FUNDAMENTAL FORMS OF SURFACES AND THE GAUSS-BONNET THEOREM

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Abstract. We define the first and second fundamental forms of surfaces, exploring their properties as they relate to measuring arc lengths and areas, identifying isometric surfaces, and finding extrema. These forms can be used to define the Gaussian curvature, which is, unlike the first and second fundamental forms, independent of the parametrization of the surface. Gauss's Egregious Theorem reveals more about the Gaussian curvature, that it depends only on the first fundamental form and thus identifies isometries between surfaces. Finally, we present the Gauss-Bonnet Theorem, a remarkable theorem that, in its global form, relates the Gaussian curvature of the surface, a geometric property, to the Euler characteristic of the surface, a topological property.

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1. Surfaces and the First Fundamental Form

We begin our study by examining two properties of surfaces in $\mathbb{R}^3$, called the first and second fundamental forms. First, however, we must understand what we are dealing with when we talk about a surface.

Definition 1.1. A **smooth surface in** $\mathbb{R}^3$ is a subset $X \subset \mathbb{R}^3$ such that each point has a neighborhood $U \subset X$ and a map $\mathbf{r} : V \to \mathbb{R}^3$ from an open set $V \subset \mathbb{R}^2$ such that

- $\mathbf{r} : V \to U$ is a homeomorphism. This means that $\mathbf{r}$ is a bijection that continously maps V into U, and that the inverse function $\mathbf{r}^{-1}$ exists and is continuous.
- $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$ has derivatives of all orders.
- At all points, the first partial derivatives $\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}$ and $\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}$ are linearly independent.

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It is often more intuitive and useful to think of surfaces as either graphs of two-variable functions, or as geometric consequences of the implicit function theorem. Locally, any surface can be expressed in such a form. The following examples demonstrate this.

Examples:

- The sphere:
$$\mathbf{r}(u,v) = a\sin u \sin v\mathbf{i} + a\cos u\sin v\mathbf{j} + a\cos v\mathbf{k}$$
Note that this surface is given by the implicit function $x^2+y^2+z^2=1$.

- The cone:
$$\mathbf{r}(u,v) = au\cos v\mathbf{i} + au\sin v\mathbf{j} + u\mathbf{k}$$
This can also be viewed as the graph of a function over x and y, where $f(x,y) = \frac{1}{a}\sqrt{x^2+y^2}$.

From here we move on to one of many properties of a surface, the first fundamental form.

Definition 1.2. The **first fundamental form** of a surface in $\mathbb{R}^3$ is the expression $\mathbf{v}\cdot\mathbf{v}$, where $\mathbf{v} = \mathbf{r}_u du + \mathbf{r}_v dv$.

This is the inner product of what one might think of as a general velocity vector with itself. If one constructed a parametrized curve, then it can be used to calculate arc length, as integrating the norm of the velocity (i.e., the square root of the inner product of the velocity vector with itself) gives us arc length.

It is more useful to compute the inner product and write the first fundamental form out as
$$Edu^2 + 2Fdudv + Gdv^2$$
where $E = \mathbf{r}_u\cdot\mathbf{r}_u, F = \mathbf{r}_u\cdot\mathbf{r}_v, G = \mathbf{r}_v\cdot\mathbf{r}_v$.

From this expression, we can see very clearly that the first fundamental form is a quadratic form. It is also a bilinear form, though we only examine it as a quadratic form here. Furthermore, its arguments come from the tangent space, so it is an inner product on the tangent space. This will be of importance later.

Examples:

- The sphere. We have
$$\mathbf{r}(u,v) = a\sin u \sin v\mathbf{i} + a\cos u\sin v\mathbf{j} + a\cos v\mathbf{k}$$
Then
$$\mathbf{r}_u = a\cos u\sin v\mathbf{i} - a\sin u\sin v\mathbf{j},\ \mathbf{r}_v = a\sin u\cos v\mathbf{i} + a\cos u\cos v\mathbf{j} - a\sin v\mathbf{j}$$
from which we obtain
$$E = \mathbf{r}_u\cdot\mathbf{r}_u = a^2\sin^2 v,\ F = \mathbf{r}_u\cdot\mathbf{r}_v = 0,\ G = \mathbf{r}_v\cdot\mathbf{r}_v = a^2$$
The first fundamental form is thus
$$a^2\sin^2 v du^2 + a^2 dv^2$$

- The cone. Here, we will take the parametrization of the cone in terms of x and y, where
$$r(x,y) = x\mathbf{i} + y\mathbf{j} + \frac{\sqrt{x^2+y^2}}{a}\mathbf{k}$$

then

$$\mathbf{r}_x = \mathbf{i} + \frac{x}{a\sqrt{x^2+y^2}}\mathbf{k},\ \mathbf{r}_y = \mathbf{j} + \frac{y}{a\sqrt{x^2+y^2}}\mathbf{k}$$

and

$$E = \mathbf{r}_x \cdot \mathbf{r}_x = 1 + \frac{x^2}{a^2(x^2+y^2)},\ F = \mathbf{r}_x \cdot \mathbf{r}_y = \frac{xy}{a^2(x^2+y^2)},$$
$$G = \mathbf{r}_y \cdot \mathbf{r}_y = 1 + \frac{x^2}{a^2(x^2+y^2)}$$

so the first fundamental form is

$$\left(1 + \frac{x^2}{a^2(x^2+y^2)}\right)dx^2 + \frac{xy}{a^2(x^2+y^2)}dxdy + \left(1 + \frac{y^2}{a^2(x^2+y^2)}\right)dy^2$$

We can see from the examples that the first fundamental form (as well as the second fundamental form that follows) is an expression that can be evaluated at a point on the surface, and we do so when integrating it, as we shall do frequently here.

We mentioned earlier that the first fundamental form has to do with how arc length is measured. We will make that a little more explicit now.

Suppose we have a smooth surface expressed as $\mathbf{r}(u,v)$. Suppose that we take a smooth curve (that is, a curve with derivatives of all orders) parametrized as $\mathbf{r}(t) = \mathbf{r}(u(t),v(t))$ along this surface, with t ranging from a to b. Then the arc length is

$$\begin{aligned}\int_a^b |\mathbf{r}'(t)|dt &= \int_a^b \sqrt{\mathbf{r}'(t)\cdot\mathbf{r}'(t)}dt \\ &= \int_a^b \sqrt{(\mathbf{r}_u u'(t) + \mathbf{r}_v v'(t))\cdot(\mathbf{r}_u u'(t) + \mathbf{r}_v v'(t))}dt \\ &= \int_a^b \sqrt{(\mathbf{r}_u\cdot\mathbf{r}_u)(u'(t))^2 + 2(\mathbf{r}_u\cdot\mathbf{r}_v)u'(t)v'(t) + (\mathbf{r}_v\cdot\mathbf{r}_v)(v'(t))^2}dt \\ &= \int_a^b \sqrt{E(u'(t))^2 + 2Fu'(t)v'(t) + G(v'(t))^2}dt\end{aligned}$$

where E, F, and G are the values they take for the first fundamental form. In this manner, we can see that the first fundamental form encodes some very important information about the length of curves across a surface.

Since the first fundamental form is used to measure arc length, it should not surprise us that it is tied closely to isometries between surfaces. Isometries are simply homeomorphisms from one surface X to another surface X' that map curves in X to curves in X' of the same length.

Theorem 1.3. *Coordinate patches of surfaces U and U' are isometric if and only if there exist parametrizations $\mathbf{r}: V \to \mathbb{R}^3$ and $\mathbf{r}': V \to \mathbb{R}^3$ of U and U', respectively, with the same first fundamental form.*

Additionally, the first fundamental form encodes information about area on the surface and angles between curves on the surface. We briefly show how the first fundamental form encodes information about area on a surface, as this will be of use to us later.

Definition 1.4. The area of an open connected set $\mathbf{r}(U) \subset \mathbb{R}^3$ in a surface is

$$\iint_U |\mathbf{r}_u \times \mathbf{r}_v| du dv$$

Theorem 1.5. *The area of an open connected set in a surface is*

$$\iint_U \sqrt{EG - F^2} du dv$$

This comes from a vector product identity:

$$|\mathbf{r}_u \times \mathbf{r}_v|^2 = (\mathbf{r}_u \cdot \mathbf{r}_u)(\mathbf{r}_v \cdot \mathbf{r}_v) - (\mathbf{r}_u \cdot \mathbf{r}_v)^2 = EG - F^2$$

Note that E, F, and G depend on the parametrization of the surface. The cone, for example, has different values of E, F, and G if it is parametrized in terms of u and v, rather than in terms of x and y. However, the du^2, $dudv$, and dv^2 terms account for the choice of parametrization, making it an invariant for the surface that does not depend on the parametrization, though its expression differs according to the parametrization. If the parametrization is changed, then changing first fundamental forms is easily accomplished with a change of variable.

When changing variables, we can use the total derivative and a clever bit of matrix multiplication to avoid starting from scratch. If we want to move from x and y to u and v, we can take the total derivatives

$$dx = x_u du + x_v dv,\ dy = y_u du + y_v dv$$

and substitute, then combine terms to get a new E', F' and G' in terms of u and v. We can write this transformation in terms of matrix multiplication.

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} x_u & y_u \\ x_v & y_v \end{pmatrix} = \begin{pmatrix} E' & F' \\ F' & G' \end{pmatrix} \tag{1.6}$$

Note that we can write this transformation out this way only because the first fundamental form is a quadratic form.

As an example, take the cone. We showed earlier that, written in terms of rectangular coordinates, the first fundamental form is

$$\left(1 + \frac{x^2}{a^2(x^2 + y^2)}\right) dx^2 + \frac{xy}{a^2(x^2 + y^2)} dxdy + \left(1 + \frac{y^2}{a^2(x^2 + y^2)}\right) dy^2$$

We can also find the first fundamental form written in terms of polar coordinates. Here, we have $x = au \cos v$ and $y = au \sin v$. This gives us the total derivatives

$$dx = a \cos v du - au \sin v dv,\ dy = a \sin v du + au \cos v dv$$

If we substitute those expressions in for dx and dy, as well as substitute in for x and y and collect all of our terms together, we get, after some messy computations

$$(1 + a^2)du^2 + a^2u^2dv^2$$

We note that this version of the first fundamental form of the cone looks a lot nicer than it did before! It shouldn't surprise us, given that polar coordinates give a much nicer parametrization of the cone. Nevertheless, they are equivalent, and we can use whichever is more convenient for our purposes.

2. The Second Fundamental Form

Now that we've explored the first fundamental form at some length, we move to the second fundamental form. Suppose we have a surface $\mathbf{r}(u, v)$, and we push or pull the surface along its normal vector. Then we get a family of surfaces, $\mathbf{R}(u, v, t) = \mathbf{r}(u, v) - t\mathbf{n}(u, v)$. While the first fundamental form encodes something about arc length on a surface, the second fundamental form encodes how the arc length changes as the surface moves along its normal vector - that is, how the first fundamental form of the surface changes as t changes.

Definition 2.1. Let

$$E(t)du^2 + 2F(t)dudv + G(t)dv^2$$

be the first fundamental form of the family of surfaces $\mathbf{R}(u, v, t) = \mathbf{r}(u, v) - t\mathbf{n}(u, v)$, where

$$E(t) = \mathbf{R}_u \cdot \mathbf{R}_u,\ F(t) = \mathbf{R}_u \cdot \mathbf{R}_v,\ G(t) = \mathbf{R}_v \cdot \mathbf{R}_v$$

for

$$\mathbf{R}_u = \mathbf{r}_u - t\mathbf{n}_u,\ \mathbf{R}_v = \mathbf{r}_v - t\mathbf{n}_v$$

Then the **second fundamental form** of $\mathbf{r}(u, v) = \mathbf{R}(u, v, 0)$ is

$$\frac{1}{2}\frac{\partial}{\partial t}(E(t)du^2 + 2F(t)dudv + G(t)dv^2)|_{t=0}$$

We can compute this expression out to make more sense of it. For example,

$$\begin{aligned} E(t) &= \mathbf{R}_u \cdot \mathbf{R}_u \\ &= (\mathbf{r}_u - t\mathbf{n}_u) \cdot (\mathbf{r}_u - t\mathbf{n}_u) \\ &= \mathbf{r}_u \cdot \mathbf{r}_u - 2t\mathbf{r}_u \cdot \mathbf{n}_u + t^2\mathbf{n}_u \cdot \mathbf{n}_u \end{aligned}$$

and so

$$\frac{\partial E}{\partial t}|_{t=0} = -2\mathbf{r}_u \cdot \mathbf{n}_u$$

Similarly,

$$\frac{\partial F}{\partial t}|_{t=0} = -2(\mathbf{r}_u \cdot \mathbf{n}_v + \mathbf{r}_v \cdot \mathbf{n}_u),\ \frac{\partial G}{\partial t}|_{t=0} = -2\mathbf{r}_v \cdot \mathbf{n}_v$$

We can make these expressions a little nicer. Note that both $\mathbf{r}_u$ and $\mathbf{r}_v$ are always orthogonal to $\mathbf{n}$. Thus,

$$\begin{aligned} 0 = (\mathbf{r}_u \cdot \mathbf{n})_u = \mathbf{r}_{uu} \cdot \mathbf{n} + \mathbf{r}_u \cdot \mathbf{n}_u &\Rightarrow \frac{\partial E}{\partial t}|_{t=0} = -2\mathbf{r}_u \cdot \mathbf{n}_u = 2\mathbf{r}_{uu} \cdot \mathbf{n} \\ 0 = (\mathbf{r}_v \cdot \mathbf{n})_v = \mathbf{r}_{vv} \cdot \mathbf{n} + \mathbf{r}_v \cdot \mathbf{n}_v &\Rightarrow \frac{\partial G}{\partial t}|_{t=0} = -2\mathbf{r}_v \cdot \mathbf{n}_v = 2\mathbf{r}_{vv} \cdot \mathbf{n} \\ 0 = (\mathbf{r}_u \cdot \mathbf{n})_v = \mathbf{r}_{uv} \cdot \mathbf{n} + \mathbf{r}_u \cdot \mathbf{n}_v &\Rightarrow \frac{\partial F}{\partial t}|_{t=0} = -2(\mathbf{r}_u \cdot \mathbf{n}_v + \mathbf{r}_v \cdot \mathbf{n}_u) = 4(\mathbf{r}_{uv} \cdot \mathbf{n}) \end{aligned}$$

The last equality holds because $\mathbf{r}_{uv} = \mathbf{r}_{vu}$. Plugging all of this in to the expression above, we get that the second fundamental form is

$$Ldu^2 + 2Mdudv + Ndv^2$$

where $L = \mathbf{r}_{uu} \cdot \mathbf{n},\ M = \mathbf{r}_{uv} \cdot \mathbf{n},\ N = \mathbf{r}_{vv} \cdot \mathbf{n}$.

Note that since the second fundamental form is calculated by taking the derivative at $t = 0$, it is the second fundamental form only for the surface $\mathbf{R}(u, v, 0) = \mathbf{r}(u, v)$. By taking the derivative of the first fundamental form at different points we can find the second fundamental form of different surfaces in the same family, but it is not necessarily the same expression. The second fundamental form measures in some way how arc length changes as the surface moves along the normal (that is, the derivative of the first fundamental form with respect to t, the coefficient of $\mathbf{n}$ in $\mathbf{R}(u, v, t)$) at a particular time $t = 0$, so it refers only to the specific surface $\mathbf{R}(u, v, 0) = \mathbf{r}(u, v)$, not a family of surfaces.

Again, we can see that the second fundamental form is also a quadratic form. In particular, it is also a quadratic form on the tangent space, though this is less obvious. For instance, $L = \mathbf{r}_{uu} \cdot \mathbf{n} = -\mathbf{r}_u \cdot \mathbf{n}_u$. $\mathbf{r}_u$ is clearly in the tangent space. Since $|\mathbf{n}|$ is constant, $\mathbf{n}_u$ is orthogonal to $\mathbf{n}$, and is thus in the tangent space, since the tangent space is precisely the space orthogonal to $\mathbf{n}$. So L takes arguments from the tangent space, and similar arguments hold for M and N.

Since, like the first fundamental form, the second fundamental form is a quadratic form, it undergoes the same transformation for a change of variables. In particular, if we move from L, M, and N in terms of x and y to L', M', and N' in terms of u and v, the transformation is

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} x_u & y_u \\ x_v & y_v \end{pmatrix} = \begin{pmatrix} L' & M' \\ M' & N' \end{pmatrix} \tag{2.2}$$

Once again, this will be important later.

Example: the sphere

We are dealing with a sphere of radius a centered at the origin, so $\mathbf{n} = \frac{1}{a}\mathbf{r}$, and accordingly $\mathbf{n}_u = \frac{1}{a}\mathbf{r}_u$ and $\mathbf{n}_v = \frac{1}{a}\mathbf{r}_v$. So $L = \mathbf{r}_u \cdot \mathbf{n}_u = \frac{1}{a}\mathbf{r}_u \cdot \mathbf{r}_u = \frac{1}{a}E$, where E is from the first fundamental form of the sphere. Similarly, $M = \frac{1}{a}F$ and $N = \frac{1}{a}G$, so the second fundamental form of the sphere is $\frac{1}{a}$ times the first fundamental form.

Theorem 2.3. *If the fundamental form vanishes at some point, then the surface is planar at that point.*

Proof. If the second fundamental theorem vanishes, then

$$0 = \mathbf{r}_u \cdot \mathbf{n}_u = \mathbf{r}_u \cdot \mathbf{n}_v = \mathbf{r}_v \cdot \mathbf{n}_u = \mathbf{r}_v \cdot \mathbf{n}_v$$

But since $\mathbf{r}_u$ and $\mathbf{r}_v$ are linearly independent and thus span the tangent plane, this means that we must have that $\mathbf{n}_u$ and $\mathbf{n}_v$ are orthogonal to the tangent plane. But we established earlier that $\mathbf{n}_u$ and $\mathbf{n}_v$ lie in the tangent plane. Therefore, $\mathbf{n}_u = \mathbf{n}_v = 0$, and so $\mathbf{n}$ is constant. But then $(\mathbf{r} \cdot \mathbf{n})_u = \mathbf{r}_u \cdot \mathbf{n} + \mathbf{r} \cdot 0 = 0 + 0 = 0$, and similarly $(\mathbf{r} \cdot \mathbf{n})_v = 0$. So $\mathbf{r} \cdot \mathbf{n}$ is constant, which gives us the plane. □

It follows immediately that the second fundamental form vanishes everywhere on the plane.

Suppose we look at the second fundamental form of a surface $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}$, the graph of a function $z = f(x, y)$. Suppose we look at a critical point, where $\mathbf{n} = \mathbf{k}$. Now $\mathbf{r}_{xx} = f_{xx}\mathbf{k}$, $\mathbf{r}_{xy} = f_{xy}\mathbf{k}$, and $\mathbf{r}_{yy} = f_{yy}\mathbf{k}$, which means that the second fundamental form gives us the Hessian:

$$\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$$

Applying the second partial derivative test from multivariable calculus, we know that the determinant of this matrix, that is, $LN - M^2 = f_{xx}f_{yy} - f_{xy}^2$ is positive if we have a local extremum (a maximum or minimum according to the sign of f_{xx} and f_{yy}), or is negative if we have a saddle point. This provides us with some useful intuition about our next property.

3. Curvature

Definition 3.1. The **Gaussian curvature K** of a surface in $\mathbb{R}^3$ is

$$K = \frac{LN - M^2}{EG - F^2}$$

Recall that the first and second fundamental form depend on parametrization of the surface. This is not the case, however, for the Gaussian curvature. Since both the first and second fundamental forms undergo the same transformation to change variables (in equations 1.6 and 2.2), the quotient of the two determinants is unchanged, so K is independent of parametrization.

Examples:

- The sphere: a sphere of radius a has first and second fundamental forms such that $L = \frac{1}{a}E$, $M = \frac{1}{a}F$, and $N = \frac{1}{a}G$, so $K = \frac{EG-F^2}{a^2(EG-F^2)} = \frac{1}{a^2}$
- The plane: the plane has a second fundamental form of 0, so the curvature is also 0. This fits with our intuition that the plane is flat, and not curved at all. However, this intuition is not entirely correct, as we explore further below.
- The parabolic cylinder: the parabolic cylinder given by the graph of the function $z(x, y) = x^2$ has a Gaussian curvature of 0. We will not use the above definition to prove this, instead appealing to alternative (but equivalent) definitions and Gauss's Egregious Theorem.

How is it that the parabolic cylinder, which appears to be curved in some way, has 0 Gaussian curvature? First, we can observe that it is isometric to the xy-plane. Intuitively, we can see this from the fact that we can just "flatten out" the parabolic cylinder to the plane. It can be shown that its first fundamental form is the same as that of the plane. Note, however, that the parametrization that gives us the correct first fundamental form is *not* the parametrization above, since the distance between (1,0,1) on the parabolic cylinder is not length 1 away from the origin, as (1,0,0) is in the xy-plane. From there, we appeal to a clever theorem proved by Gauss.

Theorem 3.2 (Gauss's Egregious Theorem)**.** *The Gaussian curvature depends only on the first fundamental form.*

The outline of the proof is as follows: Suppose that we take a smooth orthonormal basis $\{\mathbf{e^1}, \mathbf{e^2}\}$ of the tangent plane at all points of the surface, so $\mathbf{e^1}$ and $\mathbf{e^2}$ are smooth functions of the surface parameters u and v, and that $\mathbf{e^1} \times \mathbf{e^2} = \mathbf{N}$. It can be shown that

$$\mathbf{e^1}_u \cdot \mathbf{e^2}_v - \mathbf{e^2}_u \cdot \mathbf{e^1}_v = \frac{LN - M^2}{\sqrt{EG - F^2}} \tag{3.3}$$

and therefore

$$K = \frac{\mathbf{e^1}_u \cdot \mathbf{e^2}_v - \mathbf{e^2}_u \cdot \mathbf{e^1}_v}{\sqrt{EG - F^2}}$$

Furthermore, it can be shown that $\mathbf{e^1}_u \cdot \mathbf{e^2}_v - \mathbf{e^2}_u \cdot \mathbf{e^1}_v$ depends only on E, F, and G, thus proving that the Gaussian curvature depends only on the first fundamental form. We can give an explicit formula, stated without proof:

$$K = \frac{\begin{vmatrix} -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix}}{(EG - F^2)^2}$$

What this result amounts to is that surfaces that have the same first fundamental form, or, equivalently, surfaces that are isometric, have the same Gaussian curvature. This is enough to conclude that the parabolic cylinder has 0 Gaussian curvature, as it is isometric to another surface, the plane, with 0 Gaussian curvature.

Alternatively, this can also be observed from a different definition of the Gaussian curvature, which relies on the concept of principal curvatures.

Definition 3.4. The **curvature** of a curve is the signed inverse of the radius of the osculating circle, which is the circle tangent to the curve that also matches the second derivative of the curve. The sign is positive if the circle is on the same side of the surface as the normal vector, or negative if it is on the opposite side of the normal vector.

Definition 3.5. The **principal curvatures** of a point are the extremal values of the curvatures of all curves through that point obtained by intersecting the surface by all planes containing the normal vector.

Theorem 3.6. *The Gaussian curvature at a point is the product of the two principal curvatures at that point.*

Though we state this as a theorem, we can also define the Gaussian curvature to be the product of the two principal curvatures; the definitions are equivalent.

Using this definition of Gaussian curvature, it is easy to see how the parabolic cylinder has 0 Gaussian curvature. At any point, there is a plane containing the normal whose planar curve is a straight line parallel to the y-axis. This curve has curvature 0, as the osculating circle has infinite radius, and this is a minimal value, so the Gaussian curvature is 0.

4. The Gauss-Bonnet Theorem

We will present the Gauss-Bonnet Theorem in three different flavors. The Gauss-Bonnet Theorem is a profound theorem that, in its global form, connects the geometry of surfaces to the topology of surfaces. The first two variations don't quite show the profound nature of the theorem, but they are necessary buildup, and rely on the concept of geodesic curvature.

Definition 4.1. The geodesic curvature κ_g of a smooth curve is

$$\kappa_g = \mathbf{t}' \cdot (\mathbf{n} \times \mathbf{t})$$

This measures the tangential derivative of $\mathbf{t}$, that is, how much $\mathbf{t}$ changes in the tangent plane. Intuitively, the geodesic curvature measures how far a curve is from being a geodesic, the shortest path between two points. A geodesic curvature of 0 indicates that the curve is a geodesic. Accordingly, the vanishing of the geodesic curvature gives us the Euler-Lagrange equation that minimizes arc length.

Examples:

- On the plane, geodesics are straight lines. Straight lines have a constant tangent vector, so the derivative of the tangent vector is the 0 vector, and the geodesic curvature is 0. Conversely, at a point on a curve that is not a straight line, the derivative of the tangent vector points towards the center of the osculating circle, and so the geodesic curvature is non-zero. The greater the absolute value of the geodesic curvature and the smaller the radius of the osculating circle, the further the curve is from being a geodesic (recall that the osculating circle of a straight line is one of infinite radius).
- On a sphere, geodesics fall along great circles. On a great circle, the derivative of the tangent vector points directly towards the center of the sphere, with no component in the tangent plane, and so the geodesic curvature is 0. The tangent vector of a curve that does not follow a great circle will have a component of its derivative in the tangent plane, so it will have a non-zero geodesic curvature.

Theorem 4.2 (Gauss-Bonnet Theorem for Simple Closed Curves)**.** *Let γ be a positively-oriented simple closed curve on a surface enclosing a region S of that surface. Then*

$$\int_\gamma \kappa_g ds = 2\pi - \iint_S K dA$$

The proof is too lengthy to include in full, but we will provide some way of making sense of the integral of the Gaussian curvature. Suppose that $S = \mathbf{r}(R)$. We can rewrite the integral on the right above by substituting in for both Gaussian curvature and the area element (from theorem 1.5), giving us

$$\begin{aligned}\iint_S K dA &= \iint_R \frac{LN - M^2}{EG - F^2}\sqrt{EG - F^2}dudv \\ &= \iint_R \frac{LN - M^2}{\sqrt{EG - F^2}}dudv \\ &= \iint_R (\mathbf{e^1}_u \cdot \mathbf{e^2}_v - \mathbf{e^2}_u \cdot \mathbf{e^1}_v)dudv\end{aligned}$$

where the last equality holds from equation 3.3, given a smooth orthonormal basis $\{\mathbf{e^1}, \mathbf{e^2}\}$. But we can rewrite this as

$$\iint_R ((\mathbf{e^1} \cdot \mathbf{e^2}_v)_u - (\mathbf{e^2}_u \cdot \mathbf{e^1})_v)dudv$$

This puts us in a position where we can appeal to Green's theorem. From there, we have a line integral around the curve, which is somewhat easier to make sense of and gets us closer to the desired result.

If, instead of a simple closed curve, we have a piecewise smooth closed curve, which we will call a curvilinear polygon, the Gauss-Bonnet Theorem still holds, but we must include a correction for the vertices of the polygon where the curve is not smooth. We state it without proof.

Theorem 4.3 (Gauss-Bonnet Theorem for Curvilinear Polygons)**.** *Let γ be a positively oriented curvilinear n-gon on a surface, enclosing a region S of that surface, and let $\alpha_1, \alpha_2, \dots \alpha_n$ be the interior angles at its vertices. Then*

$$\int_\gamma \kappa_g ds = \sum_{i=1}^{n} \alpha_i - (n-2)\pi - \iint_S K dA \tag{4.4}$$

Though we are not done with the Gauss-Bonnet Theorem yet, this is already a profound result, and many basic theorems in geometry follow directly.

Examples:

- Suppose we have an n-gon in the plane, with straight edges. We have established that the Gaussian curvature on the plane is 0, as is the geodesic curvature of straight lines on the plane, so applying equation (4.4) gives us
$$0 = \sum_{i=1}^{n} \alpha_i - (n-2)\pi - 0$$
$$\Rightarrow \sum_{i=1}^{n} \alpha_i = (n-2)\pi$$
So the Gauss-Bonnet Theorem gives us the sum of the interior angles of polygons with straight edges on the plane.
- Suppose we have a geodesic n-gon, an n-gon with geodesic edges, enclosing a region S on the unit sphere. The Gaussian curvature on the unit sphere is 1, and the geodesic curvature of the edges is 0. Thus, equation (4.4) gives us
$$0 = \sum_{i=1}^{n} \alpha_i - (n-2)\pi - \iint_S dA$$
$$\Rightarrow Area = \sum_{i=1}^{n} \alpha_i - (n-2)\pi$$
So the Gauss-Bonnet Theorem gives us a formula for the area of geodesic polygons on the sphere, given the interior angles of the polygon.
- More generally, for a geodesic n-gon enclosing a region S on some surface, we have
$$\iint_S K dA = \sum_{i=1}^{n} \alpha_i - (n-2)\pi$$
So for a geodesic n-gon, the integral of the Gaussian curvature gives the difference of the sum of interior angles compared to that of an n-gon with straight edges in the plane.

The two versions of the Gauss-Bonnet Theorem we have given so far are local, in that they focus on just a section of the surface, namely that enclosed by a curve. But the third version of the Gauss-Bonnet Theorem is global, providing information about the entire surface, provided that it is compact.

It is a fact, which we shall not prove here, that any compact surface S can be triangulated into finitely many triangles, where a triangulation is a collection of curvilinear triangles such that every point of S is in at least one of the curvilinear triangles, any two curvilinear triangles are either disjoint, or their intersection is one common edge or one common vertex, and each edge is an edge of exactly two curvilinear triangles.

Given a compact surface and a triangulation, we can calculate a property called the Euler characteristic.

Definition 4.5. The **Euler Characteristic** χ of a triangulation of a compact surface S is

$$\chi = V - E + F$$

where V is the number of vertices, E is the number of edges, and F is the number of faces of the triangulation.

Theorem 4.6. *Given a surface S, the Euler characteristic is independent of the triangulation. Thus, we say that χ is the Euler characteristic of the surface.*

This theorem is not necessary to prove the Gauss-Bonnet Theorem; in fact, as we will show later, the Gauss-Bonnet Theorem proves it instead!

We are now prepared to state the fully-fledged version of the Gauss-Bonnet Theorem.

Theorem 4.7 (Gauss-Bonnet Theorem for Compact Surfaces)**.** *Given a compact surface S with Euler characteristic χ,*

$$\iint_S K dA = 2\pi\chi \tag{4.8}$$

Proof. Take a triangulation of S. We start from equation (4.4) from the Gauss-Bonnet Theorem for Curvilinear Polygons. We take each local term for a single polygon, and replace it with the corresponding global term, for the entire surface.

First, consider $\int_\gamma \kappa_g ds$. Each edge will be traversed twice, since each edge is an edge of exactly two triangles. Since we take all line integrals in the positive (counterclockwise) direction, the edge will be traversed in opposite directions. Therefore, the sum of this integral over all triangles will be 0.

Now consider $\displaystyle\sum_{i=1}^{n} \alpha_i$, the sum of the interior angles of the polygon. Summing over all triangles gives us the sum of all interior angles. But this is simply equal to the number of vertices V times 2π, the sum of the angles around each vertex.

Now we look at the $(n-2)\pi$ term, where n is the number of edges on the face. Summing over all faces, we get $(2E - 2F)\pi$, since n summed over all the faces is twice the number of edges, as the edges are double-counted, while we simply multiply 2 by the number of faces.

Finally, the sum of all of the integrals $\iint_R K dA$ over faces R of the surface is simply the integral over the entire surface, $\iint_S K dA$.

So, replacing the local terms in equation (4.4) by their global terms gives us

$$0 = 2\pi V - 2\pi E + 2\pi F - \iint_S K dA$$
$$\Rightarrow \iint_S K dA = 2\pi\chi$$

□

This beautiful theorem expresses a profound relationship between the geometric and topological properties of a surface. Given any surface, we can instantly determine the integral of the Gaussian curvature over a compact surface simply by calculating its Euler characteristic. We can deform the surface, pushing or pulling it, and while this may change the Gaussian curvature locally, as long as we do not change the topology of the surface, the integral of the Gaussian curvature over the entire surface remains the same.

We also state a few interesting corollaries.

Corollary 4.9. *Given a surface S, the Euler characteristic is independent of the triangulation. Thus, we say that χ is the Euler characteristic of the surface.*

We stated this earlier without proof as theorem 4.6, but it follows directly from the Gauss-Bonnet Theorem. The left side of equation (4.8) is independent of the triangulation, so the right side (taking χ to be the Euler characteristic of the triangulation, instead of the surface) is also independent of the triangulation, making the Euler characteristic a property of the surface.

Corollary 4.10. *Any surface with Euler characteristic $\chi < 0$ has a point of negative Gaussian curvature (which is saddle shaped, according to theorem 3.6).*

Proof. For a surface with Euler characteristic $\chi < 0$, the right side of equation (4.8) will be negative. Therefore, the left side must be negative, so there must be a point where the Gaussian curvature is negative. □

Finally, it is worth noting that, although we have dealt with surfaces as they sit in $\mathbb{R}^3$, this result holds for any abstract surface, provided that we have a first fundamental form also defined abstractly.

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