d$ with both endpoints in $V(\partial B)$.

**Notation 3.22.** Given $n \in \mathbf{N}$, we define $\Lambda(n)$ to be the box generated by $[-n, n]^d$.

**Definition 3.23.** For each edge $e \in \mathbb{E}^d$, we have as in Notation 2.20 the projection function $p_e$, which we can consider as a random variable on $\Omega$. Define $\mathscr{L}(n) = \sigma(\{p_e \,|\, e \in \Lambda(n)\})$, the $\sigma$-algebra of events that depend only on the configurations of the edges inside $\Lambda(n)$. Let $\mathscr{L} = \bigcup_{n\in\mathbf{N}} \mathscr{L}(n)$.

**Fact 3.24.** *The set of events $\mathscr{L}$ is an algebra, and $\mathscr{F} = \sigma(\mathscr{L})$.*

The extra structure provided by the lattice lets us consider translations. Note that the lattice itself is invariant under translation: none of the points of the lattice are singled out.

**Definition 3.25.** Let $z \in \mathbf{Z}^d$.

1. If $G$ is a subgraph of $\mathbb{L}^d$, the *translation of $G$ by $z$* is the subgraph $G + z$ of $\mathbb{L}^d$ with vertex set $V(G + z) = \{x + z \,|\, x \in \mathbf{Z}^d\}$ and edge set consisting of the edges of $G$ translated similarly.
2. Let $\omega$ be a configuration of $\mathbb{L}^d$. The *translation of $\omega$ by $z$* is the configuration given by $(\omega + z)(e) = \omega(e - z)$ for all $e \in \mathbb{E}^d$.
3. Suppose that $A$ is an event in the probability space $(\Omega = \{0,1\}^{\mathbb{E}^d}, \mathscr{F} = \mathscr{P}(\Omega), \mathbf{P}_p)$. The *translation of $A$ by $z$* is the event $A + z = \{\omega \,|\, \omega - z \in A\}$. We say that $A$ is *translation-invariant* if for all $z \in \mathbf{Z}^d$, we have $A + z = A$.
4. A measure $\mathbf{P}$ is *translation-invariant* if $\mathbf{P}(A) = \mathbf{P}(A + z)$ for all events $A$ and all $z \in \mathbf{Z}^d$.

**Examples 3.26.** The following events are translation-invariant:

1. The event $A = \{\text{all edges are open in } \omega\}$.
2. The event $A = \{\text{there is an infinite open cluster in } \omega\}$.

The event $B = \{\text{there is an infinite open cluster containing the origin in } \omega\}$, however, is not translation-invariant, since translating a configuration in which the origin is part of an infinite open cluster may move the cluster away from the origin.

**Fact 3.27.** *Product measure $\mathbf{P}_p$ is translation-invariant by definition.*

**Theorem 3.28** (Zero-one law)**.** *If $A$ is a translation-invariant event, then either $\mathbf{P}_p(A) = 0$ or $\mathbf{P}_p(A) = 1$.*

*Proof.* Given $e \in \mathbf{E}^d$, define the random variable $e^{**} : \Omega \to \{0,1\}$ by $e^{**}(\omega) = \omega(e)$. Note that the $\sigma$-algebra $\mathscr{L}_n = \sigma(\{p_e \,|\, e \in \Lambda(n)\})$ is the $\sigma$-algebra of events that only depend on the edges in $\Lambda(n)$.

Fix $\varepsilon > 0$. By Proposition 2.5, we know that $\bigcup_{n\in\mathbf{N}} \mathscr{L}_n$ is an algebra. Furthermore, because any event can be expressed as a countable union or intersection of events only depending on the edges in finite boxes (see Proposition 2.21), we have that $A \in \sigma\left(\bigcup_{n\in\mathbf{N}} \mathscr{L}_n\right)$. Therefore, by Theorem 2.17, there is an $n \in \mathbf{N}$ and an event $U \in \mathscr{L}_n$ so that $\mathbf{P}_p(A \triangle U) < \varepsilon$, which implies $|\mathbf{P}_p(A) - \mathbf{P}_p(U)| < \varepsilon$ and

$$|\mathbf{P}_p(A)^2 - \mathbf{P}_p(U)^2| = |\mathbf{P}_p(A) + \mathbf{P}_p(U)| \cdot |\mathbf{P}_p(A) - \mathbf{P}_p(U)| < 2\varepsilon. \tag{3.29}$$

Let $z = (n+1, 0, 0, \ldots, 0) \in \mathbf{Z}^d$ and let $V = U + z$. Clearly, $U$ and $V$ are independent and have identical probabilities under the product measure, so $\mathbf{P}_p(U \cap V) = \mathbf{P}_p(U)\mathbf{P}_p(V) = \mathbf{P}_p(U)^2$. Substituting into (3.29), we have

$$|\mathbf{P}_p(A)^2 - \mathbf{P}_p(U \cap V)| < 2\varepsilon. \tag{3.30}$$

Note that, by several applications of Definition 3.25.3 and by the translation-invariance of $A$, we have

$$\begin{aligned} A \triangle V &= \{\omega_1 \,|\, \omega_1 \in A \text{ and } \omega_1 \notin V\} \cup \{\omega_1 \,|\, \omega_1 \in V \text{ and } \omega_1 \notin A\} \\ &= \{\omega_1 \,|\, \omega_1 \in A \text{ and } \omega_1 \notin U + z\} \cup \{\omega_1 \,|\, \omega_1 \in U + z \text{ and } \omega_1 \notin A\} \\ &= \{\omega_1 \,|\, \omega_1 \in A \text{ and } \omega_1 - z \notin U\} \cup \{\omega_1 \,|\, \omega_1 - z \in U \text{ and } \omega_1 \notin A\} \\ &= \{\omega_2 - z \,|\, \omega_2 + z \in A \text{ and } \omega_2 \notin U\} \cup \{\omega_2 - z \,|\, \omega_2 \in U \text{ and } \omega_2 + z \notin A\} \\ &= \{\omega_2 - z \,|\, \omega_2 \in A \text{ and } \omega_2 \notin U\} \cup \{\omega_2 - z \,|\, \omega_2 \in U \text{ and } \omega_2 \notin A\} \\ &= (A \triangle U) - z, \end{aligned}$$

and thus $\mathbf{P}_p(A \triangle V) = \mathbf{P}_p(A \triangle U)$ by the translation-invariance of $\mathbf{P}_p$. Note also that

$$\begin{aligned} A \triangle (U \cap V) &= (A \cap (U \cap V)^c) \cup (A^c \cap U \cap V) \\ &= (A \cap (U^c \cup V^c)) \cup (A^c \cap U \cap V) \\ &= (A \cap U^c) \cup (A \cap V^c) \cup (A^c \cap U \cap V) \\ &\subseteq (A \cap U^c) \cup (A^c \cap U) \cup (A \cap V^c) \cup (A^c \cap V) \\ &= (A \triangle U) \cup (A \triangle V). \end{aligned}$$

Therefore, $\mathbf{P}_p(A \triangle (U \cap V)) \leq \mathbf{P}_p(A \triangle U) + \mathbf{P}_p(A \triangle V) = 2\mathbf{P}_p(A \triangle U) < 2\varepsilon$, so

$$|\mathbf{P}_p(A) - \mathbf{P}(U \cap V)| < 2\varepsilon. \tag{3.31}$$

Combining (3.30) and (3.31) by the triangle inequality, we have that $|\mathbf{P}_p(A) - \mathbf{P}_p(A)^2| < 4\varepsilon$. But this holds for all $\varepsilon > 0$, so $\mathbf{P}_p(A) = \mathbf{P}_p(A)^2$, so $\mathbf{P}_p(A)$ must be equal to 0 or to 1. □

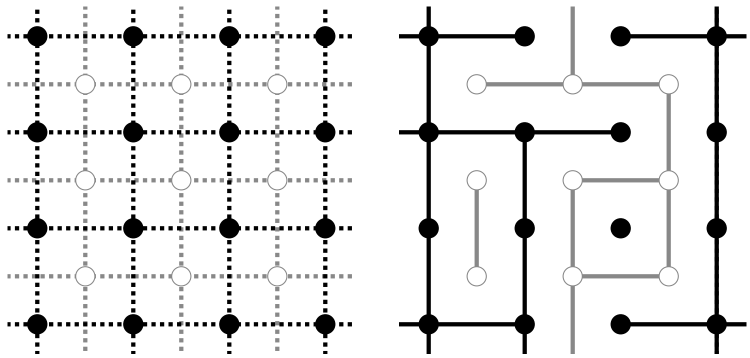
*Remark* 3.32. The key mechanism in the above proof is the application of Theorem 2.17. The translation-invariant event $A$ certainly cannot depend on any specific region of the lattice, but Theorem 2.17 says that it can be approximated in measure by an event that depends *only* on some specific region of the lattice. This can happen only if $A$ happens almost always or almost never.

3.4. **Planar Duality.** An important peculiarity of the two-dimensional square lattice $\mathbb{L}^2$ is that the lattice is "dual" to a copy of itself translated by $(1/2, 1/2)$. This allows us to use arguments about the connectedness of the dual lattice to draw conclusions about the connectedness of the original lattice.

**Definition 3.33.** We define the *dual two-dimensional square lattice* by $\mathbb{L}^2_{\mathrm{d}} = \mathbb{L}^2 + (1/2, 1/2)$ (Figure 1a).

*Remark* 3.34. The dual lattice $\mathbb{L}^2_{\mathrm{d}}$ has a vertex at the center of every square in $\mathbb{L}^2$. Every edge $e$ of $\mathbb{L}^2$ intersects exactly one edge $e_{\mathrm{d}}$ of $\mathbb{L}^2$ and vice versa, so the map $e \mapsto e_{\mathrm{d}}$ is a bijection.

**Definition 3.35.** A configuration $\omega$ of $\mathbb{L}^2$ gives rise to a configuration $\omega_{\mathrm{d}}$ of $\mathbb{L}^2_{\mathrm{d}}$ given by $\omega_{\mathrm{d}}(e_{\mathrm{d}}) = \omega(e)$, with $e_{\mathrm{d}}$ defined as in Remark 3.34. We will sometimes omit the subscript $_{\mathrm{d}}$ and speak of the configuration $\omega$ of $\mathbb{L}^2_{\mathrm{d}}$.



(a) Nodes and edges of $\mathbb{L}^2$ (dark) and $\mathbb{L}^2_{\mathrm{d}}$ (light).

(b) Two open clusters (dark) in a configuration of $\mathbb{L}^2$ are "separated" by a *closed* path (light) in $\mathbb{L}^2_{\mathrm{d}}$.

Figure 1. The lattice and dual lattice.

*Remark* 3.36. Informally, if $x$ and $y$ are members of two different clusters of $\mathbb{L}^2$ or of a subset of $\mathbb{L}^2$, then there must be a closed path in $\mathbb{L}^2_{\mathrm{d}}$ "separating" $x$ from $y$ (Figure 1b). We omit the formalization and proof of this statement, but will use this type of argument frequently in analyzing percolation on $\mathbb{L}^2$.

## 4. Percolation

We consider percolation on the $d$-dimensional square lattice $\mathbb{L}^d$. We will consider the sample space $\Omega$ consisting of all configurations $\omega$ of $\mathbb{L}^d$, equipped with the product $\sigma$-algebra $\mathscr{F}$ and product measure $\mathbf{P}_p$. With respect to $\mathbf{P}_p$, the individual edge-states $\omega(e)$ are independent, identically-distributed random variables satisfying the law $\mathbf{P}_p(\omega(e) = 1) = p$.

4.1. **Basic theory and the critical probability.** Sometimes it will be more convenient to think about the size of the cluster containing a node in terms of the lengths of paths originating at that node, a concept we formalize next.

**Definition 4.1.** Let $G = (V, E)$ be a graph. Given $n \in \mathbf{N}$ and $x \in V$, a *self-avoiding walk* of length $n$ starting at $x$ is a sequence $W = (x = x_1, x_2, x_3, \ldots, x_n)$ of length $n$ such that $\langle x_i, x_{i+1}\rangle \in E$ for all $i = 1, \ldots, N$ and also that $x_i = x_j$ implies $i = j$. Given a configuration $\omega$ of $G$, we say that $W$ is *open* in $\omega$ if $\omega(\langle x_i, x_{i+1}\rangle) = 1$ for all $i = 1, \ldots, N$.

**Proposition 4.2.** *Given a configuration $\omega \in \Omega$, there is an open self-avoiding walk of length $n$ starting at $0$ for all $n \in \mathbf{N}$ if and only if $|C_0(\omega)| = \infty$.*

*Proof.* If there is an open self-avoiding walk $W_n$ of length $n$ starting at 0 for all $n \in \mathbf{N}$, then each of the infinite number elements of $\bigcup_{n\in\mathbf{N}} W_n$ is certainly connected to 0. Conversely, if $|C_0(\omega)| = \infty$, then for any $n \in \mathbf{N}$ the open cluster containing the origin must contain a point with a graph-theoretic distance of $n$ from the origin, and the self-avoiding walk (which can be attained by erasing the cycles from an arbitrary path) from that point to the origin must have length at least $n$. $\square$

**Definition 4.3.** Given $\Omega$ with product measure $\mathbf{P}_p$, the *percolation probability* $\theta(p)$ is defined by $\theta(p) = \mathbf{P}_p(|C_0| = \infty)$.

**Proposition 4.4.** *The percolation probability function $\theta : [0,1] \to [0,1]$ is nondecreasing.*

*Proof.* The function $f : \Omega \to \mathbf{R}$ given by $f(\omega) = \mathbf{1}_{\{|C_0)|=\infty\}}(\omega)$ is certainly nondecreasing, since a configuration with an infinite cluster containing the origin will continue to have an infinite cluster containing the origin if more edges are opened. Therefore, by Theorem 3.7, $\theta(p) = \mathbf{E}_p[f]$ is nondecreasing in $p$. □

**Definition 4.5.** The *critical probability* $p_c$ is defined by $p_c = \sup(\theta^{-1}(0))$.

*Remark* 4.6. By Proposition 4.4, $\theta(p) = 0$ for all $p < p_c$ and $\theta(p) > 0$ for all $p > p_c$. It has been shown that $\theta(p_c) = 0$ for $d = 2$ and $d \geq 19$, and it is believed that this result holds for $3 \leq d \leq 18$, but the latter statement has not been proven.

Much of the remainder of this paper will be devoted to proving results about the nature of percolation when $p < p_c$ and when $p > p_c$. However, some of these results would be vacuous if it turned out that $p_c = 0$ or $p_c = 1$. The next proposition and two theorems show that while $d = 1$ is an uninteresting case, we have $0 < p_c < 1$ whenever $d \geq 2$. This establishes the existence of criticality in percolation: the behavior of the system changes abruptly at $p = p_c$.

**Proposition 4.7.** *For $d = 1$, we have $p_c = 1$.*

*Proof.* Let $p < 1$. For any $\omega \in \Omega$, we have $|C_0(\omega)| = \infty$ if and only if either $\omega(\langle n-1, n\rangle) = 1$ for all $n \in \mathbf{N}$ or $\omega(\langle -n+1, -n\rangle) = 1$ for all $n \in \mathbf{N}$. Both of these events have probability $\lim_{n\to\infty} p^n = 0$, so $\theta(p) = 0$. □

**Theorem 4.8.** *For $d \geq 2$, we have $p_c > 0$.*

*Proof.* We show that $p_c > 0$ by finding a $p > 0$ such that $\theta(p) = 0$. By Proposition 4.2, $|C_0(\omega)| = \infty$ if and only if for each $n \in \mathbf{N}$, there is an open self-avoiding walk of length $n$ starting at 0. Let $\mathscr{W}_n$ be the set of all self-avoiding walks of length $n$ starting at 0, and let $\sigma_n = |\mathscr{W}_n|$. Let $N_n(\omega)$ be the number of open self-avoiding walks of length $n$ in $\omega$ starting at 0. Then, $\theta(p) = \mathbf{P}_p(\bigcap_{n\in\mathbf{N}}\{N_n \geq 1\})$, so for all $n \in \mathbf{N}$,

$$\begin{aligned}\theta(p) \leq \mathbf{P}_p(N_n \geq 1) = \mathbf{E}_p[\mathbf{1}_{N_n\geq 1}]\\ \leq \mathbf{E}_p[N_n] = \mathbf{E}_p \sum_{W\in\mathscr{W}_n} \mathbf{1}_{\{W \text{ open}\}} = \sum_{W\in\mathscr{W}_n} \mathbf{P}_p(W \text{ open}) = \sigma_n p^n.\end{aligned}$$

All that remains is to bound $\sigma_n$. The first step of a self-avoiding walk starting at the origin can be in any of $2d$ directions, and all subsequent steps can go in at most $2d-1$ directions, since a self-avoiding walk certainly cannot backtrack its previous step. Therefore, $\sigma_n \leq 2d(2d-1)^{n-1}$, so $\theta(p) \leq 2d(2d-1)^{n-1}p^n = \frac{2d}{2d-1}(p(2d-1))^n$ for all $n \in \mathbf{N}$. If $p \in (0, \frac{1}{2d-1})$, this implies that $\theta(p) = 0$, so $p_c \geq 1/(2d-1) > 0$. □

**Theorem 4.9.** *For $d \geq 2$, we have $p_c < 1$.*

*Proof.* We must find a $p < 1$ so that $\theta(p) > 0$. If $\theta(p) > 0$ for $d = 2$, then $\theta(p) > 0$ for all $d \geq 2$. This is because we can define an embedding $\ell : \mathbb{L}^2 \to \mathbb{L}^d$ by $\ell(x_1, x_2) = (x_1, x_2, 0, \ldots, 0)$ and mapping edges in $\mathbb{L}^2$ to corresponding edges in $\mathbb{L}^d$. If there is an open cluster in $\ell(\mathbb{L}^2) \subset \mathbb{L}^d$ containing the origin (which, by the properties of product measure, happens with probability $\theta(p)$ for $d=2$), then there is certainly an open cluster in $\mathbb{L}^d$ containing the origin. Therefore, we can restrict our argument to $d=2$, which allows us to use a planar duality argument to show the desired result.

Note that, as in Remark 3.36, we have $|C(\omega)| < \infty$ if and only if $\omega_{\mathrm{d}}$ has a connected "circuit" of closed edges encircling the origin in $\mathbb{L}^2_{\mathrm{d}}$. Let $\mathcal{Q}_n$ be the set of all circuits of length $n$ encircling the origin in $\mathbb{L}^2_{\mathrm{d}}$. If $Q \in \mathcal{Q}_n$, then $Q$ must contain a node in the set $\{(k+1/2, 1/2) \,|\, k \in \{0, \ldots, n-1\}\}$, which we can consider the "starting point" of $Q$. Then, $M$ must have $n$ edges, and there can be at most 4 (in all but the first case, no more than 3, but the better estimate is unnecessary) possibilities for the direction of each edge. Therefore, $|\mathcal{Q}_n| \le n4^n$.

If we define the random variable $M_n$ by $M_n(\omega) = |\{Q \in \mathcal{Q}_n \,|\, Q \text{ is closed in } \omega\}|$, we have

$$\begin{aligned}\mathbf{P}_p(|C| < \infty) &= \mathbf{P}_p\left(\bigcup_{n=4}^{\infty}\{M_n \ge 1\}\right) = \mathbf{P}_p\left(\sum_{n=4}^{\infty} M_n \ge 1\right) = \mathbf{E}_p\left(\mathbf{1}_{(\sum_{n=4}^{\infty} M_n \ge 1)}\right) \\ &\le \mathbf{E}_p\left(\sum_{n=4}^{\infty} M_n\right) = \sum_{n=4}^{\infty}\sum_{M \in \mathcal{M}_n} \mathbf{E}_p[\mathbf{1}_{(M \text{ closed})}] \\ &= \sum_{n=4}^{\infty}\sum_{M \in \mathcal{M}_n} \mathbf{P}_p(M \text{ closed}) = \sum_{n=4}^{\infty} |\mathcal{Q}_n|(1-p)^n \le \sum_{n=4}^{\infty} n(4(1-p))^n.\end{aligned}$$

By making $p \in (0,1)$ sufficiently close to 1, we can make this last sum less than 1, which is what we want. □

4.2. **Subcritical percolation: exponential decay.** Percolation is *subcritical* when $p < p_c$. In this case, there will almost never be an infinite cluster, and thus it is natural to ask what we can say about the size of the cluster $C_0$ containing the origin. The chief result is that the probability of $C_0$ extending to a box of radius $n$ decays exponentially in $n$.

**Definition 4.10.** Given $p \in [0,1]$, the *mean cluster size* of $\mathbb{L}^d$ is defined by $\chi(p) = \mathbf{E}_p[|C_0|]$, the expectation of the size of the cluster containing the origin.

**Fact 4.11.** *If $p > p_c$, then there is a nonzero probability of an infinite cluster, so $\chi(p) = \infty$.*

An important result for establishing exponential decay, the proof[3] of which we omit, is the following theorem.

**Theorem 4.12.** *If $p < p_c$, then $\chi(p) < \infty$.*

We can think of Theorem 4.12 as a statement about the coincidence of the critical points of $\theta$ and of $\chi$. Assuming this result, we prove the exponential decay of subcritical percolation.

**Theorem 4.13.** *If $p < p_c$, then there is a $\kappa(p) > 0$ so that, for all $n \ge 1$, we have*

$$\mathbf{P}_p(0 \leftrightarrow \partial\Lambda(n)) \le e^{-n\kappa(p)}.$$

*Proof.* By Theorem 4.12, we have $\chi(p) = \mathbf{E}_p[|C_0|] < \infty$.

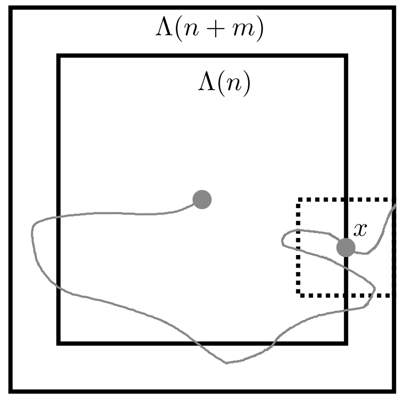



Figure 2. Illustration of the disjoint-paths argument that lets us use the BK inequality.

We first show that we can bound $\mathbf{P}_p(0 \leftrightarrow \partial\Lambda(n+m))$ by $K_p(m)\mathbf{P}_p(0 \leftrightarrow \partial\Lambda(n))$ for a function $K_p$. Moreover, we can find an $m$ so that $K_p(m) < 1$, which allows us to use the division algorithm to establish exponential decay.

Let $m, n \in \mathbf{N}$. In a configuration $\omega$ in which there is a path from 0 to $\partial\Lambda(n+m)$, there are also disjoint paths from 0 to some $x \in \partial\Lambda(n)$ and from $x$ to the translated box boundary $x + \partial\Lambda(m)$ (Figure 2). Because these paths are disjoint, we can find a subset $F \subset E$ so that $\omega_F \in \{0 \leftrightarrow x\}$ and $\omega_{E\setminus F} \in \{x \leftrightarrow x + \partial\Lambda(m)\}$. Therefore, in the notation of Definition 3.13, we have

$$\{\omega \leftrightarrow \partial\Lambda(n+m)\} \subseteq \bigcup_{x\in\partial\Lambda(n)} [\{0 \leftrightarrow x\} \circ \{x \leftrightarrow x + \partial\Lambda(m)\}],$$

so the BK inequality (Theorem 3.16) says that

$$\begin{aligned}
\mathbf{P}_p(0 \leftrightarrow \partial\Lambda(m+n)) &\le \sum_{x\in\partial\Lambda(n)} \mathbf{P}_p(0 \leftrightarrow x)\mathbf{P}_p(x \leftrightarrow x + \partial\Lambda(m)) \\
&= \sum_{x\in\partial\Lambda(m)} \mathbf{P}_p(0 \leftrightarrow x)\mathbf{P}_p(0 \leftrightarrow \partial\Lambda(m)) \\
&= \mathbf{P}_p(0 \leftrightarrow \partial\Lambda(m)) \sum_{x\in\partial\Lambda(m)} \mathbf{P}_p(0 \leftrightarrow x) \\
&= \mathbf{P}_p(0 \leftrightarrow \partial\Lambda(m))\mathbf{E}_p[|\,\{x \in \Lambda(m) \mid 0 \leftrightarrow x\}\,|].
\end{aligned}$$

We thus choose $K_p(m) = \mathbf{E}_p[|\,\{x \in \Lambda(m) \mid 0 \leftrightarrow x\}\,|]$, and we want to analyze $K_p$ to find an $M$ such that $K_p(M) < 1$. In fact, summing over all $m$, we have

$$\begin{aligned}
\sum_{m=0}^{\infty} K_p(m) &= \sum_{m=0}^{\infty} \left( \sum_{x\in\partial\Lambda(n)} \mathbf{P}_p(0 \leftrightarrow x) \right) \\
&= \sum_{x\in\mathbf{Z}^d} \mathbf{P}_p(0 \leftrightarrow x) = \mathbf{E}_p[|C_0|] = \chi(p) < \infty
\end{aligned}$$

by hypothesis. Thus, since $\sum_{n=0}^{\infty} K_p(m)$ converges, $\lim_{m\to\infty} K_p(m) = 0$, and in particular, there is an $M \in \mathbf{N}$ so that $K_p(M) < 1$, which we fix for the remainder of the proof.

Let $n \in \mathbf{N}$. By the division algorithm, we have $q, r \in \mathbf{N}$ (with $0 \le r < M$) so that $n = qM + r$. Since a path from 0 to $\partial\Lambda(n) = \partial\Lambda(qM + r)$ certainly intersects $\partial\Lambda(qM)$, we have

$$\begin{aligned}\mathbf{P}_p(0 \leftrightarrow \partial\Lambda(n)) &\le \mathbf{P}_p(0 \leftrightarrow \partial\Lambda(qM)) \\ &\le K_p(m)\mathbf{P}_p(0 \leftrightarrow \partial\Lambda((q-1)M) \\ &\vdots \\ &\le (K_p(m))^q = e^{q\log(K_p(m))}.\end{aligned}$$

Since $K_p(m) < 1$, we have $\log(K_p(m)) < 0$. Also, since $n < (q+1)M$, we have that $q > -1 + n/M = n(\frac{1}{M} - \frac{1}{n}) \ge n(\frac{1}{M} - \frac{1}{M+1})$ for all $n \ge M + 1$. Therefore, letting

$$\kappa'(p) = -\left(\frac{1}{M} - \frac{1}{M+1}\right)\log(K_p(M)),$$

we have $\kappa'(p) > 0$ and

$$q\log(K_p(m)) < -n\kappa'(p)$$

for all $n \ge M + 1$, which implies that $\mathbf{P}_p(0 \leftrightarrow \partial\Lambda(n)) < e^{-n\kappa'(p)}$ for all $n \ge M + 1$. Since there are only a finite number of $n \le M$, we can choose an $S$ so that $\mathbf{P}_p(0 \leftrightarrow \partial\Lambda(n)) \le e^{-nS}$ for all $n \le M$, and then letting $\kappa(p) = \max\{\kappa'(p), S\}$ gives the desired inequality. □

4.3. **Supercritical percolation: uniqueness of the infinite cluster.** If $p > p_c$, then there is a nonzero probability of an infinite cluster containing the origin. By the zero-one law (Theorem 3.28), for a given $p$ there is either almost surely or almost never an infinite cluster. Thus, if $p > p_c$, there is almost surely an infinite cluster. The goal of this section is to show that this infinite cluster is almost surely *unique*.

We first must establish a technical result regarding the number of ways of partitioning a set, which we will use in the proof of the theorem to establish a relationship between special points in the interior of a box and points on the boundary of the box.

**Definition 4.14.**

1. Let $Y$ be a set. A *partition* of $Y$ is a collection $P = \{P^{(1)}, P^{(2)}, P^{(3)}\}$ of three nonempty, disjoint subsets of $Y$ such that $P^{(1)} \cup P^{(2)} \cup P^{(3)} = Y$. (Note that we consider two partitions of $Y$ to be equal if they consist of the same three subsets of $Y$, regardless of the ordering of the three subsets.)
2. Two partitions $P$ and $Q$ of a set $Y$ are *compatible* if there are orderings of $P$ and $Q$ so that $P^{(1)} \supseteq Q^{(2)} \cup Q^{(3)}$.
3. A collection of partitions is *compatible* if the partitions are pairwise compatible.

**Lemma 4.15.** *Suppose that $\mathbf{S} = \{S_i\}_i$ is a compatible collection of partitions of a set $Y$. Then $|\mathbf{S}| \le |Y| - 2$.*

*Proof.* We proceed by induction on $n = |Y|$. Because elements of a partition must be nonempty, there can only be one partition of $Y$ if $n = 3$, so the lemma holds in this case. For the rest of the proof, we will fix $n \geq 4$ and assume that the lemma holds for all $Z$ with $|Z| < n$.

Let $S \in \mathbf{S}$ be an arbitrary partition. Fix an ordering $S = \left\{S^{(1)}, S^{(2)}, S^{(3)}\right\}$ of the elements of $S$. Because $\mathbf{S}$ is compatible, for each $T \in \mathbf{S} \setminus S$ there is an ordering of $T$ and a unique $i \in \{1, 2, 3\}$ so that $S^{(i)} \supseteq T^{(2)} \cup T^{(3)}$; therefore, we can write $\mathbf{S} \setminus S$ as the disjoint union $\mathbf{S} = \mathbf{S}^{(1)} \cup \mathbf{S}^{(2)} \cup \mathbf{S}^{(3)}$, where $T \in \mathbf{S}^{(i)}$ whenever there is an ordering of $T$ such that $S^{(i)} \supseteq T^{(2)} \cup T^{(3)}$.

Fix $i \in \{1, 2, 3\}$. Let $Z_i = S^{(i)} \cup \{\Delta\}$, where $\Delta$ is an arbitrary object (not an element of $Y$). Note that $|Z_i| = |S^{(i)}| + 1 \leq n - 1 < n$, so there can be no more than $|S^{(i)}| - 1$ partitions of $Z_i$ by the inductive hypothesis. If $T \in \mathbf{S}^{(i)}$, then we can define a partition $T'$ of $Z_i$ by

$$T' = \left\{(T^{(1)} \cap S^{(i)}) \cup \{\Delta\}, T^{(2)}, T^{(3)}\right\}.$$

It is clear that the map that takes $T$ to $T'$ is injective. Thus, since there can be at most $|S^{(i)}| - 1$ partitions of $Z_i$, we have that $|\mathbf{S}^{(i)}| \leq |S^{(i)}| - 1$. This holds for all $i \in \{1, 2, 3\}$, so $|\mathbf{S}| = |\mathbf{S}^{(1)}| + |\mathbf{S}^{(2)}| + |\mathbf{S}^{(3)}| + 1 \leq (|S^{(1)}| - 1) + (|S^{(2)}| - 1) + (|S^{(3)}| - 1) + 1 = n - 2$. □

We can now prove the main theorem of this section.

**Theorem 4.16** (Uniqueness of the infinite cluster)**.** *If $p > p_c$, then there is almost surely exactly one infinite cluster. In other words, if $N(\omega)$ is the number of infinite clusters in a configuration $\omega$, then $\mathbf{P}_p(N(\omega) = 1 : \omega \in \Omega) = 1$.*

*Proof.* We will use the notation $\omega^F$ and $\omega_F$ from 3.3.

Let $S(n)$ be the diamond of radius $n$ around the origin in $\mathbb{L}^d$, so $V(S(n)) = \left\{v \in \mathbb{L}^d \,\middle|\, \|v\|_1 \leq n\right\}$ and $E(S(n))$ is the set all edges between adjacent elements of $V(S(n))$.

For any $k \in \mathbf{N}$, the event $E_k = \{N(\omega) = k\}$ is translation-invariant, so by the zero-one law (Theorem 3.28), we have $\mathbf{P}_p(E_k) \in \{0, 1\}$. Thus, there is a $k \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$ so that $\mathbf{P}_p(E_k) = 1$, and our goal is to show that $k = 1$. By the assumption that $p > p_c$, we know that $k \geq 1$.

Suppose first that $k \in \mathbf{Z}$. For any natural number $n$, both of the two events $O_n = \left\{\omega \in \Omega \,\middle|\, \omega = \omega^{S(n)}\right\}$ and $C_n = \left\{\omega \in \Omega \,\middle|\, \omega = \omega_{S(n)}\right\}$ are cylinders and thus have nonzero measure under $\mathbf{P}_p$. Let $N_{S(n)}(\omega)$ be the number of distinct infinite open clusters in $\omega$ that intersect $S(n)$. Note that closing a *finite* number of edges can only increase the number of infinite clusters intersecting $S(n)$, so $N_{S(n)}(\omega) \leq N_{S(n)}(\omega_{S(n)})$ for all $n \in \mathbf{N}, \omega \in \Omega$.

Suppose that $\mathbf{P}_p(N(\omega) \neq k \,|\, \omega \in O_n) > 0$. Then

$$\begin{aligned}\mathbf{P}_p(N(\omega) \neq k) &\geq \mathbf{P}_p(N(\omega) \neq k \text{ and } \omega \in O_n) \\ &= \mathbf{P}_p(N(\omega) \neq k \,|\, \omega \in O_n)\mathbf{P}_p(\omega \in O_n) > 0,\end{aligned}$$

contradicting the definition of $k$. Therefore (using an analogous argument for $C_n$), $\mathbf{P}_p(N(\omega) = k \,|\, \omega \in O_n) = \mathbf{P}_p(N(\omega) = k \,|\, \omega \in C_n) = 1$, which implies that

$$\mathbf{P}_p(N(\omega^{S(n)}) = k) = \mathbf{P}_p(N(\omega_{S(n)}) = k) = 1.$$

The intersection of two almost-sure events is also almost-sure, so we have

$$\mathbf{P}_p(N(\omega^{S(n)}) = k = N(\omega_{S(n)})) = 1.$$

But if $\omega$ is such that $N(\omega^{S(n)}) = N(\omega_{S(n)})$, then $\omega$ can have no more than one open cluster intersecting $S(n)$, since otherwise opening all of the vertices of $\omega_{S(n)}$ (to form $\omega^{S(n)}$) would decrease the total number of open clusters. Therefore, $\mathbf{P}_p(N_{S(n)}(\omega) \geq 2) = 0$.

If $\omega \in \Omega$ is such that $N(\omega) \geq 2$, then there is an $n \in \mathbf{N}$ so that $N_{S(n)}(\omega) \geq 2$, so $\{\omega \in \Omega \,|\, N(\omega) \geq 2\} = \bigcup_{n\in\mathbf{N}} \left\{\omega \in \Omega \,\middle|\, N_{S(n)}(\omega) \geq 2\right\}$. Therefore, by continuity of measures, $\mathbf{P}_p(N(\omega) \geq 2) = \lim_{n\to\infty} \mathbf{P}_p(N_{S(n)}(\omega) \geq 2) = 0$, which implies that $k = 1$ if $k \in \mathbf{Z}$.

We now need only to dismiss the case $k = \infty$. Given a configuration $\omega \in \Omega$, we say that $x \in \mathbb{L}^d$ is a *trifurcation* of $\omega$, and write $T_x(\omega)$, if the following three conditions hold:

1. $x$ is a member of an infinite cluster of $\omega$.
2. $x$ is an endpoint of exactly three open edges $e_1 = \langle x, y_1\rangle, e_2 = \langle x, y_2\rangle, e_3 = \langle x, y_3\rangle$.
3. In the configuration $\omega_{\{e_1,e_2,e_3\}}$ created by declaring the edges surrounding $x$ to be closed, we have that $y_1, y_2, y_3$ are members of three distinct infinite clusters.

By translation-invariance of $\mathbf{P}_p$, it is clear that $\mathbf{P}_p(T_x) = \mathbf{P}_p(T_0)$ for all $x \in \mathbb{L}^d$. Therefore,

$$\mathbf{E}_p\left[\sum_{x\in S(n)} \mathbf{1}_{T_x}\right] = \sum_{x\in S(n)} \mathbf{P}_p(T_x) = |S(n)|\mathbf{P}_p(T_x).$$

We wish to show that, if $k = \infty$, the expectation $\mathbf{E}_p\left[\sum_{x\in S(n)} \mathbf{1}_{T_x}\right]$ grows in proportion to $|S(n)|$ as $n \to \infty$, which means we must prove that $\mathbf{P}_p(T_0) \neq 0$. This conclusion will lead to a contradiction, since we will show that $\mathbf{E}_p\left[\sum_{x\in S(n)} \mathbf{1}_{T_x}\right]$ in fact cannot grow faster than $|\partial S(n)|$, which grows more slowly than $|S(n)|$.

We assume that $k = \infty$, so $\mathbf{P}_p(N(\omega) = \infty) = 1$, which certainly implies that $\mathbf{P}_p(N(\omega) \geq 3) = 1$. Given a configuration $\omega$ such that $N(\omega) \geq 3$, there is clearly an $n \in \mathbf{N}$ so that $N_{S(n)}(\omega) \geq 3$, which implies that $N_{S(n)}(\omega_{S(n)}) \geq 3$. Therefore, $\{\omega \,|\, N(\omega) \geq 3\} \subseteq \bigcup_{n\in\mathbf{N}} \left\{\omega \,\middle|\, N_{S(n)}(\omega_{S(n)}) \geq 3\right\}$, so

$$1 = \mathbf{P}_p(N(\omega) \geq 3) \leq \lim_{n\to\infty} \mathbf{P}_p(N_{S(n)}(\omega_{S(n)}) \geq 3) \leq 1,$$

so $\lim_{n\to\infty} \mathbf{P}_p(N_{S(n)}(\omega_{S(n)}) \geq 3) = 1$, so there is an $m \in \mathbf{N}$ such that

$$\mathbf{P}_p(N_{S(m)}(\omega_{S(m)}) \geq 3) \geq 1/2,$$

with $1/2$ chosen as an arbitrary constant in $(0, 1)$.

Suppose $\omega$ is a configuration such that $N_{S(m)}(\omega_{S(m)}) \geq 3$. Then there are three points $x(\omega), y(\omega), z(\omega) \in \partial S(m)$ so that $x(\omega), y(\omega), z(\omega)$ are in different infinite clusters of $\omega$. It can be shown geometrically that there is a set of edges $E_\omega \subseteq E(S(m))$ connecting $x(\omega), y(\omega), z(\omega)$ through disjoint paths to the origin so that 0 is a trifurcation of the configuration $\omega^{E_\omega}_{E(S(m))\setminus E_\omega}$ (which is given by taking all edges of $E_\omega$ to be open and all other edges of $S(m)$ to be closed). Now

$$\begin{aligned}\mathbf{P}_p(T_0(\omega)) &\geq \mathbf{P}_p\left(N_{S(m)}(\omega_{S(m)}) \geq 3 \text{ and } \omega = \omega^{E_\omega}_{E(S(m))\setminus E_\omega}\right)\\ &= \mathbf{P}_p\left(N_{S(m)}(\omega_{S(m)}) \geq 3\right) \mathbf{P}_p\left(\omega = \omega^{E_\omega}_{E(S(m))\setminus E_\omega} \,\middle|\, N_{S(m)}(\omega_{S(m)}) \geq 3\right)\\ &\geq \frac{1}{2}\mathbf{P}_p\left(\omega = \omega^{E_\omega}_{E(S(m))\setminus E_\omega} \,\middle|\, N_{S(m)}(\omega_{S(m)}) \geq 3\right)\\ &> 0,\end{aligned}$$

since $\left\{\omega \,\middle|\, \omega = \omega^{E_\omega}_{E(S(m))\setminus E_\omega}\right\}$ is a cylinder event.

Thus, we can write $\mathbf{E}_p\left[\sum_{x\in S(n)} \mathbf{1}_{T_x}\right] = K|S(n)|$, where $K = \mathbf{P}_p(T_x) > 0$. However, we will use Lemma 4.15 to show that this is absurd, because for any $\omega \in \Omega$, we must have $\sum_{x\in S(n)} \mathbf{1}_{T_x}(\omega) < |\partial S(n)|$.

Let $\omega \in \Omega$ and $n \in \mathbf{N}$. Fix an infinite cluster $C$ that intersects $S(n)$. Let $U_C = \partial S(n) \cap C$. If $x \in C \cap S(n)$ is a trifurcation of $\omega$, then closing the edges around $x$ separates $C$ into three infinite clusters $C_x^{(1)}, C_x^{(2)}, C_x^{(3)}$ and thus gives rise to a partition $P_x = \left\{P_x^{(i)} = C_x^{(i)} \cap U_C\right\}_{i=1,2,3}$ of $U_C$. Given two trifurcations $x, x' \in C\cap S(n)$ of $\omega$, ordering $P_x$ and $P_{x'}$ so that $P_x^{(1)} \ni x'$ and $P_{x'}^{(1)} \ni x$ ensures that $P_x^{(1)} \supseteq P_{x'}^{(2)} \cup P_{x'}^{(3)}$. Thus, in the language of Definition 4.14, the class of partitions $\mathbf{P} = \{P_x \,|\, x \in C \cap S(n) \text{ and } x \text{ is a trifurcation of } \omega\}$ is compatible, so $|\mathbf{P}| < |U_C| - 2$ by Lemma 4.15. Moreover, by the definition of trifurcation, the map that takes $x$ to $P_x$ is a bijection, so $\sum_{x\in S(n)\cap C} \mathbf{1}_{T_x}(\omega) \leq |U_C| - 2 < |U|$. Summing over all clusters $C$, we have that $\sum_{x\in S(n)} \mathbf{1}_{T_x}(\omega) < |\partial S(n)|$ since the $U_C$s are disjoint. But this means that for large $n$, we cannot possibly have $\mathbf{E}_p\left[\sum_{x\in S(n)} \mathbf{1}_{T_x}\right] = K|S(n)|$, since $|S(n)|$ grows in proportion to $n^d$ while $|\partial S(n)|$ grows in proportion to $n^{d-1}$. This dismisses the case $k = \infty$, so we have proved that $k = 1$. □

4.4. **The critical value in two dimensions.** In general, it is very difficult to compute exactly the critical probability $p_c$ for a given lattice. For the two-dimensional square lattice $\mathbb{L}^2$, however, we can use planar duality and the results of the previous two sections to prove that $p_c = 1/2$. We use separate arguments to show that $p_c \geq 1/2$ and that $p_c \leq 1/2$.

**Theorem 4.17.** *On $\mathbb{L}^2$, we have $p_c \geq 1/2$.*

*Proof.* Let $p = 1/2$. Suppose that $p_c < 1/2$, so $\theta(1/2) > 0$. For each $n \in \mathbf{N}$, define the event $A^n = \{\partial\Lambda(n) \leftrightarrow \infty\}$, and note that the $A^n$s form an increasing sequence. Since $p > p_c$, there is almost surely an infinite cluster in any configuration, and the box $\Lambda(n)$ will intersect this infinite cluster if $n$ is sufficiently large, so $\bigcup_n A^n = \Omega$. Therefore, $\lim_{n\to\infty} \mathbf{P}_{1/2}(A^n) = 1$, so there is an $N \in \mathbf{N}$ so that

$$\mathbf{P}_{1/2}(\partial\Lambda(n) \leftrightarrow \infty) = \mathbf{P}_{1/2}(A^n) \geq 1 - (1/8)^4 \tag{4.18}$$

for all $n \geq N$. Fix $n = N + 1$. Let $A^N$, $A^S$, $A^E$, and $A^W$ be the events that the north, south, east, and west sides of $\Lambda(n)$, respectively, are members of infinite open clusters. Note that these events have equal probability by symmetry. We have that

$$\mathbf{P}_{1/2}(\Lambda(n) \not\leftrightarrow \infty) = \mathbf{P}_{1/2}((A^N)^c \cap (A^S)^c \cap (A^E)^c \cap (A^W)^c) \geq [\mathbf{P}_{1/2}((A^N)^c)]^4$$

by Corollary 3.11 (since $(A^N)^c, (A^S)^c, (A^E)^c, (A^W)^c$ are decreasing events), so

$$\mathbf{P}_{1/2}((A^N)^c) \leq [1 - \mathbf{P}_{1/2}(\Lambda(n) \leftrightarrow \infty)]^{1/4} \leq 1/8$$

by (4.18).

Consider next the dual box $\Lambda(n)_\mathrm{d}$ in $\mathbb{L}^2_\mathrm{d}$ with vertex set $[-n, n-1]^2 + (1/2, 1/2)$. Let $A^N_\mathrm{d}, A^S_\mathrm{d}, A^E_\mathrm{d}, A^W_\mathrm{d}$ be the events that the north, south, east, and west sides of $\Lambda(n)_\mathrm{d}$ are members of infinite *closed* clusters in $\mathbb{L}^2_\mathrm{d}$. The dual box $\Lambda(n)_\mathrm{d}$ has side length $n - 1 = N$, so (4.18) applies to $\Lambda(n)_\mathrm{d}$ as well, and thus $\mathbf{P}_{1/2}((A^N_\mathrm{d})^c) \leq 1/8$.

Let $A = A^N \cap A^S \cap A^E_\mathrm{d} \cap A^W_\mathrm{d}$. If $A$ occurs, then by Theorem 4.16, there must be an open path in $\mathbb{L}^2$ joining the north and south sides of $A$ and a closed path in $\mathbb{L}^2_\mathrm{d}$ joining the east and west sides of $A$. But this is impossible since it would require the paths to cross. Therefore, $\mathbf{P}_{1/2}(A) = 0$. On the other hand, we have

$$\begin{aligned}
\mathbf{P}_{1/2}(A^c) &= \mathbf{P}_{1/2}((A^N)^c \cup (A^S)^c \cup (A^E_\mathrm{d})^c \cup (A^W_\mathrm{d})^c) \\
&\leq \mathbf{P}_{1/2}((A^N)^c) + \mathbf{P}_{1/2}((A^S)^c) + \mathbf{P}_{1/2}((A^E_\mathrm{d})^c) + \mathbf{P}_{1/2}((A^W_\mathrm{d})^c \\
&\leq 1/2,
\end{aligned}$$

so $\mathbf{P}_{1/2}(A) \geq 1/2$, contradicting the fact proven above that $\mathbf{P}_{1/2}(A) = 0$. Therefore, $p_c \geq 1/2$, as desired. □

**Theorem 4.19.** *On $\mathbb{L}^2$, we have $p_c \leq 1/2$.*

*Proof.* Let $p = 1/2$. Suppose for the sake of contradiction that $p_c > 1/2$. By Theorem 4.13, there is a $\kappa(1/2) > 0$ so that $\mathbf{P}_{1/2}(0 \leftrightarrow \partial\Lambda(n)) \leq \exp(-n\kappa(1/2))$ for all $n \in \mathbf{N}$.

Let $B(n)$ be the modified box given by removing the edges on the east and west sides of the box given by $[0, n+1] \times [0, n]$. Let $A_n$ be the event that there is an open path from the west side of $B(n)$ to the east side of $B(n)$. We consider the dual box $B(n)_\mathrm{d}$ in $\mathbb{L}^2_\mathrm{d}$ generated by $[0, n] \times [0, n+1] + (1/2, -1/2)$, with the edges on the north and south sides removed. Let $A'_n$ be the event that there is a closed path from the north side of $B(n)_\mathrm{d}$ to the south side of $B(n)_\mathrm{d}$. It is clear that $\mathbf{P}_{1/2}(A_n) = \mathbf{P}_{1/2}(A'_n)$ and that exactly one of $A_n$ and $A'_n$ must occur (see Remark 3.36), so

$$\mathbf{P}_{1/2}(A_n) = 1/2. \tag{4.20}$$

However, we also have

$$A_n \subseteq \bigcup_{i=1}^{n+1} \{(0, i) \leftrightarrow i + \Lambda(n)\},$$

which implies

$$\begin{aligned}
\mathbf{P}_{1/2}(A_n) &\leq \mathbf{P}_{1/2}\left(\bigcup_{i=1}^{n+1} \{(0, i) \leftrightarrow i + \Lambda(n)\}\right) \\
&\leq \sum_{i=1}^{n+1} \mathbf{P}_p((0, i) \leftrightarrow i + \Lambda(n)) \\
&= (n+1)\mathbf{P}_p(0 \leftrightarrow \Lambda(n)) \\
&\leq (n+1)\exp(-\kappa(1/2)n),
\end{aligned}$$

which will eventually fall below $1/2$ as $n$ grows large, contradicting (4.20). □

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