EXPLORING GRAPHS OF TRIANGULATED N-GONS

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ABSTRACT. A quite notable graph theory problem, resolved by Sleator-Tarjan-Thurston, serves as the motivation for this research.Our research uses graph theory to understand the geometry of the space for triangulated n-gons with one interior vertex. Our main focus of this project to find out if $R_{n,1}$ is connected and to find an upper bound on its diameter.

CONTENTS

1. GRAPHS: THEORY & DEFINITIONS

Graph theory is a branch of mathematics that focuses on the study of points and lines. In particular, it involves the ways in which sets of points, called vertices, can be connected by lines or arcs, called edges. Graphs in this context differ from the coordinate plots that portray mathematical relations and functions. There are many different types of graphs. Essentially, a graph G is defined as an ordered pair, $G = (N, E)$, where N is a set of vertices, and each edge is a pair of nodes such that $E \subset N \times N$.

Graph theory is quite significant to mathematics and other fields such as economics, psychology, and biology [2]. The vast range of applications for graph theory extends from a simple representation of a soccer tournament, to the design of integrated circuits for electronic devices as well as the study of biochemistry. For instance, in computational biochemistry, there are many situations where a biochemist tries

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to resolve conflicts between sequences in a sample by excluding some of the sequences. Of course, exactly what constitutes a conflict must be precisely defined in the biochemical context. Thus, it is possible to define a conflict graph where the nodes represent the sequences in the sample and there is an edge between two nodes if and only if there is a conflict between the corresponding sequences. The aim is to remove the fewest possible sequences that will eliminate all conflicts [3]. Moreover, the combinatorial methods found in graph theory have also been used to prove significant and well-known results in a variety of areas in mathematics [3].

The study of the graph $R_{n,0}$ resolved a graph theory problem that entails finding the maximum rotation distance between any pair of n-node binary trees. In their findings, Sleator-Tarjan-Thurston explored the connection between the maximum rotation distance of binary trees and the graph $R_{n,0}$. Given the connected graph $R_{n,0}$ such that the nodes of this graph are triangulated n-gons, Sleator-Tarjan-Thurston proved that the diameter of $R_{n,0}$ is $2n - 10$, for $n \geq 13$ [4]. As an extension of the works by Sleator-Tarjan-Thurston, we study the graph $R_{n,1}$. Given the graph $R_{n,1}$ such that its nodes are triangulated n-gons with one interior vertex, this study explores the connectivity and bounds on the diameter. This research will explore properties of the graph $R_{n,1}$ and uses graph theory to understand the geometry of the space for triangulations of n-gons with one interior vertex.

The nodes of $R_{n,1}$ are triangulated n-gons with one interior vertex, for n greater than or equal to 3. In general, a triangulation of an n-gon with interior vertices is the decomposition of the n-gon with triangles. In a triangulation, if an interior edge of the n-gon is a common edge of two triangles whose union is a 4-gon, then there are two non-adjacent vertices of that 4-gon that does not have a connected edge [1]. Therefore, one can flip the common edge of the two triangles so that it connects to the two non-adjacent edges of the 4-gon. This method of flipping edges produce new triangulations-nodes of $R_{n,1}$. When a new vertex is produced by this method, we say that the two nodes are connected by a flipped edge.

First, we say that $R_{n,1}$ is connected if every vertex is connected to every other vertex by an arc [2]. Such implies that $R_{n,1}$ is connected if it is possible to get from one triangulation, T_1, to another, T_2, by flipping an interior edge of the triangulated n-gon. An arc in $R_{n,1}$ is obtained by a sequence of flipped edges from one triangulated n-gon, T_i to another triangulated n-gon. One result of this paper is that $R_{n,1}$ is connected by flipped edges. Additionally, this project explores the furthest distance between any two triangulations T_1 and T_2 so that we can find an upper bound on the diameter of the graph $R_{n,1}$. The diameter of a graph is the maximum length of the distance between two nodes. The distance between two vertices in a graph is computed by finding the arc, α, of minimum length between those two vertices. Thus, $diam(G) = maxdist(T_1, T_2) = max(minlength(\alpha))$; where T_1 and T_2 are nodes of $R_{n,1}$. Through this research, we can have a better understanding of properties of triangulation of n-gons with one interior vertex which has potential application to problems in computer science. As a result of the properties of the graph $R_{n,1}$, we generalized our findings to study the graph $R_{n,i}$. Let $R_{n,i}$ be the graph such that its nodes are triangulated n-gons with i interior vertices and two nodes are connected by an edge if they differ by a flipped interior edge, we also showed that $R_{n,i}$ is connected.

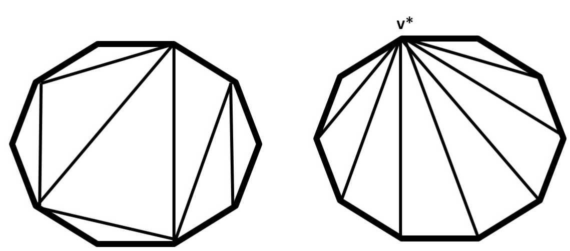
1.1. **Definitions.**

Definition 1.1. A *graph*, G, is defined as an ordered pair, $G = (N, E)$, where N is a set of nodes, $E \subset N \times N$ is a set of edges such that each pair of nodes is an edge.

Definition 1.2. An *edge* of a graph is a line segment that connects nodes.

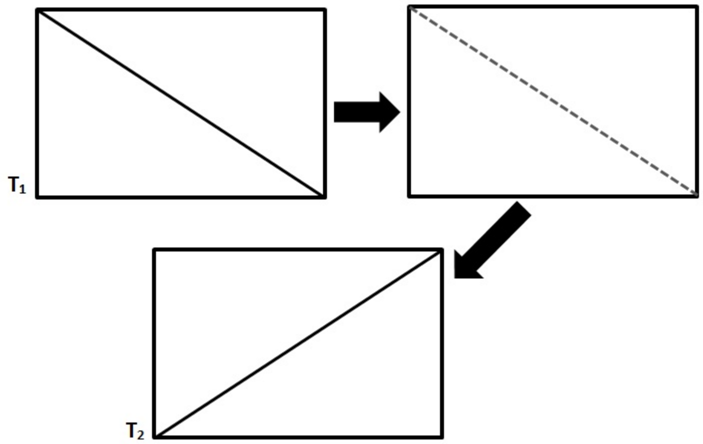
Definition 1.3. An *arc* is a path in a graph that passes no nodes twice and no edges twice.

Definition 1.4. For $n{\geq}3$, a *triangulation* of an n-gon is the decomposition of the n-gon into triangles. A triangulation of an n-gon has the following properties: each triangle in the triangulation of the n-gon has three distinct vertices and three distinct edge and any two adjacent triangles can only share one edge.
We say that a triangulation of an n-gon is a special triangulation if all of its interior edges are connected to some vertex of the n-gon, v^*.



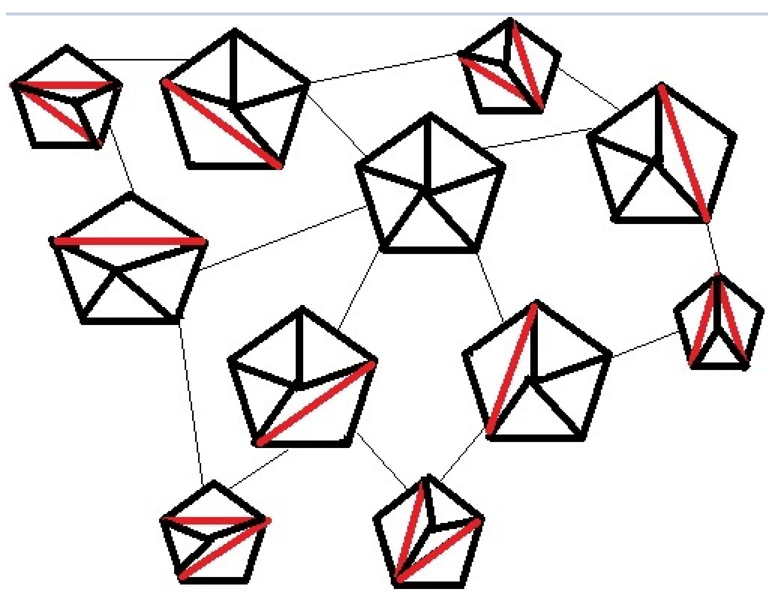
Definition 1.5. An *elementary move* is the act of flipping an edge in the triangulated n-gon.

In a triangulation, if an interior edge of the n-gon is a common edge of two triangles whose union is a 4-gon, then there are two non-adjacent vertices of that 4-gon that does not have a connected edge. Therefore, one can remove then flip the common edge of the two triangles so that it connects to the two non-adjacent edges of the 4-gon. This *method of flipping* edges produce new triangulations-nodes of $R_{n,1}$. When a new node is produced by this method, we say that the two nodes are connected by a flipped edge.



Definition 1.6. $R_{n,0}$ is a graph that has triangulated n-gons with no interior vertices as nodes and its nodes are connected if the triangulations differ by a flipped interior edge.

Definition 1.7. $R_{n,1}$ is a graph that has triangulated n-gons with one interior vertex as nodes and its nodes are connected if the triangulations differ by a flipped interior edge.



Graph of $R_{5,1}$

Definition 1.8. We say that a graph is *connected* if every node is connected to every other node by an arc. The connected components are the nodes and edges in the connected graph.

Definition 1.9. The *diameter* of a graph is the maximum length of the distance between two nodes. The distance between two nodes in a graph is computed by finding the arc, α, of minimum length between those two nodes. Thus, $diam(G) = \max dist(T_1, T_2) = \max (\min \text{length}(\alpha))$; where T_1 and T_2 are nodes of the graph.

One may think of the graph G as a map, such that the nodes are junctions and the edges are roads. The arc is like a path between two junctions so that one only passes each junction and road once. As for the diameter, one may think of it as the shortest path between the two furthest junctions, therefore, the path that one is most likely to take if they want to minimize the amount of time when traveling.

1.2. **Problems.** I wish to study the connectivity and diameter for graphs of triangulated n-gons. The objective of this research is to understand the geometry of space for $R_{n,1}$.

Problem I: Is the graph $R_{n,1}$ connected?

Problem II: What is the diameter of the graph $R_{n,1}$?

2. Properties of Graphs of Triangulated n-gons

2.1. Triangulation Properties: $R_{n,0}$ & $R_{n,1}$.

Proposition 2.1. *The number of triangles in a triangulation of an n-gon with n-sides and no interior vertices is n-2, for* $n \geq 3$.

Proof. We preceded by induction on n.
Base Case: By definition, the number of triangles in the triangulation of a 3-gon is $3 - 2 = 1$.

Assume that it is true that for any k-gon, the number of triangles in the triangulation of a k-gon is $k - 2$. We want to show that for any $(k + 1) - gon$, the number of triangles in the triangulation is $(k + 1) - 2 = k - 1$.
There exists a connected edge E that connects two non-adjacent vertices in the (k+1)-gon. Divide the (k+1)-gon along that edge, E, into two polygons L_1 and L_2 such that L_1 has l_1 sides and L_2 has l_2 sides.
Thus, the number of sides of the (k+1)-gon is: $k+1 = (l_1-1)+(l_2-1) = l_1+l_2-2$.

Let E be an edge that divides the $(k + 1)$-gon into L_1 and L_2 at two non-adjacent vertices such that L_1 or L_2 has at least three sides. Since L_1 has l_1 sides and L_2 has l_2 sides. The number of triangles in a triangulation of the (k+1)-gon is the sum of the triangles in the triangulation of L_1 and L_2.

Sum of triangles in L_1 and L_2:

$$\begin{aligned}(l_1 - 2) + (l_2 - 2) &= l_1 + l_2 - 2 - 2 \\ &= (k + 1) - 2 \\ &= k - 1\end{aligned}$$

By the Principle of Mathematical Induction, the number of triangles in the triangulation of an n-gon with no interior vertex is n-2, for n≥3. □

Proposition 2.2. *The number of triangles in a triangulation of an n-gon with one interior vertex is n, for n≥3.*

Proof. For any n-gon with one interior vertex, there exists a triangulation of the n-gon such that there is an edge that connects from the interior vertex to a vertex on the perimeter on the n-gon. Cut along that edge such that we now have an $n+2$-gon with no interior vertex. Thus, the number of triangles in that triangulated $n+2$-gon with no interior vertex is $n+2-2=n$. □

2.2. **Connectivity.**

Theorem 2.3. $R_{n,0}$ *is connected*[4].

$R_{n,0}$ is connected if every node in $R_{n,0}$ is connected to every other node by an arc. The nodes of $R_{n,0}$ are triangulated n-gons with no interior vertex, for n≥3. Sleator-Tarjan-Thurston proved that $R_{n,0}$ is connected. We created an algorithm that shows that it is possible to get from any triangulation of $R_{n,0}$ to the special triangulation by a sequence of flips. The algorithm shows that by performing the elementary move just once to any interior edge of the n-gon that is not connected to v^*, we can get to a special triangulation. Thus, we can conclude that short paths between nodes of $R_{n,0}$ passes through the special triangulations. The algorithm is quite similar to the one that is included in this paper for the graph $R_{n,1}$.

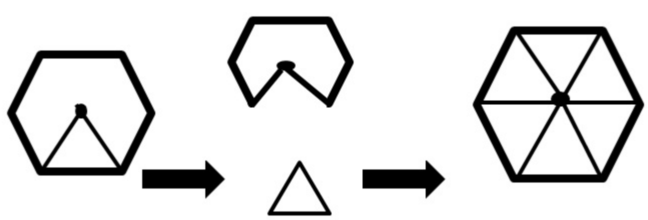
Proposition 2.4. $R_{n,1}$ *is connected.*

Proof. We proved that the graph $R_{n,1}$ is connected by showing that every node in $R_{n,1}$ is connected to the special node T^* by flipped edges. For $n=3$, it is true that for a $3-gon$ with one interior vertex, $\exists$ a triangulation of the $3-gon$ such that all inner edges are connected to the interior vertex, v^*. By definition, that triangulation is the only node of $R_{3,1}$ and the special node. Thus $R_{3,1}$ is connected.

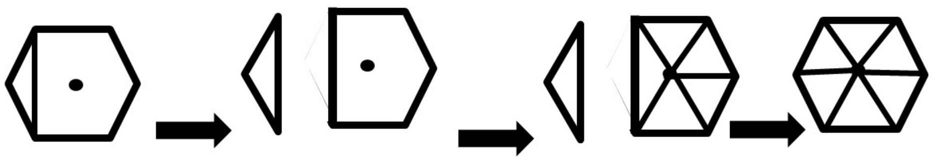
Assume that is is true that for any triangulation, T of a k-gon, with one interior vertex, it is possible to get from T to T^* by a sequence of flips. Thus, all k interior edges will be connected to v^*.I want to show that $P(k) \Rightarrow P(k+1)$.

Let T be any triangulation of a (k+1)-gon with one interior vertex. I want to show that $\exists$ a sequence of flips taking T to T^*. For any (k+1)-gon in $R_{k,1}$ there is a triangle contained in the (k+1)-gon. Consider the following cases

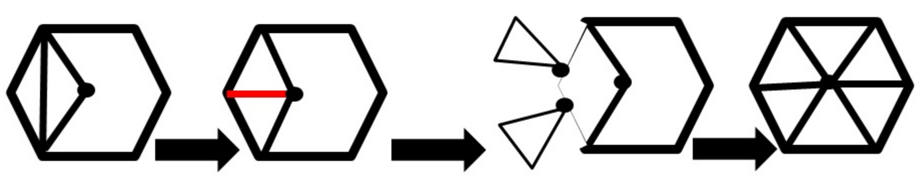
Case 1: One edge of the triangle is on the perimeter of the (k+1)-gon with one interior vertex, v^* and the other two edges connect to v^*.
Let E be an edge of the triangle that divides the (k+1)-gon into a triangle and a k-gon with one interior vertex. By the induction hypothesis, triangulate the k-gon such that all inner edges are connected to v^*. Rejoin the triangle with the special triangulation of the k-gon, then flip the edge of the triangle that is not connected to v^*. Thus all k+1 inner edges are connected to v^*, and any triangulation this (k+1)-gon is connected to T^* by flips.



Case 2: Two edges of the triangle are on the perimeter of the (k+1)-gon with one interior vertex, v^* and one edge is on the inside connecting the two edges of the triangle that are on the perimeter.
Separate the triangle from the (k+1)-gon. We now have a triangle and a polygon, L, with no interior vertex. Since $R_{n,0}$ is connected, there exists a triangulation of L such that all inner edges connects to some vertex on the perimeter of L. Let the vertex for the special triangulation of L be v^*, the vertex that the triangle and L has in common. Rejoin the triangle and L to get the special triangulation of the (k+1)-gon, T^*.



Case 3: All 3 edges of the triangle are contained in the (k+1)-gon with one interior vertex.
If all three edges,E_1, E_2, E_3, of the triangle are contained in the (k+1)-gon of $R_{(k+1),1}$. There exists one edge of the triangle, E_1,that is not connected to v^*. E_1 is contained in a 4-gon that is contained in the (k+1)-gon such that v^* is one of the vertices of the 4-gon. Flip E_1 such that it connects to v^* and a vertex on the perimeter of the (k+1)-gon. Divide the (k+1)-gon into three polygons P_1, P_2, and P_3 of $R_{n,0}$ such that all three polygons have one common vertex v^*. Since $R_{n,0}$ is connected, there exists a triangulation of P_1, P_2, and P_3 such that all inner edges connects to some vertex on the perimeter of the polygons. Let the vertex for the special triangulation of P_1, P_2, and P_3 be v^*, the vertex that all the polygons have in common. Rejoin P_1, P_2, and P_3 to get the special triangulation of the (k+1)-gon, T^*.



By the Principle of Mathematical Induction, since $P(k)$ implies $P(k+1)$ is true, then $P(n)$ holds n$\geq$3. □

2.3. **Diameter Bounds.** The fact that $R_{n,1}$ is connected is quite useful to finding the diameter. We know that every node in $R_{n,1}$ is connected to the special node T^* by flips. If we knew the distance of the shortest path from any node to the special node then we can find the diameter bounds of $R_{n,0}$.

Lemma 2.5. *Let G be a finite connected graph. If there exists the special node, T^*, such that they all nodes in G have an arc to T^* of length at most L, then the diameter of the graph is at most $2L$.*

Proof. Let T_i and T_j be two nodes that are furthest apart. By assumption, there is an arc from T_i to T^* and its length is at most L. Thus, dist(T_i, T^*)$\leq$ L. There is also an arc from T_j to T^* and its length is at most L. Thus, dist(T_j, T^*)$\leq$ L. Hence,

$$\begin{aligned} diameter(G) &= dist(T_i, T_j) \\ &\leq dist(T_i, T^*) + dist(T_j, T^*) \\ &\leq (L + L) \\ &\leq 2L \end{aligned}$$

□

Since the edges of $R_{n,0}$ and $R_{n,1}$ connects two triangulations, T_1 and T_2 that differ by a flipped edge, the shortest path from T_1 to T_2 will be a path that requires any edge of T_1 that is not an edge of T_2 to be flipped once. Also, if T_1 and T_2 have an edge in common then the shortest path between T_1 and T_2 is the path such that the common edge of both triangulations is not flipped [4].

What is the distance from any node T to the special node T^*?

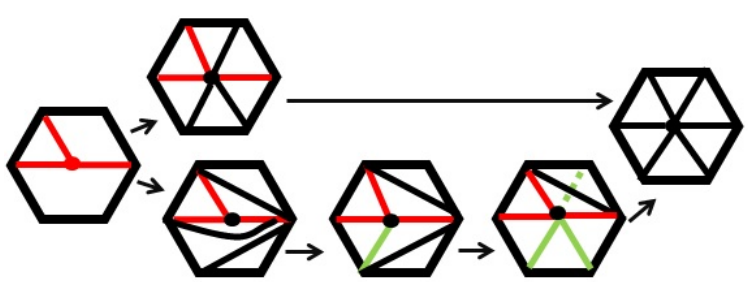
Proposition 2.6. *The distance from any node T to the special node T^* of $R_{n,1}$ is at most n-3.*

Proof. We created an algorithm that shows that one can get from any triangulation T to the special node T^* by flipping each edge of T that is not an edge of T^* just once.

Algorithm:

(1) Find the interior vertex, v^*
(2) If the edges that are not connected to v^* are edges of the perimeter of the n-gon and n interior edges are connected to v^*, then stop.
(3) If the edges that are not connected to v^* are not edges of the perimeter of the n-gon, then they are interior edges $e_1, e_2 \dots e_i \dots$
(4) Find an interior edge e_i such that e_i is not connected to v^* and e_i is contained in a 4-gon where 2 edges of the 4-gon are connected to v^*.
(5) Flip e_i so that it connects to v^*.
(6) Repeat

□



The proof of the diameter bounds of $R_{n,1}$ is a corollary of this algorithm and the previous lemma.

Corollary 2.7. *Claim: The diameter of $R_{n,1}$ is at most 2(n-3).*

Proof. By the lemma, since the distance from any triangulation T to T^* is at most $n-3$ in $R_{n,1}$, dist$(T, T^*) \leq n-3$. Thus, the diameter of $R_{n,1} \leq 2(n-3)$. □

Proposition 2.8. *The diameter $R_{n,1}$ is at least n.*

Proof. We show that given any two triangulations T_1 and T_2 in $R_{n,1}$ such that no edge of T_1 is an edge of T_2, then the distance from T_1 to T_2 is at least n. Let T_1 and T_2 be two triangulations of $R_{n,1}$ that have no common edges. Then on the path from T_1 to T_2, we must flip each edge of T_1 at least once to get to T_2. Since there are n edges in any triangulation of $R_{n,1}$, then the distance between T_1 and T_2 is at least n.

□

3. Discussion

3.1. **Future Directions with Diameter Bounds.** We have shown that a lower bound on the diameter of $R_{n,1}$ is n and that an upper bound on the diameter bounds for $R_{n,1}$ is at most $2(n-3)$. However, is there a better diameter bound? An improvement on the bounds of the diameter will help us to find the actual diameter of the graph $R_{n,1}$.

3.2. **Generalize Properties for the Graph $R_{n,i}$.** Let $R_{n,i}$ be the graph such that its nodes are triangulated n-gons with i interior vertices and two nodes are connected by an edge if they differ by a flipped interior edge. Using a similar argument and an algorithm, we have also proven that $R_{n,i}$ is connected. In the future, we would like to find other properties of $R_{n,i}$ including its diameter bounds. Below is an outline of the proof for the connectedness of $R_{n,i}$.

Proposition 3.1. *Claim: $R_{n,i}$ is connected.*

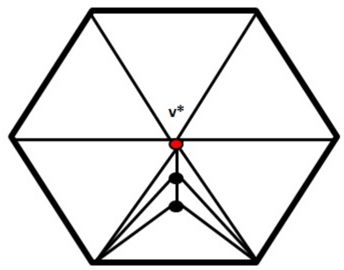
Proof. We proceeded by induction on i.

Base Case: $R_{n,0}$ is connected.

Let T^* be a special node, then it has the following properties:

(1) there exists i interior vertices in the n-gon: v^* and $i-1$ interior vertices
(2) the n-gon is triangulated such that each of the n vertices on the perimeter of the n-gon has a connected edge to v^*
(3) the other $i-1$ vertices are contained in a triangle who has two edges connected to v^* and one edge is on the perimeter of the n-gon
(4) that 3-gon with $i-1$ interior vertices is triangulated has such that:
    (a) any three adjacent triangles share a common interior vertex and at least two of the three triangles also share a common vertex on the perimeter of the 3-gon

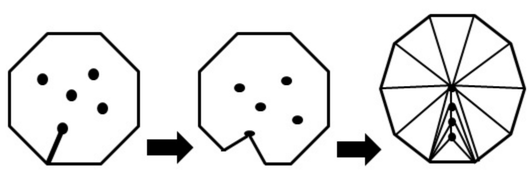
(b) $(2(i-1))$ triangles are symmetric about a series of i aligned line segments such that $(i-1)$ interior vertices are connected to each other and there is an edge that connects them to v^*.



Graph of $R_{6,3}$

Induction Hypothesis: For $P(k)$, let $k = i - 1$. Assume that it is true that for any n-gon, $R_{n,i-1}$ is connected. Fix n, for any n-gon in $R_{n,i}$, we show that $R_{n,i}$ is connected. This proof has two parts. In part one, we show that for any node in $R_{n,i}$, we can get to a special node by flips.

Fix n for some triangulation of an n-gon with i interior vertices, $T \in R_{n,i}$. There exists an edge that connects an interior vertex of T to a vertex on the perimeter of the n-gon. Cut along that edge such that we now have an triangulation of an $n+2$-gon with $(i-1)$ interior vertices. By the induction hypothesis, $R_{n+2,i-1}$ is connected.



In part two, we use an algorithm to show that all special nodes in $R_{n,i}$ are connected by flips.

□

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