THE CLASSIFICATION OF SIMPLE COMPLEX LIE ALGEBRAS

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Abstract. This paper introduces Lie groups and their associated Lie algebras. With the goal of describing simple Lie groups, we analyze semisimple complex Lie algebras by their root systems to classify simple Lie algebras. We assume a background in linear algebra, differential manifolds, and covering spaces.

Contents

1. Introduction

At the intersection of differential topology and algebra, Lie groups are smooth manifolds with a compatible group structure. In the first section we introduce Lie groups with several canonical examples. While many classical Lie groups are matrix groups, we shall develop the theory in its full generality. The primary aim of this paper will be to describe simple Lie groups, or connected Lie groups possessing no proper, nontrivial analytic normal subgroups. In contrast to the colossal project constituting the classification of simple finite groups, the classification of simple Lie groups is greatly simplified by exploiting the manifold structure. In particular, every Lie group has an associated Lie algebra consisting of its tangent space at the identity equipped with a bracket operation. A theme of this paper will be to investigate the relationship between Lie groups and their Lie algebras. We develop tools like the exponential map and adjoint representation to translate information between the two objects. For example, we show that a Lie group is simple if and only if its Lie algebra is simple, or contains no proper, nontrivial ideals.

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While an arbitrary Lie group is not completely determined by its Lie algebra, there is a bijective correspondence between Lie algebras and simply-connected Lie groups. Moreover, any Lie group can be realized as the quotient of its universal covering group by a discrete central subgroup. This reduces the problem of finding simple Lie groups to classifying simple simply-connected Lie groups and thereby to classifying simple Lie algebras. With this end in mind, the rest of the paper focuses on developing the theory of Lie algebras. We introduce representations of semisimple Lie algebras to decompose them into their root spaces. A root system, encoded in its associated Dynkin diagram, bears all the information about its Lie algebra. As the roots of semisimple Lie algebras satisfy several restrictive geometrical properties, we can classify all irreducible root systems by a brief series of combinatorial arguments. After unwinding the equivalences between this cast of objects, the result will finally provide a classification of simple complex Lie algebras and a major step in classifying simple Lie groups.

2. Lie Groups and Lie Algebras

Definition 2.1. A *Lie group* G is a smooth manifold with a group operation such that multiplication $m : G \times G \to G$ and inversion $i : G \to G$ are smooth.

Example 2.2. The most straightforward example of a Lie group is $\mathbb{R}^n$ with smooth structure given by the identity and group operation given by vector addition. Similarly, $\mathbb{C}^n$ is a Lie group of dimension $2n$.

Another Lie group is the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ with smooth structure given by projections of small neighborhoods in $\mathbb{R}^n$ and group operation given by addition modulo the integer lattice.

The *general linear group* $GL_n(\mathbb{R}) = \{X \in M_{n\times n}(\mathbb{R}) \mid \det(X) \neq 0\}$ representing linear automorphisms of $\mathbb{R}^n$ is an open subset of $\mathbb{R}^{n^2}$ and therefore a manifold of dimension n^2. Matrix multiplication and inversion are rational functions in the coordinates that are well-defined on $GL_n(\mathbb{R})$, so the group operations are smooth. Similarly, $GL_n(\mathbb{C}) = \{X \in M_{n\times n}(\mathbb{C}) \mid \det(X) \neq 0\}$ is a Lie group of dimension of $2n^2$. Many classical Lie groups are closed subgroups of $GL_n(\mathbb{R})$ or $GL_n(\mathbb{C})$.

The *special linear group* $SL_n(\mathbb{R}) = \{X \in GL_n(\mathbb{R}) \mid \det(X) = 1\}$ represents volume and orientation preserving automorphisms of $\mathbb{R}^n$. Using elementary methods from the theory of smooth manifolds, one can show $SL_n(\mathbb{R})$ is a Lie group of dimension $n^2 - 1$.

The *orthogonal group* $O_n = \{X \in GL_n(\mathbb{R}) \mid XX^t = I_n\}$ representing automorphisms of $\mathbb{R}^n$ which preserve the standard inner product is a closed subgroup of dimension $\frac{n(n-1)}{2}$. The group consists of two connected components depending on the sign of the determinant. The *special orthogonal group* $SO_n = O_n \cap SL_n(\mathbb{R})$ is a connected Lie group of dimension $\frac{n(n-1)}{2}$.

Of course, we also have the corresponding Lie groups over $\mathbb{C}$, such as $SL_n(\mathbb{C})$, $SO_n(\mathbb{C})$, and $Sp_{2n}(\mathbb{C})$. They can be embedded as subgroups of $GL_{2n}(\mathbb{R})$.

The *unitary group* $U(n) = \{X \in GL_n(\mathbb{C}) \mid XX^* = I_n\}$ representing automorphisms of $\mathbb{C}^n$ which preserve the Hermitian inner-product is a Lie group of dimension n^2. The *special unitary group* $SU_n = SL_n(\mathbb{C}) \cap U(n)$ is a Lie group of dimension $n^2 - 1$.

Let $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. The *symplectic group* $Sp_{2n}(\mathbb{R}) = \{X \in GL_{2n}(\mathbb{R}) \mid X^tJX = J\}$ represents automorphisms of $\mathbb{R}^{2n}$ preserving the nondegenerate skew-symmetric inner-product represented by the matrix J. It is a Lie group of dimension $2n^2 + n$.

We first observe the interaction of the algebraic and topological properties of a Lie group G by considering its tangent bundle. For any x in G the left action $L_x : G \to G$ sending an element y to the product xy is a diffeomorphism. Hence the collection of differential maps $d_1L_x : T_1(G) \to T_x(G)$ smoothly rotate the tangent space at the identity to any point along G. The tangent space at the identity therefore describes the entire tangent bundle. For example, the maps d_1L_x rotate any fixed vector $v \in T_1(G)$ along G to define a vector field. We say that a vector field X is *left-invariant* if $X_x = d_1L_x(v)$ for some fixed $v \in T_1(G)$. Clearly the subspace of left-invariant vector fields is isomorphic to $T_1(G)$.

A vector field X on G acts on a smooth real-valued function $f : G \to \mathbb{R}$ at a point $x \in G$ by $X_x(f) = d_xf(X_x)$. As this action defines a new smooth function $X(f)$, we apply another vector field Y to define the smooth function $YX(f)$. There generally does not exist a vector field corresponding to the action of YX. However, an easy calculation in coordinates shows that there is a unique vector field corresponding to the commutator $XY - YX$. Define a bracket operation on vector fields by choosing $[X, Y]$ to be the unique vector field satisfying $[X, Y](f) = (XY - YX)(f)$ for all smooth functions f.

If X and Y are left-invariant vector fields on G, then so is the bracket $[X, Y]$. Using the correspondence between left-invariant vector fields and vectors in $T_1(G)$, we obtain a bracket operation on $T_1(G)$.

Definition 2.3. For a Lie group G, the tangent space at the identity $T_1(G)$ equipped with the described bracket $[\cdot, \cdot] : T_1(G) \times T_1(G) \to T_1(G)$ is the *Lie algebra of* G. We denote the Lie algebra of G as $\mathfrak{g}$.

By inspection, for any Lie group G the bracket operation on the Lie algebra of G satisfies the following properties:

(1) Bilinearity
(2) Antisymmetry: $[x, x] = 0$ for any $x \in \mathfrak{g}$.
(3) The Jacobi identity: $[[x, y], z] + [[z, x], y] + [[y, z], x] = 0$ for all $x, y, z \in \mathfrak{g}$.

The value of the above relations is that they characterize general properties of the bracket operation expressible purely in terms of vectors in $\mathfrak{g}$. We can use this to define the notion of an abstract Lie algebra.

Definition 2.4. A *Lie algebra* $\mathfrak{g}$ is a vector space with an antisymmetric bilinear bracket operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity.

There is a robust theory of Lie algebras which is a priori independent of their role in the study of Lie groups. However, a goal of this paper is to investigate the extent to which the structure of a Lie group is encoded in its Lie algebra, whose linear structure will generally be more amenable to analysis. This will provide a means of describing Lie groups by passing to their respective Lie algebras.

Example 2.5. We identify the Lie algebras of the Lie groups given in example (2.2).

The Lie algebra of the Lie group $\mathbb{R}$ is just the vector space $\mathbb{R}$. Since the Lie algebra is one-dimensional, the antisymmetry of the bracket operation implies $[x, y] = 0$ for all $x, y \in \mathbb{R}$. In general, we say a Lie algebra $\mathfrak{g}$ is *abelian* if $[x, y] = 0$ for all $x, y \in \mathfrak{g}$. Similarly, the Lie algebra of $\mathbb{T}$ is also abelian. In fact, it is not hard to show $\mathbb{R}^n$ and $\mathbb{T}^n$ have abelian Lie algebras.

As $GL_n(\mathbb{R})$ is an open set in $\mathbb{R}^{n^2}$, any matrix $x \in M_{n\times n}(\mathbb{R}) \simeq \mathbb{R}^{n^2}$ can be realized as the velocity vector at the identity of a small enough curve in $GL_n(\mathbb{R})$. Therefore the associated Lie algebra $\mathfrak{gl}_n(\mathbb{R})$ is isomorphic to $M_{n\times n}(\mathbb{R})$. It is convenient to note that the bracket of two matrices x and y in $\mathfrak{gl}_n(\mathbb{R})$ determined by their left-invariant vector fields coincides with the commutator $xy - yx$. This also holds for the Lie algebra of any closed subgroup of $GL_n(\mathbb{R})$.

Using elementary methods we can identify the tangent spaces of the remaining Lie groups:

$$\mathfrak{sl}_n(\mathbb{R}) = \{X \in M_{n\times n}(\mathbb{R}) \mid Tr(X) = 0\}$$
$$\mathfrak{so}_n = \{X \in M_{n\times n}(\mathbb{R}) \mid X^t = -X\}$$
$$\mathfrak{u}_n = \{X \in M_{n\times n}(\mathbb{C}) \mid X^* = -X\}$$
$$\mathfrak{su}_n = \{X \in M_{n\times n}(\mathbb{C}) \mid X^* = -X, Tr(X) = 0\}$$
$$\mathfrak{sp}_{2n}(\mathbb{R}) = \{X \in M_{2n\times 2n}(\mathbb{R}) \mid X^t J = -JX\}$$

We also have the corresponding Lie algebras over $\mathbb{C}$, such as $\mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{so}_n(\mathbb{C})$, and $\mathfrak{sp}_{2n}(\mathbb{C})$.

3. The Exponential Map and Adjoint Representation

The first goal of this paper is to illustrate how a Lie group and its Lie algebra are related. As motivation, we say that a subgroup H of a Lie group G is an *analytic subgroup* if it is a connected group and the inclusion $i : H \hookrightarrow G$ is a smooth embedding. We say that a connected group G is *simple* if has no proper, nontrivial analytic normal subgroups. Note that this contrasts with the definition of simple abstract groups since a simple Lie group may still have normal subgroups. Just as we study finite simple groups to develop a firm foundation from which to study more complicated compositions, so would we also like to describe simple Lie groups. To extend subgroups and normal subgroups to analogous notions in Lie algebras, we say that a subspace $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a *subalgebra* if it is closed under the bracket with itself, and we say it is an *ideal* if it is closed under the bracket with any element in $\mathfrak{g}$. A Lie algebra is *simple* if it has no proper, nontrivial ideals. It is natural to ask how subgroups and normal subgroups correspond to subalgebras and ideals. To address these questions we define the exponential and adjoint maps.

For notational convenience, given two subspaces α and β of a Lie algebra, let $\alpha + \beta = \{x + y \mid x \in \alpha, y \in \beta\}$, and let $[\alpha, \beta]$ denote the span of the set $\{[x, y] \mid x \in \alpha, y \in \beta\}$. For example, it is easy to show that if α and β are ideals, then so are $\alpha + \beta$, $\alpha \cap \beta$, and $[\alpha, \beta]$. Also, for subsets H and K of a Lie group G, let $HK = \{hk \mid h \in H \text{ and } k \in K\}$. For example, for an element $g \in G$ and subset $U \subset G$ we have $gU = L_g(U)$.

A *Lie algebra homomorphism* $\psi : \mathfrak{g} \to \mathfrak{h}$ is a linear transformation satisfying $\psi([X, Y]) = [\psi(X), \psi(Y)]$ for any X and Y in $\mathfrak{g}$. Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. A *Lie group homomorphism* is a smooth group homomorphism

$\phi : G \to H$. Then ϕ induces a linear map $d_1\phi : \mathfrak{g} \to \mathfrak{h}$. In fact, ϕ preserves left-invariant vector-fields, and $d_1\phi$ commutes with the bracket operation. Thus $d_1\phi$ is a Lie algebra homomorphism. Since we are only concerned with the derivative at the identity, we will henceforth abbreviate it as $d\phi$.

Since the Lie algebra $\mathfrak{g}$ of a Lie group G is the tangent space at the identity, we can regard elements as velocity vectors of smooth curves in G. We construct the exponential map by considering a special class of such curves.

Definition 3.1. A *one-parameter subgroup* of a Lie group G is a Lie group homomorphism $\sigma : \mathbb{R} \to G$.

We can associate a one-parameter subgroup σ with the vector $d\sigma(1) \in \mathfrak{g}$. It turns out that this correspondence is bijective.

Proposition 3.2. *For each x in $\mathfrak{g}$ there exists a unique one-parameter subgroup σ_x such that $d\sigma_x(1) = x$.*

Proof. Given $x \in \mathfrak{g}$, a standard theorem from differential equations (see page 72 of [7]) ensures that for the left-invariant vector field X on G corresponding to x there exists a unique smooth function $\phi : I \to G$ defined on some open interval I containing 0 such that $\phi(0) = 1$ and

$$d_t\phi(1) = X_{\phi(t)}.$$

In other words, ϕ gives a smooth path corresponding to the flow of the vector field X. We claim ϕ acts like a group homomorphism in the sense that for real numbers s, t and $s + t$ in I, $\phi(t + s) = \phi(t)\phi(s)$. Indeed, fix s and let t vary to define curves α and β by $\alpha(t) = \phi(s)\phi(t)$ and $\beta(t) = \phi(s + t)$. Rewriting $\alpha(t) = L_{\phi(s)} \circ \phi(t)$, we have

$$\begin{aligned} d\alpha_t(1) &= dL_{\phi(s)} \circ d_t\phi(1) \\ &= dL_{\phi(s)} \circ X_{\phi(t)} \\ &= dL_{\phi(s)} \circ dL_{\phi(t)}x \\ &= dL_{\phi(t)\phi(s)}x \\ &= X_{\alpha(t)}. \end{aligned}$$

Similarly, $d\beta_t(1) = X_{\beta(t)}$. Note also $\alpha(0) = \beta(0) = 1$. By the uniqueness statement above this implies $\alpha = \beta$. Now, ϕ extends to a map ψ defined on all of $\mathbb{R}$ as follows: for a given $y \in \mathbb{R}$ find an integer n such that $\frac{y}{n} \in I$. Define

$$\psi(y) = \phi(\frac{y}{n})^n.$$

This is well-defined, for if m is another such integer then the fact that $\phi(t + s) = \phi(t)\phi(s)$ for s and t in I implies

$$\phi(\frac{y}{n})^n = \phi(\frac{y}{mn})^{mn} = \phi(\frac{y}{m})^m.$$

Similarly, it is easy to check ψ is a group homomorphism. Thus ψ gives the one-parameter subgroup associated to x. □

Definition 3.3. Define the *exponential map* $\exp_G : \mathfrak{g} \to G$ of a Lie group G by $\exp_G(x) = \sigma_x(1)$.

The exponential map of G is typically abbreviated as exp, the group being understood. The uniqueness of the one-parameter subgroups shows $\sigma_{tv}(1) = \sigma_v(t)$ for a scalar $t \in \mathbb{R}$, which implies the exponential map is well-defined. Moreover, consisting of solutions to a smoothly varying system of differential equations, the exponential map is smooth.

Example 3.4. In the case of $GL_n(\mathbb{R})$, the exponential map applied to a matrix x in its Lie algebra $\mathfrak{gl}_n(\mathbb{R})$ is defined by the sum $\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, where $x^0 = I_n$. Indeed, it is easy to show that this defines a well-defined map into the space of invertible matrices such that the one-parameter subgroup $\sigma_x : \mathbb{R} \to G$ defined by $\sigma_x(t) = \exp(tx)$ satisfies $dc(1) = x$. For a closed subgroup of $GL_n(\mathbb{R})$ with Lie algebra $\mathfrak{g}$, the exponential map is just the restriction of this sum to $\mathfrak{g}$.

It is clear by definition that the induced map $d(\exp) : \mathfrak{g} \to \mathfrak{g}$ is the identity. By the inverse function theorem, the exponential map sends a neighborhood U of the origin diffeomorphically onto a neighborhood V of the identity in G. By translating the open set V around G, it follows that the group generated by $\exp(\mathfrak{g})$ is both open and closed in G and therefore generates the identity component. This marks the first crucial property of the exponential map. The second crucial property is that the exponential map is natural with maps of Lie groups.

Proposition 3.5. *If $\phi : G \to H$ is a map of Lie groups, then* $\exp \circ d\phi = \phi \circ \exp$. *In other words, the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{d\phi} & \mathfrak{h} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow[\phi]{} & H \end{array}$$

Proof. Given $x \in \mathfrak{g}$, let σ_x denote its one-parameter subgroup. By the chain rule

$$d(\phi \circ \sigma_x)(1) = d\phi \circ d\sigma_x(1) = d\phi(x).$$

This implies $\phi \circ \sigma_x$ is the one-parameter subgroup corresponding to $d\phi(x)$. Thus

$$\exp(d\phi(x)) = \sigma_{d\phi(x)}(1) = \phi \circ \sigma_x(1) = \phi \circ \exp(x).$$

□

Example 3.6. If H is an analytic subgroup of G with inclusion $i : H \hookrightarrow G$, then clearly $di : \mathfrak{h} \hookrightarrow \mathfrak{g}$ embeds $\mathfrak{h}$ as a subalgebra of $\mathfrak{g}$. Naturality implies

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{di} & \mathfrak{g} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ H & \xrightarrow[i]{} & G \end{array}$$

Thus exp maps $\mathfrak{h}$ into H. In particular, by the inverse function theorem $\exp(\mathfrak{h})$ contains an open set in H, which implies H equals the group generated by $\exp(\mathfrak{h})$.

Now, there is not a correspondence between between general subgroups and subalgebras. For example, $\mathbb{Q}$ is a proper subgroup of $\mathbb{R}$ which is not even a manifold. However, the preceding example describes how any analytic subgroup has an associated subalgebra. It turns out that this association is bijective.

Theorem 3.7. *For a Lie algebra $\mathfrak{g}$ of a Lie group G, every subalgebra $\mathfrak{h}$ is the Lie algebra of a unique analytic subgroup H.*

Proof. We sketch a proof. Consult page 119 of [2] for details.

As shown in example (3.6), if H exists then it is uniquely determined as the subgroup generated by $\exp(\mathfrak{h})$.

To show existence, the idea is to again consider the subgroup generated by $\exp(\mathfrak{h})$. To avoid generating a subgroup with too large a Lie algebra, first choose an open ball B around 0 on which exp is a diffeomorphism. Let $H' = \exp(B \cap \mathfrak{h})$. Note that H' is open in $\exp(\mathfrak{h})$ and hence generates the same subgroup H. One can show that in a neighborhood $G' = \exp(B)$ of the identity $H'H' \cap G' \subset H'$, which implies that generating the subgroup H contributes no new elements around the identity. Then H has a natural smooth structure with charts given by the translates hH' for each h in H. This makes H the required analytic subgroup. □

To make full use of the exponential map, we now introduce the adjoint representation. Any element x in G provides a diffeomorphism $\phi_x : G \to G$ sending an element y to the conjugate xyx^{-1}. Then $d\phi_x$ induces an automorphism of $\mathfrak{g}$, giving a map $Ad : G \to Aut(\mathfrak{g})$ which sends x to $d\phi_x$. The map Ad defines a smooth representation of G on $\mathfrak{g}$ called the *adjoint representation.* This representation interacts meaningfully with the bracket structure on $\mathfrak{g}$ in the sense that the derivative $d(Ad) : \mathfrak{g} \to End(\mathfrak{g})$ satisfies $d(Ad)(x)(y) = [x, y]$ for all $x, y \in \mathfrak{g}$. See page 79 of [6] for details. We shall denote the derivative as ad, and for notational clarity we abbreviate $Ad(x)$ as Adx and $ad(x)$ as adx.

Example 3.8. The naturality of the exponential map implies that for $g \in G$ and $x \in \mathfrak{g}$ we have

$$\exp(Adg(x)) = \exp \circ d\phi_g(x) = \phi_g \circ \exp(x) = g \exp(x) g^{-1}.$$

Similarly, we also have the commuting diagram:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{ad} & End(\mathfrak{g}) \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow[\phi]{} & Aut(\mathfrak{g}) \end{array}$$

Proposition 3.9. *Given a connected group G with Lie algebra $\mathfrak{g}$ and an analytic subgroup H with Lie algebra $\mathfrak{h}$, then H is normal in G if and only if $\mathfrak{h}$ is an ideal in $\mathfrak{g}$.*

Proof. Suppose $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. Since $\exp(\mathfrak{h})$ generates H and $\exp(\mathfrak{g})$ generates G, it suffices to show $\exp(g)\exp(h)\exp(g)^{-1} \in H$ for any $h \in \mathfrak{h}$ and $g \in \mathfrak{g}$. Extend a basis of $\mathfrak{h}$ to a basis of $\mathfrak{g}$. Let $\mathfrak{s}$ denote the subspace spanned by the added vectors so that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$ as a vector space. Since $\mathfrak{h}$ is an ideal, then with respect to this ordered decomposition we have

$$adg = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

Thus $Ad\exp(g) = \exp(adg)$ is also of this form and sends $\mathfrak{h}$ back into itself. Then for $h \in \mathfrak{h}$ we have

$$\exp(g)\exp(h)\exp(g)^{-1} = \exp(Adg(h)) \subset \exp(\mathfrak{h}) \subset H.$$

Conversely, suppose H is normal in G. Note that whether or not the bracket of two elements of $\mathfrak{g}$ lies in $\mathfrak{h}$ is independent of scalar multiples of the arguments since the bracket is bilinear. Consider a ball B around the origin in $\mathfrak{g}$ on which exp is a diffeomorphism. Then exp maps a neighborhood $B \cap \mathfrak{h}$ of the origin in $\mathfrak{h}$ diffeomorphically onto a neighborhood V of the identity in H. For any $t \in \mathbb{R}$ we have

$$\exp(Ad\exp(tg)(h)) = \exp(tg)\exp(h)\exp(tg)^{-1} \in H.$$

By continuity, we can scale g and h to be small enough so that this product lands in V, which implies $Ad\exp(tg)(h)$ lies in $\mathfrak{h}$ for every $|t| < 1$. Hence $Ad(\exp(tg))$ sends $\mathfrak{h}$ into itself, so it is of the form given above with respect to the ordered decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$. Then $adg = \frac{d}{dt}\mid_{t=0} Ad(\exp(tg))$ is also of this form and hence sends $\mathfrak{h}$ into itself. □

Corollary 3.10. *A connected group is simple if and only if its Lie algebra is simple.*

Therefore the classification of simple Lie groups begins with the classification of simple Lie algebras.

4. Covering Groups

Before a detailed study of Lie algebras, it is worth asking whether this association between Lie groups and Lie algebras is bijective. While a Lie group determines its Lie algebra, we saw in example (2.5) that $\mathbb{R}^n$ and $\mathbb{T}^n$ have isomorphic Lie algebras with an isomorphism given by the projection. Evidently, the theory of Lie groups does not perfectly coincide with the theory of Lie algebras, yet there is something that can be said about Lie groups with isomorphic Lie algebras.

Proposition 4.1. *If G and H are two connected groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, then a map $\phi : G \to H$ inducing an isomorphism $\psi : \mathfrak{g} \to \mathfrak{h}$ is a covering map.*

Proof. By the inverse function theorem ϕ maps a neighborhood U of the identity in G diffeomorphically onto a neighborhood V of the identity in H. By translating V by elements of H, the connectedness of H implies ϕ is surjective. Also, for any $y \in H$ and $x \in G$ mapping to y, ϕ maps xU diffeomorphically onto yV. □

This hints toward a partial converse which we shall now develop. It is useful to discuss the relationship between Lie groups and their universal covering spaces, which is closely related to quotients of Lie groups. Let G be a Lie group with closed subgroup H, and let $\pi : G \to G/H$ denote the projection onto the coset space. Equip the coset space with the quotient topology induced by π. In particular, if H is normal in G then this defines a topological group G/H admitting the structure of a Lie group such that the projection π is a Lie group homomorphism. It is also an open map, for if U is open in G then $\pi^{-1}(\pi(U)) = \bigcup_{h \in H} hU$ is also open in G, which implies $\pi(U)$ is open in the quotient G/H. In the special case when H is discrete, the projection is a covering map.

Proposition 4.2. *Let G be a connected group. Then G is a covering space of G/H with a covering map given by the projection π if and only if H is discrete.*

Proof. If π is a covering map, then $H = \pi^{-1}(1)$ is clearly discrete. If H is discrete, there exists a neighborhood V around the identity intersecting H only at the identity. We can choose V to be small enough so that $V \cap hV = \emptyset$ for any $h \in H$. Clearly the restriction of π to one of the sets hV is a homeomorphism onto its image. Then for a coset $yH \in G/H$, the preimage of the neighborhood $(yH)\pi(V)$ consists of the union $\bigcup_{h\in H} yhV$. □

Note that the projection $\pi : \mathbb{R}^n \to \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is a special case. We will focus on discrete subgroups since they admit covering maps onto their quotients. They are also often easy to identify by the following proposition:

Proposition 4.3. *Any discrete normal subgroup H of a connected group G is central.*

Proof. Given $h \in H$ define a map $\phi : G \to H$ by $\phi(g) = ghg^{-1}$. Since G is connected, the image under ϕ is connected in H. Since H is discrete, it consists of the single point $h = \phi(1)$. □

Proposition (4.2) shows that if a Lie group G can be realized as the quotient of a group H by a discrete central subgroup, then since the projection is a smooth covering map and thus a diffeomorphism in a neighborhood of the identity it must induce an isomorphism of Lie algebras. Conversely, it is worth considering whether a given Lie algebra $\mathfrak{g}$ corresponds to a Lie group G such that all Lie groups with Lie algebra $\mathfrak{g}$ can be realized as quotients of G by discrete central subgroups. This question has two parts. First, to show that each Lie algebra can be realized as the Lie algebra of some group, we quote Ado's theorem, a proof of which can be found in [2] on page 500.

Theorem 4.4. *(Ado) Any Lie algebra $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}_n(\mathbb{R})$ for some n.*

Then by proposition (3.7), $\mathfrak{g}$ is the Lie algebra of a corresponding subgroup of $GL_n(\mathbb{R})$. Now that $\mathfrak{g}$ is realized by some Lie group G, to answer the second part of the question it is natural to consider its universal cover $\tilde{G}$ with covering map $p : \tilde{G} \to G$.

Theorem 4.5. *(1) Let $\tilde{G}$ denote the universal cover of a Lie group G. Let $\tilde{1} \in p^{-1}(1)$. Then there exists a unique multiplication on $\tilde{G}$ with identity $\tilde{1}$ such that $\tilde{G}$ is a Lie group, and the covering map p is a group homomorphism.*

(2) Moreover, there exists a discrete central subgroup H of $\tilde{G}$ such that $G = \tilde{G}/H$ and the projection $\pi : \tilde{G}/H \to G$ is a covering map.

Proof. We shall sketch a proof. See page 88 of [6] for details.

(1) Let $m : G \times G$ denote multiplication. Let $\phi : \tilde{G} \times \tilde{G} \to G$ denote the composition $m \circ (p, p)$. Then by the simple-connectedness of $\tilde{G}$

$$1 = \phi_*(\pi_1(\tilde{G} \times \tilde{G}, \tilde{1} \times \tilde{1})) \subset p_*(\pi_1(\tilde{G}, \tilde{1})).$$

By a standard lifting theorem from the theory of covering spaces (see page 61 of [3]), we obtain a unique continuous map $\tilde{\phi} : \tilde{G} \times \tilde{G} \to \tilde{G}$ satisfying $\phi = p \circ \tilde{\phi}$ and $\tilde{\phi}(\tilde{1}, \tilde{1}) = \tilde{1}$. One can check this defines a group multiplication on $\tilde{G}$ for which p is a group homomorphism. With charts given by the covering map, $\tilde{G}$ inherits the structure of a manifold in such a way that the covering map is smooth, which implies that multiplication and inversion in $\tilde{G}$ are smooth.

(2) Let H denote the kernel of the covering map p, which is a closed and normal subgroup of G. It is also discrete as the preimage of a single point by a covering map. By proposition (4.3), H is a central subgroup. By the standard isomorphism theorem $G = \tilde{G}/H$. □

Thus the class of groups with a fixed Lie algebra contains at least one simply-connected representative. It turns out that this simply-connected group is unique.

Lemma 4.6. *If G is connected then a map $\phi : G \to H$ is determined by its induced Lie algebra map $d\phi = d\psi : \mathfrak{g} \to \mathfrak{h}$.*

Proof. The naturality of the exponential map implies ϕ is determined by $d\phi$ on $\exp(\mathfrak{g})$. But the connectedness of G implies $\exp(\mathfrak{g})$ generates G. □

Proposition 4.7. *If G is simply-connected and H is connected with a map $\lambda : \mathfrak{g} \to \mathfrak{h}$, then there exists a map of Lie groups $\phi : G \to H$ such that $d\phi = \lambda$.*

Proof. Let $\mathfrak{s}$ denote the graph of γ in the Lie algebra of $G \times H$. By (3.7), this subalgebra corresponds to an analytic subgroup S of $G \times H$. Let $\psi_G : G \times H \to G$ and $\psi_H : G \times H \to H$ denote the projections, which are clearly Lie group homomorphisms. Since $d\psi_G$ is an isomorphism of Lie algebras, by proposition (4.1) ψ_G is a covering map. Since G is simply-connected, ψ_G must be an isomorphism. Thus we obtain a map $\phi : G \to H$ defined by $\phi = \psi_H \circ \psi_G^{-1}$. □

Corollary 4.8. *Any two simply-connected groups with isomorphic Lie algebras are isomorphic.*

Proof. Suppose G and H are simply-connected Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. By Proposition 4.7 an isomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ induces a map $\Phi : G \to H$. The inverse $\psi : \mathfrak{h} \to \mathfrak{g}$ also induces a map $\Psi : H \to G$. The uniqueness in Lemma 4.6 implies ϕ and ψ are double-sided inverses. □

Lie algebras coincide bijectively with simply-connected Lie groups, and any subgroup with a given Lie algebra can be realized as a quotient of its simply-connected representative by a discrete central subgroup. Able to extend any knowledge of simple Lie algebras to a description of simple Lie groups, we are now ready to turn our attention squarely on the theory of Lie algebras.

5. Fundamentals of Lie Algebras

We will restrict our attention to Lie algebras over the algebraically closed field $\mathbb{C}$. As an aside, the classification of simple real Lie algebras follows, albeit with some work, by considering complexifications, which we will not cover in the limited scope of this paper. To begin, we consider typical constructions with Lie algebras.

Definition 5.1. Given Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, their *direct sum* $\mathfrak{g} \oplus \mathfrak{h}$ consists of the vector space direct sum with a bracket operation restricting to the original brackets on $\mathfrak{g}$ and $\mathfrak{h}$ and satisfying $[\mathfrak{g}, \mathfrak{h}] = 0$.

Definition 5.2. Let $\mathfrak{h} \subset \mathfrak{g}$ be an ideal and $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ denote projection onto the vector space quotient. Then the bracket $[\pi(x), \pi(y)] = \pi[x, y]$ is a well-defined Lie algebra bracket. This defines the *quotient algebra* $\mathfrak{g}/\mathfrak{h}$.

Familiar isomorphism theorems from the theory of groups and rings have analogs for Lie algebras. For example, given a Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$, the kernel $\ker(\phi)$ is an ideal and $\phi(\mathfrak{g}) \simeq \mathfrak{g}/\ker(\phi)$. Also, if α and β are ideals, then $(\alpha + \beta)/\alpha \simeq \beta/\alpha \cap \beta$.

It will be useful to consider other classes of Lie algebras which naturally arise in the classification of simple Lie algebras.

Definition 5.3. Define the *commutator series* of $\mathfrak{g}$ as follows: let $\mathfrak{g}^0 = \mathfrak{g}$, $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$, ... , $\mathfrak{g}^{n+1} = [\mathfrak{g}^n, \mathfrak{g}^n]$. We say $\mathfrak{g}$ is *solvable* if $\mathfrak{g}^n = 0$ for some n.

Definition 5.4. Define the *lower central series* of $\mathfrak{g}$ as follows: let $\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$, ... , $\mathfrak{g}_{n+1} = [\mathfrak{g}, \mathfrak{g}_n]$. We say $\mathfrak{g}$ is *nilpotent* if $\mathfrak{g}_n = 0$ for some n.

Since by induction $\mathfrak{g}^i \subset \mathfrak{g}_i$, any nilpotent Lie algebra is also solvable. It is also easy to see that any subalgebra or homomorphic image of a nilpotent (resp. solvable) Lie algebra is nilpotent (resp. solvable).

Example 5.5. For any vector space V, the space of endomorphisms $End(V)$ possesses a Lie algebra bracket given by the commutator. The Heisenberg algebra consisting of strictly upper triangular matrices is nilpotent. The Borel algebra consisting of upper triangular matrices is solvable but not nilpotent

Definition 5.6. A Lie algebra $\mathfrak{g}$ is *semisimple* if it contains no nontrivial solvable ideals.

It is evident that any simple Lie algebra is also semisimple, and any semisimple Lie algebra cannot also be solvable or nilpotent. The *center* of $\mathfrak{g}$ is the subspace consisting of all elements x such that $[x, y] = 0$ for all y in $\mathfrak{g}$. Since the center is clearly a solvable ideal, any semisimple Lie algebra has trivial center.

Example 5.7. If $\mathfrak{g}_1, \mathfrak{g}_2$ are simple Lie algebras, then it is easy to check $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ contains only the ideals 0, $\mathfrak{g}_1$, $\mathfrak{g}_2$, and $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, which implies it is semisimple. Similarly, any direct sum of simple Lie algebras is semisimple.

We invoke several classical results about solvable, nilpotent, simple, and semisimple Lie algebras. First, it is natural to consider the adjoint map $ad_{\mathfrak{g}} : \mathfrak{g} \to End(\mathfrak{g})$ defined by $ad(x)(y) = ad[x, y]$. As before, we shall denote $ad_{\mathfrak{g}}(x)$ as $ad_{\mathfrak{g}}x$. Also, we omit the subscript $\mathfrak{g}$ unless otherwise ambiguous.

Proposition 5.8. *The map ad is a Lie algebra homomorphism. Thus a Lie algebra $\mathfrak{g}$ is nilpotent or solvable if and only if $ad(\mathfrak{g})$ is nilpotent or solvable.*

Proof. Bilinearity of the bracket implies ad is linear. To show that ad preserves the bracket we use antisymmetry and the Jacobi identity. Given $x, y, z \in \mathfrak{g}$,

$$\begin{aligned}
[adx, ady](z) &= (adxady - adyadx)(z) \\
&= [x, [y, z]] + [y, [x, z]] \\
&= -[z, [x, y]] \\
&= [[x, y], z] \\
&= ad[x, y](z)
\end{aligned}$$

One can show by induction $ad\mathfrak{g}^n = (ad\mathfrak{g})^n$ and $ad\mathfrak{g}_n = (ad\mathfrak{g})_n$. Noting that the kernel of ad is the center of $\mathfrak{g}$, which is trivially nilpotent and solvable, this implies the result. □

Theorem 5.9. *(Engel) Let V be a vector space. If $\mathfrak{g} \subset End(V)$ is a subalgebra consisting of nilpotent endomorphisms, then there exists a common eigenvector $v \neq 0 \in V$ such that $\mathfrak{g}(v) = 0$. Thus $\mathfrak{g}$ is nilpotent, and there exists a basis of V for which all elements of $\mathfrak{g}$ are strictly upper triangular.*

Proof. See page 46 of [6]. □

Corollary 5.10. *If adx is nilpotent for every x in $\mathfrak{g}$ then $\mathfrak{g}$ is nilpotent.*

Proof. By Engel's theorem, there exists a basis in which $ad\mathfrak{g}$ consists of strictly upper triangular matrices. Since the bracket of two matrices is given by their commutator, it is clear from inspection that $ad(\mathfrak{g})$ is nilpotent. By proposition (5.8) $\mathfrak{g}$ is also nilpotent. □

Theorem 5.11. *(Lie) Let $V \neq 0$ be a vector space. If $\mathfrak{g} \subset End(V)$ is solvable, then there exists a common eigenvector $v \neq 0 \in V$ such that for each $x \in \mathfrak{g}$ there exists a scalar $\lambda(x) \in \mathbb{C}$ with $x(v) = \lambda(x)v$. Hence there exists a basis of V for which all elements of $\mathfrak{g}$ are upper triangular.*

Proof. See page 42 of [6]. □

Corollary 5.12. *If $\mathfrak{g}$ is solvable, then adx is nilpotent for any x in $[\mathfrak{g}, \mathfrak{g}]$. Thus $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.*

Proof. Since $\mathfrak{g}$ is solvable, so is $ad\mathfrak{g}$. By Lie's theorem there is a basis in which all the elements of $ad\mathfrak{g}$ are upper triangular. Clearly the commutator of any two upper triangular matrices is strictly upper triangular, which implies any element of $[ad(\mathfrak{g}), ad(\mathfrak{g})] = ad[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. Then $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent by Engel's theorem. □

6. Semisimple Lie Algebras and the Killing Form

We now turn to studying the structure of semisimple Lie algebras by their finite-dimensional representations. We begin by considering a class of endomorphisms which behave nicely with respect to the bracket operation.

Definition 6.1. For a Lie algebra $\mathfrak{g}$, a *derivation* is a map $D \in End(\mathfrak{g})$ satisfying $D[x, y] = [Dx, y] + [x, Dy]$. Denote the subspace of derivations as $Der(\mathfrak{g})$.

A straightforward computation shows that the bracket of two derivations is also a derivation, from which it follows that the subspace of derivations is a subalgebra of $End(\mathfrak{g})$.

An especially important subalgebra of the space of derivations is given by the adjoint map $ad : \mathfrak{g} \to End(\mathfrak{g})$. Indeed, for any $x, y, z \in \mathfrak{g}$ the Jacobi identity and antisymmetry of the bracket imply

$$[z, [x, y]] = [[z, x], y] + [x, [z, y]],$$

which shows adz is a derivation. We also showed in proposition (5.8) that ad is a Lie algebra homomorphism from $\mathfrak{g}$ to $End(\mathfrak{g})$.

Observe that the adjoint map allows a Lie algebra to act on itself. We can generalize this action by considering any finite-dimensional vector space V and a Lie algebra homomorphism $\phi : \mathfrak{g} \to End(V)$.

Definition 6.2. Such a pair (V, ϕ) is a *representation* of the Lie algebra $\mathfrak{g}$.

We often suppress the map ϕ. To simplify matters we would like to have a way of decomposing any arbitrary representation into elementary pieces. An *invariant subspace* W of a representation V is a subspace preserved under the action of $\mathfrak{g}$. We say that a representation V is *irreducible* if it has no proper, non-trivial subrepresentations. A representation V is *completely reducible* if there exist irreducible invariant subspaces U_i such that $V = \bigoplus_i U_i$. Note that the representation of a direct sum is the direct sum of the individual representations. That is, for representations V and W, an element $x \in \mathfrak{g}$ acts on $(v, w) \in V \oplus W$ by $x(v, w) = (xv, xw)$.

A theorem due to Weyl shows that representations are well-behaved for semisimple Lie algebras.

Theorem 6.3. *(Weyl) Any representation of a semisimple Lie algebra is completely reducible.*

Proof. See page 481 of [2]. □

The preceding theorem reduces the task of studying the representations of a semisimple Lie algebra $\mathfrak{g}$ to merely identifying the irreducible representations. Because of this we shall restrict our attention to semisimple Lie algebras. Since we are only interested in representations of semisimple Lie algebras, we would like to have a means of identifying them. A useful tool for recognizing and working with semisimple Lie algebras is the Killing form.

Definition 6.4. Define the *Killing form* $K(\cdot, \cdot)$ on a Lie algebra $\mathfrak{g}$ by $K(x, y) = Tr(adxady)$.

The Killing form is clearly a symmetric bilinear form due to the bilinearity of *ad* and the identity $Tr(xy) = Tr(yx)$ for transformations x and y. By the same token, one can show it also satisfies associativity with respect to the bracket operation, or $K([x, y], z) = K(x, [y, z])$ for all $x, y, z \in \mathfrak{g}$.

Definition 6.5. Given a bilinear form C on a vector space V, denote the *radical* $radC = \{v \in V \mid C(v, u) = 0 \text{ for all } u \in V\}$. A form C is *nondegenerate* if $radC = 0$.

Since a semisimple Lie algebra is defined as having no solvable nontrivial ideals, we first use the Killing form to characterize solvable Lie algebras.

Theorem 6.6. *(Cartan's Solvability Criterion) A Lie algebra $\mathfrak{g}$ is solvable if and only if its Killing form K satisfies $K(x, y) = 0$ for any $x \in \mathfrak{g}$ and $y \in [\mathfrak{g}, \mathfrak{g}]$.*

Proof. See page 50 of [6]. □

Corollary 6.7. *For any Lie algebra $\mathfrak{g}$, $radK$ is a solvable ideal of $\mathfrak{g}$.*

Proof. First we show $radK$ is an ideal. If $h \in radK$ and $x, y \in \mathfrak{g}$ then by associativity $K([h, x], y) = K(h, [x, y]) = 0$, which implies $[h, x] \in radK$.

To show $radK$ is solvable, let K' denote the Killing form on $radK$. Let $\mathfrak{s}$ denote the subspace spanned by the added vectors so that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$ as a vector space. Let $x \in radK$. Since $radK$ is an ideal, then with respect to this ordered decomposition we have

$$adx = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}$$

Thus if x and y are in $radK$ then $K'(x, y) = Tr(adxady \mid_{radK}) = Tr(adxady) = K(x, y) = 0$. By Cartan's solvability criterion, $radK$ is solvable. □

Theorem 6.8. *(Cartan's Semisimplicity Criterion) A Lie algebra $\mathfrak{g}$ is semisimple if and only if its Killing form is nondegenerate.*

Proof. If $\mathfrak{g}$ is semisimple then K is nondegenerate by the preceding corollary.

Suppose K is nondegenerate and $\mathfrak{g}$ has a nontrivial solvable ideal β. Let n denote the smallest integer such that $\beta^n = 0$. Then $\alpha = \beta^{n-1}$ is a nontrivial abelian ideal. If $x \in \alpha$ and $y \in \mathfrak{g}$, then $adyadx$ sends $\mathfrak{g}$ to α and hence $(adyadx)^2 = 0$ since α is abelian, which implies $K(x, y) = Tr(adxady) = 0$. Thus $\alpha \subset radK$, contradicting the nondegeneracy of K. □

Finally we end with a structure theorem for semisimple Lie algebras which can be stated independently from the Killing form.

Theorem 6.9. *A Lie algebra $\mathfrak{g}$ is semisimple if and only if $\mathfrak{g} = \bigoplus_i g_i$ for simple Lie algebras g_i.*

Proof. If $\mathfrak{g} = \bigoplus_i g_i$ for simple Lie algebras g_i, it is easy to check that the ideals are of the form $\alpha = \bigoplus_k g_{i_k}$, or sums of some of the g_i, and these ideals are clearly not solvable.

Suppose $\mathfrak{g}$ is semisimple. We induct on the dimension of $\mathfrak{g}$. Let α denote a minimal nonzero ideal. Form the subspace $\alpha^\perp$ relative to the Killing form. Then $\alpha^\perp$ is an ideal by associativity of the Killing form.

Now $\alpha \cap \alpha^\perp$ is an ideal of $\mathfrak{g}$ on which the restriction of the Killing form is trivial. By Cartan's solvability criterion, $\alpha \cap \alpha^\perp$ is solvable and hence 0 since $\mathfrak{g}$ is semisimple. Since the Killing form is nondegenerate on both $\mathfrak{g}$ and α, elementary linear algebra shows $\mathfrak{g} = \alpha \oplus \alpha^\perp$ as a vector space. Clearly this is also a direct sum of Lie algebras.

The orthogonal decomposition shows that any ideal of α is also an ideal of $\mathfrak{g}$, and similarly for $\alpha^\perp$. Since α is a minimal ideal in $\mathfrak{g}$, this implies α is simple. Also, $\alpha^\perp$ may not contain any solvable ideals, so we may apply the inductive hypothesis to $\alpha^\perp$. □

As a corollary, the structure theorem for semisimple Lie algebras implies that a quotient of a semisimple Lie algebra is also semisimple. By the isomorphism theorem, this implies that a homomorphic image of any semisimple Lie algebra is also semisimple. We can use this to identify structures of semisimple Lie algebras which are invariant under representations. In particular, we obtain a decomposition of elements of $\mathfrak{g}$ analogous to the Jordan canonical form. First we recall the Jordan canonical form from standard linear algebra.

Theorem 6.10. *(Jordan Canonical Form) A linear endomorphism x of a vector space V over an algebraically closed field can be decomposed uniquely as a sum $x = x_s + x_n$ such that x_s is diagonalizable, x_n is nilpotent, and x_s and x_n commute.*

Proof. See page 217 of [4]. □

We call x_s and x_n the semisimple and nilpotent parts of x.

Proposition 6.11. *(1) For an element x of a semisimple Lie subalgebra $\mathfrak{g}$ of $End(V)$, the nilpotent and semisimple parts of x are contained in $\mathfrak{g}$.*

(2) For a semisimple Lie algebra $\mathfrak{g}$, an element $x \in \mathfrak{g}$ can be decomposed uniquely as a sum $x = x_s + x_n$ such that $adx_s = (adx)_s$ and $adx_n = (adx)_n$ and $[x_n, x_s] = 0$. This is the abstract Jordan form of x.

(3) If $\mathfrak{g}$ is a semisimple subalgebra of $End(V)$, then the abstract and usual Jordan forms coincide.

(4) For any semisimple Lie algebra $\mathfrak{g}$ *and representation* $\phi : \mathfrak{g} \to End(V)$, $\phi(x_s) = \phi(x)_s$ *and* $\phi(x_n) = \phi(x)_n$.

Proof. (1) See page 29 of [5].

(2) For an arbitrary semisimple Lie algebra $\mathfrak{g}$, recall that the kernel of $ad : \mathfrak{g} \to End(\mathfrak{g})$ is the center of $\mathfrak{g}$, which is trivial since $\mathfrak{g}$ is semisimple. For any x in $\mathfrak{g}$, apply the Jordan canonical form to adx to obtain $adx = (adx)_s + (adx)_n$. By the injectivity of the adjoint map and (1) applied to the semisimple Lie algebra $ad\mathfrak{g}$, there exist unique x_s and x_n in $\mathfrak{g}$ such that $adx_s = (adx)_s$ and $adx_n = (adx)_n$. Note $ad[x_n, x_s] = [adx_n, adx_s] = 0$ since adx_n and adx_s commute. Then $[x_n, x_s] = 0$ by the injectivity of ad.

(3) For an element x of a semisimple Lie algebra $\mathfrak{g} \subset End(V)$, denote the usual Jordan form $x = x_n + x_s$. It is easy to show since x_n is nilpotent, then so is adx_n. Similarly, adx_s is diagonalizable. Also, adx_n and adx_s commute since $[adx_s, adx_n] = ad[x_s, x_n] = 0$. This shows that $adx_s = (adx)_s$ and $adx_n = (adx)_n$. By the uniqueness of the abstract Jordan form, this implies that the usual and abstract Jordan forms coincide on $\mathfrak{g}$.

(4) Let $\phi : \mathfrak{g} \to End(V)$ be a representation of $\mathfrak{g}$. Note that $\phi(\mathfrak{g})$ is a semisimple subalgebra of $End(V)$, so we can consider the adjoint map $ad_{\phi(\mathfrak{g})} : \phi(\mathfrak{g}) \to End(\phi(\mathfrak{g}))$. For any $\phi(y)$ in $\phi(\mathfrak{g})$ the adjoint map satisfies

$$ad_{\phi(\mathfrak{g})}\phi(x_n)(\phi(y)) = [\phi(x_n), \phi(y)] = \phi[x_n, y] = \phi(ad_{\mathfrak{g}}x_n(y)),$$

from which it is clear that $ad_{\phi(\mathfrak{g})}\phi(x_n)$ is nilpotent and $ad_{\phi(\mathfrak{g})}\phi(x_s)$ is diagonalizable. Finally, $[\phi(x_n), \phi(x_s)] = \phi[x_n, x_s] = 0$ shows that $\phi(x_n)$ and $\phi(x_s)$ commute. We see that $ad_{\phi(\mathfrak{g})}\phi(x_n)$ and $ad_{\phi(\mathfrak{g})}\phi(x_s)$ are the nilpotent and diagonalizable parts of $ad_{\phi(\mathfrak{g})}\phi(x)$. Thus the abstract Jordan form of $\phi(x)$ is $\phi(x) = \phi(x_n) + \phi(x_s)$. Then (3) implies the result. □

Henceforth we will have no need to refer to the Killing form by the name K. Since the only bilinear that will concern us is the Killing form, there will be no ambiguity in abbreviating the Killing form $K(\cdot, \cdot)$ as $(\cdot, \cdot)$.

7. Representations of $\mathfrak{sl}_2(\mathbb{C})$

As an illustrative and useful example, we shall consider representations of $\mathfrak{sl}_2(\mathbb{C})$.

Let E_{ij} denote the matrix with 1 in the (i, j) position and 0 elsewhere. We will work in the basis given by $y = E_{12}, x = E_{21}$, and $h = E_{11} - E_{22}$. This basis satisfies the relations

$$[h, x] = 2x, [h, y] = -2y, [x, y] = h.$$

First we must check $\mathfrak{sl}_2(\mathbb{C})$ is semisimple. It is in fact simple, for each basis element generates the entire Lie algebra. Indeed, by the above calculation it is immediately obvious the subalgebra generated by h contains x and y, which then implies it also contains h. Also, the ideal generated by x or y contains h, so it includes the entire Lie algebra. Finally, given an arbitrary element $ax+by+ch$, then if $b \neq 0$ the subalgebra it generates contains $[[ax+by+ch, x], x] = [bh+2cx, x] = 2bx$, which generates the Lie algebra. Similarly if $a \neq 0$ then it contains $-2ay$, and if $c \neq 0$ it contains $-2cy$.

Now let $\phi : \mathfrak{sl}_n(\mathbb{C}) \to End(V)$ be an irreducible representation. The bracket relations show that h acts diagonally on $\mathfrak{g}$ by the adjoint representation. Hence $\phi(h)$ is diagonalizable by the preservation of Jordan form (proposition (6.11)), so V

decomposes as a direct sum of eigenspaces $V_\alpha = \{v \in V \mid h \cdot v = \alpha v\}$ for eigenvalues $\alpha \in \mathbb{C}$. We call the eigenvalues the *weights* and the eigenspaces the *weight spaces.*

We now consider how the other basis vectors act on the weights.

Lemma 7.1. *If $v \in V_\alpha$ then $xv \in V_{\alpha+2}$ and $yv \in V_{\alpha-2}$.*

Proof.

$$\begin{aligned} h(x(v)) &= x(h(v)) + [h,x]v \\ &= \alpha x(v) + 2x(v) \\ &= (\alpha+2)x(v). \end{aligned}$$

A similar argument applies to the action of y. □

Corollary 7.2. *The weights form an unbroken string of values α, $\alpha+2$,...,$\alpha+2m$ starting from some initial weight $\alpha \in \mathbb{C}$.*

Proof. Fix a weight α. The subspace $W = \bigoplus_{n\in\mathbb{Z}} V_{\alpha+2n}$ is invariant under $\mathfrak{g}$. But V is irreducible. □

Since V is finite-dimensional, there is a maximal weight β occurring in the string. Choose $v \in V_\beta$. Then $xv = 0$ and $y^n v \in V_{\beta-2n}$. Let k denote the largest positive integer such that $\beta - 2k$ is a weight.

Lemma 7.3. *The subspace W spanned by the vectors $\{v, yv, ..., y^k v\}$ is invariant under the action of $\mathfrak{g}$. Thus this collection of vectors comprises a basis for V.*

Proof. We shall show the action by each basis vector of $\mathfrak{g}$ preserves W. First, it is trivial to check $y(y^n(v)) = y^{n+1}(v)$ and $h(y^n v) = (\beta - 2n)y^n v$.

The only case requiring some care is the action of x. We shall show by induction

$$x(y^n v) = n(\beta - n + 1)y^{n-1}v.$$

Recall $xv = 0$, which satisfies the base case. Assuming the statement holds for n,

$$\begin{aligned} x(y^{n+1}v) &= xy(y^n v) \\ &= ([x,y] + yx)y^n v \\ &= h(y^n v) + y(xy^n v) \\ &= (\beta - 2n)y^n v + y(n(\beta - n + 1)y^{n-1}v) \\ &= (n+1)(\beta - (n+1) + 1)y^n v. \end{aligned}$$

□

The proof of the preceding lemma provides a comprehensive description of the representation. Therefore a representation is determined by its maximum weight β. To determine β, we see

$$0 = x(y^{k+1}v) = (k+1)(\beta - k)y^k v,$$

which implies $\beta = k \in \mathbb{N}$.

To summarize the result:

Proposition 7.4. *Any irreducible representation V of $\mathfrak{sl}_2(\mathbb{C})$ is of the form $V = \bigoplus_{n=0}^{\beta} V_{-\beta+2n}$, determined by the maximal weight $\beta \in \mathbb{N}$.*

Corollary 7.5. *The number of irreducible representations U_i in the decomposition of an arbitrary representation $V = \bigoplus_i U_i$ of $\mathfrak{sl}_2(\mathbb{C})$ is the sum of the dimensions of the 0 and 1 weight spaces in the decomposition $V = \bigoplus V_\alpha$.*

8. Root Space Decomposition of Semisimple Lie Algebras

We can generalize some of this procedure to determine the representations of any semisimple Lie algebra $\mathfrak{g}$. We shall be particularly concerned with the action of $\mathfrak{g}$ on itself by the adjoint representation. As in the case of $\mathfrak{sl}_2(\mathbb{C})$, we begin by finding a subalgebra which acts on $\mathfrak{g}$ diagonally. From this there arises a decomposition of $\mathfrak{g}$ into its weight spaces called the root space decomposition. Motivated by the question of to what extent a Lie algebra is determined by its roots, in this section we shall utilize the structure of representations of $\mathfrak{sl}_2(\mathbb{C})$ to determine key properties of the roots of semisimple Lie algebras.

We commence with the observation that if $\mathfrak{g}$ contains only nilpotent elements, then by Engel's theorem $\mathfrak{g}$ is nilpotent and hence cannot be semisimple. Thus any semisimple Lie algebra contains semisimple elements. Since the span of a single semisimple element is an abelian subalgebra by antisymmetry of the bracket, $\mathfrak{g}$ must contain subalgebras consisting of semisimple elements. We call such a subalgebra a *toral subalgebra.*

Lemma 8.1. *A toral subalgebra $\mathfrak{h}$ is abelian.*

Proof. Let $x, y \in \mathfrak{h}$. Since $ad_{\mathfrak{h}}x$ is diagonalizable, it suffices to show $ad_{\mathfrak{h}}x$ has no nonzero eigenvalues. Suppose for a contradiction there exists an eigenvector y with $ad_{\mathfrak{h}}x(y) = [x, y] = ay$ for some $a \neq 0 \in \mathbb{C}$. Then by antisymmetry $ad_{\mathfrak{h}}y(x) = -ay$, and so $ad_{\mathfrak{h}}y(ad_{\mathfrak{h}}y(x)) = -a ad_{\mathfrak{h}}y(y) = 0$. On the other hand, since $ad_{\mathfrak{h}}y$ is diagonalizable, x is a linear combination of eigenvectors of $ad_{\mathfrak{h}}y$. One of the eigenvalues of y must be nonzero since $ad_{\mathfrak{h}}y(x) = -ay \neq 0$. But this contradicts $ady(ady(x)) = 0$. □

We call a maximal toral subalgebra a *Cartan subalgebra.* For example, $\mathfrak{sl}_2(\mathbb{C})$ has a Cartan subalgebra consisting of the span of $h = E_{11} - E_{22}$. By standard linear algebra, if $W \subset End(V)$ is a subspace consisting of commuting diagonalizable linear operators, then the elements of W are simultaneously diagonalizable. In particular, this implies the action of a Cartan subalgebra $\mathfrak{h}$ is simultaneously diagonalizable. Then $\mathfrak{g}$ admits a decomposition as a direct sum of eigenspaces $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)c\}$, for linear functionals α in the dual space $\mathfrak{h}^*$. In the special case of the adjoint representation, we call the nonzero eigenvalues α the *roots* of $\mathfrak{g}$ and the associated eigenspaces the *root spaces.* Denote the set of roots by R. We thus obtain the *root space decomposition* of the Lie algebra $\mathfrak{g}$:

$$\mathfrak{g} = C_{\mathfrak{g}}(\mathfrak{h}) \oplus (\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha),$$

where $C_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} \mid [x, h] = 0 \text{ for all } h \in \mathfrak{h}\}$ denotes the *centralizer* of $\mathfrak{h}$. Note $\mathfrak{h}$ is a subspace of $C_{\mathfrak{g}}(\mathfrak{h})$ since it is abelian. We now investigate properties of the roots with a view towards understanding how they characterize the Lie algebra.

Proposition 8.2. *(1) If α and β are roots, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.*
(2) If $x \in \mathfrak{g}_\alpha$ then $ad(x)$ is nilpotent.
(3) If $\alpha + \beta \neq 0$ then $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal under the Killing form.

Proof. (1) Let $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$. Then for $h \in \mathfrak{h}$ we have

$$[h, [x, y]] = [[h, x], y] + [x, [h, y]] = (\alpha + \beta)(h)[x, y]$$

(2) follows from (1) since adx translates a root space V_λ to $V_{\lambda+\alpha}$. Since $\mathfrak{g}$ is finite-dimensional, $V_{\lambda+n\alpha} = 0$ for some integer n, which implies $(adx)^n = 0$.

(3) Similarly, $adxady$ translates a root space V_γ to $V_{\gamma+(\alpha+\beta)}$. Since $V_{\gamma+n(\alpha+\beta)} = 0$ for some integer n, $(adxady)^n = 0$. □

In particular, since $\mathfrak{h}$ is orthogonal to $\mathfrak{g}_\alpha$ for each root α, the nondegeneracy of the Killing form on $\mathfrak{g}$ implies that its restriction to $C_\mathfrak{g}(\mathfrak{h})$ is also nondegenerate.

Proposition 8.3. *If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, then $C_\mathfrak{g}(\mathfrak{h}) = \mathfrak{h}$.*

Proof. We provide a sketch-proof. See page 36 of [5] for details.

The idea of the proof is to first show $C_\mathfrak{g}(\mathfrak{h})$ contains its semisimple and nilpotent parts. The maximality of $\mathfrak{h}$ implies that all the semisimple elements of $C_\mathfrak{g}(\mathfrak{h})$ are contained in $\mathfrak{h}$. We can also show that the nondegeneracy of the Killing form on $C_\mathfrak{g}(\mathfrak{h})$ implies that the restriction of the Killing form to $\mathfrak{h}$ is also nondegenerate. We then use this to prove $C_\mathfrak{g}(\mathfrak{h})$ is abelian. Thus if C is not contained in $\mathfrak{h}$, then it has a nonzero nilpotent element x. Note that if x and y are commuting linear transformations and x is nilpotent, then xy is also nilpotent, which implies $Tr(xy) = 0$. This shows x is orthogonal to $C_\mathfrak{g}(\mathfrak{h})$, contradicting the nondegeneracy of the Killing form. □

The nondegeneracy of the Killing form on $\mathfrak{h}$ provides an isomorphism between $\mathfrak{h}$ and its dual $\mathfrak{h}^*$. That is, for any $h \in \mathfrak{h}$ we obtain a functional α_h defined by $\alpha_h(x) = (h, x)$. Bilinearity of the form implies that the the map $h \mapsto \alpha_h$ is linear, and nondegeneracy implies that it is an isomorphism. Then any functional $\alpha \in \mathfrak{h}^*$ has a unique associated element $t_\alpha \in \mathfrak{h}$ satisfying $\alpha(h) = (t_\alpha, h)$ for any $h \in \mathfrak{h}$. We extend the Killing form on $\mathfrak{h}^*$ by defining $(\alpha, \beta) = (t_\alpha, t_\beta)$ for functionals α and β. This can also be written as $(\alpha, \beta) = \beta(t_\alpha)$ or $\alpha(t_\beta)$.

Now we aim to utilize the classification of the representations of $\mathfrak{sl}_2(\mathbb{C})$.

Proposition 8.4. *(1) The set of roots R spans the dual space $\mathfrak{h}^*$.*

(2) If α is a root, then $-\alpha$ is a root.

(3) The space $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is the 1-dimensional span of t_α.

(4) If α is a root, then $(\alpha, \alpha) \neq 0$.

(5) Given a root α and $x_\alpha \in \mathfrak{g}_\alpha$ there exists $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that x_α, y_α, and $h_\alpha = [x_\alpha, y_\alpha]$ span a three-dimensional subalgebra $\mathfrak{s}_\alpha$ of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

Proof. (1) Suppose there exists $h \in \mathfrak{h}$ such that $\alpha(h) = 0$ for each root α. Then h acts trivially on all the root spaces of $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, it must have trivial center, which implies $h = 0$.

(2) Given a nonzero vector x_α in $\mathfrak{g}_\alpha$, the nondegeneracy of the Killing form implies there exists an element y_α such that $(x_\alpha, y_\alpha) \neq 0$. Since $\mathfrak{g}_\alpha$ is orthogonal to $\mathfrak{g}_\beta$ for $\alpha + \beta \neq 0$, then y_α must be a nonzero element of $\mathfrak{g}_{-\alpha}$.

(3) Let $x_\alpha \in \mathfrak{g}_\alpha$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$. For any $h \in \mathfrak{h}$

$$(h, [x_\alpha, y_\alpha]) = ([h, x_\alpha], y_\alpha) = \alpha(h)(x_\alpha, y_\alpha) = (h, t_\alpha)(x_\alpha, y_\alpha)$$

which implies $(h, [x_\alpha, y_\alpha] - (x_\alpha, y_\alpha)t_\alpha) = 0$. Then $[x_\alpha, y_\alpha] = (x_\alpha, y_\alpha)t_\alpha$ by the nondegeneracy of the Killing form.

(4) Choose $x_\alpha \in \mathfrak{g}_\alpha$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[x_\alpha, y_\alpha] = (x_\alpha, y_\alpha)t_\alpha \neq 0$. By (3) this collection spans a three-dimensional subalgebra $\mathfrak{s}_\alpha$ of $\mathfrak{g}$. If $(t_\alpha, t_\alpha) = \alpha(t_\alpha) = 0$ then t_α acts trivially on $\mathfrak{s}_\alpha$, which implies $\mathfrak{s}_\alpha^1 = [\mathfrak{s}_\alpha, \mathfrak{s}_\alpha] = \mathbb{C}t_\alpha$ and $\mathfrak{s}_\alpha^2 = 0$. Since $\mathfrak{s}_\alpha$ is solvable, corollary (5.12) implies adt_α is nilpotent, contradicting the fact that t_α is a nonzero element of the Cartan subalgebra $\mathfrak{h}$.

(5) Since $\alpha(h_\alpha) \neq 0$ and hence $\alpha(t_\alpha) = (t_\alpha, t_\alpha) \neq 0$, by scaling appropriately we choose x_α and y_α so that the resulting basis for $\mathfrak{s}_\alpha$ has a bracket corresponding exactly to that of $\mathfrak{sl}_2\mathbb{C}$. That is, having chosen x_α scale y_α so that $(x_\alpha, y_\alpha) = \frac{2}{(t_\alpha, t_\alpha)}$. Then $h_\alpha = \frac{2t_\alpha}{(t_\alpha, t_\alpha)}$ and $\alpha(h_\alpha) = 2$. □

The utility of these subalgebras $\mathfrak{s}_\alpha$ is immediately apparent by our past calculation of all irreducible representations of $\mathfrak{sl}_2\mathbb{C}$. For example, we know that the weights of any representation of s_α must be a non-interrupted string of integers symmetric about the origin.

The symmetry about the the origin can be expressed in the fact that the roots are invariant under reflections. For a root α define a reflection W_α on $\mathfrak{h}^*$ by $W_\alpha(\beta) = \beta - \frac{2\beta(h_\alpha)}{\alpha(h_\alpha)}\alpha = \beta - \beta(h_\alpha)\alpha$. This is just the reflection of β over the hyperplane $\Omega_\alpha = \{\beta \in \mathfrak{h}^* \mid \beta(h_\alpha) = 0\}$. The group generated by these reflections is the *Weyl group*.

Proposition 8.5. *(1) The roots of $\mathfrak{g}$ are invariant under the Weyl group.*
(2) For any roots α and β, their inner-product $\beta(h_\alpha)$ is an integer.
(3) If α is a root then the multiples of α which are roots are $\pm\alpha$.
(4) Each of the root spaces $\mathfrak{g}_\alpha$ is one-dimensional.
(5) $\mathfrak{h}$ is spanned by the h_α. Thus $\mathfrak{g}$ is generated by the root spaces $\mathfrak{g}_\alpha$.

Proof. (1) For roots α and β, it suffices to show that the roots congruent to β modulo α are invariant under the reflection W_α associated to α. First observe that the subspace $W = \bigoplus_{n\in\mathbb{Z}} V_{\beta+n\alpha}$ is a representation for the subalgebra $\mathfrak{s}_\alpha$. Let

$$\beta + n\alpha, \beta + (n+1)\alpha, ..., \beta + m\alpha$$

denote the root string of the nontrivial root spaces. Then the root string for $\mathfrak{s}_\alpha$ consists of the string evaluated at h_α:

$$\beta(h_\alpha) + 2n, ..., \beta(h_\alpha) + 2m.$$

By symmetry about the origin, $\beta(h_\alpha) = -(m+n) \in \mathbb{Z}$. To show invariance of the roots,

$$\begin{aligned} W_\alpha(\beta + (n+k)\alpha) &= \beta + (n+k)\alpha - (\beta + (n+k)\alpha)(h_\alpha)\alpha \\ &= \beta + (n+k)\alpha - (-(m+n) + 2(n-k))\alpha \\ &= \beta + (m-k)\alpha \end{aligned}$$

(2) follows from the preceding calculation. It is also useful to note

$$\beta(h_\alpha) = (t_\beta, h_\alpha) = \left(t_\beta, \frac{2t_\alpha}{(t_\alpha, t_\alpha)}\right) = 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}.$$

(3) Suppose α is a root. Then consider the representation $V = \mathfrak{h} \oplus (\bigoplus_{\lambda\in\mathbb{C}} \mathfrak{g}_{\lambda\alpha})$ of $\mathfrak{s}_\alpha$. By Weyl's theorem, consider the decomposition into irreducible subrepresentations. For example, $\mathfrak{s}_\alpha$ is an irreducible subrepresentation of V. Also, $\mathfrak{s}_\alpha$ acts trivially on $\ker(\alpha) \subset \mathfrak{h}$, so any subspace of $\ker(\alpha)$ is a subrepresentation. Recall that an irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ has either even or odd integer weights symmetric about 0. Since $\mathfrak{h} = h_\alpha \oplus \ker(\alpha)$, if U is an irreducible representation of $\mathfrak{s}_\alpha$ with 0 as a weight, then it must contain an element of $\mathfrak{h}$. Thus either $U = \mathfrak{s}_\alpha$ if $U \cap \mathfrak{h}$ is contained in the span of h_α, or else $U \subset \ker(\alpha)$. This implies the only even weights are 0 and ± 2. Since 2α has weight 4 as a weight space of $\mathfrak{s}_\alpha$, it cannot be a root. This shows that in general, twice a root is never a root. Thus if α is a

root, then $\frac{1}{2}\alpha$ cannot also be a root. This implies that 1 cannot be a weight of V. Since any irreducible representation of $\mathfrak{s}_\alpha$ must have a 0 or 1 weight space, conclude $V = \ker(\alpha) \oplus \mathfrak{s}_\alpha$.

(4) follows from the fact that $\mathfrak{s}_\alpha$ is the only nontrivial subrepresentation with 0 as a weight.

(5) Since h_α and t_α have the same span, it suffices to prove the claim for the t_α. If the t_α do not span $\mathfrak{h}$ then they lie in the kernel of some linear functional ϕ. But then $0 = \phi(t_\alpha) = \alpha(t_\phi)$ for each root α, which contradicts the fact that the roots span the dual $\mathfrak{h}^*$. □

Proposition 8.6. *The Killing form is positive-definite on the roots and therefore positive-definite on the real subspace spanned by the roots in* $\mathfrak{h}^*$*.*

Proof. First we shall show that the Killing form is positive-definite on each $h_\alpha \in \mathfrak{h}$. Then this will extend to $\mathfrak{h}^*$ since for a root $\alpha \neq 0$

$$(\alpha, \alpha) = (t_\alpha, t_\alpha) = \left(\frac{2h_\alpha}{(h_\alpha, h_\alpha)}, \frac{2h_\alpha}{(h_\alpha, h_\alpha)} \right) = \frac{4}{(h_\alpha, h_\alpha)}.$$

Now, $(h_\alpha, h_\alpha) = Tr(adh_\alpha adh_\alpha)$. We shall sum over the root spaces. For x_β spanning $\mathfrak{g}_\beta$ we have $adh_\alpha(adh_\alpha(x_\beta)) = adh_\alpha(\beta(h_\alpha)x_\beta) = \beta(h_\alpha)^2$. Thus $Tr(adh_\alpha adh_\alpha) = \sum_{\beta \in R} \beta(h_\alpha)^2$. Since $\beta(h_\alpha) \in \mathbb{Z}$, each of the summands is non-negative. The sum equals 0 if and only if $\beta(h_\alpha) = 0$ for each root β. Since the roots span $\mathfrak{h}^*$, this happens precisely when $\alpha = 0$. □

9. Root Systems

The key properties satisfied by the roots of semisimple Lie algebras severely limit their possible geometries. They motivative the definition of an abstract structure called a root system. In this section we introduce root systems and their basic properties with the goal of establishing a correspondence between root systems and Lie algebras.

Definition 9.1. A *root system* is a collection of roots R in a Euclidean space E satisfying the following properties:

(1) R is finite and spans E.

(2) if α is a root, then the multiples of α which are roots are $\pm\alpha$.

(3) the roots are invariant under the Weyl group; in other words, for roots α and β, the reflection $W_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$ is a root.

(4) for roots β and α, the number

$$n_{\beta\alpha} = 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}$$

is an integer.

We investigate the possible geometries of root systems. If θ is the angle between roots α and β then $n_{\beta\alpha} = 2cos\theta\frac{||\beta||}{||\alpha||}$ and thus $n_{\alpha\beta}n_{\beta\alpha} = 4cos^2\theta$ is an integer. Suppose $||\beta|| \geq ||\alpha||$. Omitting the case when $\cos\theta = \pm 1$, which occurs if and only $\beta = \pm\alpha$, we obtain the following possibilities for the angle and size ratio between

α and β:

$cos\theta$	θ	$n_{\beta\alpha}$	$\frac{\|\|\beta\|\|}{\|\|\alpha\|\|}$
$\frac{\sqrt{3}}{2}$	$\frac{\pi}{6}$	3	$\sqrt{3}$
$\frac{\sqrt{2}}{2}$	$\frac{\pi}{4}$	2	$\sqrt{2}$
$\frac{1}{2}$	$\frac{\pi}{3}$	1	1
0	$\frac{\pi}{2}$	0	$*$
$\frac{-1}{2}$	$\frac{2\pi}{3}$	-1	1
$\frac{-\sqrt{2}}{2}$	$\frac{3\pi}{4}$	-2	$\sqrt{2}$
$\frac{-\sqrt{3}}{2}$	$\frac{5\pi}{6}$	-3	$\sqrt{3}$

Since the roots are symmetrical with respect to the origin, it is reasonable to partition them by choosing a direction, or a linear functional $l : E \to \mathbb{R}$ irrational with respect to the roots. Let R^+ denote the *positive roots* α satisfying $l(\alpha) > 0$, and likewise let R^- denote the *negative roots*. We say that a positive (resp. negative) root is *simple* if it cannot be written as the sum of positive (resp. negative) roots.

Example 9.2. We shall find the root system associated to $\mathfrak{sl}_n(\mathbb{C})$.

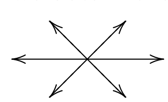
The Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ has a basis consisting of the $n^2 - 1$ matrices E_{ij} for $i \neq j$ and $H_{i(i+1)} = E_{ii} - E_{(i+1)(i+1)}$ for $1 \leq i \leq n-1$. Let $\mathfrak{h}$ denote the subalgebra spanned by the matrices $H_{i(i+1)}$. For example, in the case of $\mathfrak{sl}_2(\mathbb{C})$ we let $\mathfrak{h}$ denote the span of the single diagonal matrix h. It is easy to see this is a Cartan subalgebra.

Let $L_k : \mathfrak{sl}_n(\mathbb{C}) \to \mathbb{C}$ denote the linear functional corresponding to E_{kk}. The dual of $\mathfrak{h}$ is spanned by linear combinations $\sum_{i=1}^n a_i L_i$ such that $\sum_{i=0}^n a_i = 0$. By inspection we see $[H_{ij}, E_{km}] = (L_k - L_m)(H_{ij})E_{km}$, which implies the roots are $L_k - L_m$ for all combinations $1 \leq k \neq m \leq n$.

We shall choose a direction defined by the functional $l(\sum_{i=1}^n a_i L_i) = \sum_{i=1}^n b_i a_i$ with $\sum_{i=0}^n b_i = 0$ and $b_1 > b_2 > ... > b_n$. Then the positive roots are $L_k - L_r$ with $k > r$. We see that the simple positive roots are $L_1 - L_2, L_2 - L_3, ..., L_{n-1} - L_n$. Pairs of consecutive simple roots are separated by the angle $\frac{2\pi}{3}$, and all other pairs are orthogonal.

In the case of $\mathfrak{sl}_2(\mathbb{C})$ there is a single positive root $L_2 - L_1$ which spans the one-dimensional dual space. We can represent the root space pictorially:

In the case of $\mathfrak{sl}_3(\mathbb{C})$ there are three positive roots $L_3 - L_2$, $L_3 - L_1$, and $L_2 - L_1$. The simple roots $L_3 - L_2$ and $L_2 - L_1$ are offset by the angle $\frac{2\pi}{3}$, and their sum provides the remaining positive root. Thus we obtain the following root space diagram:



We can think of simple roots as a generating set for the root system, as clarified by the following proposition.

Proposition 9.3. *(1) If α, β are roots with $\beta \neq \pm\alpha$ then the α-string through β given by $\beta - p\alpha, ..., \beta - \alpha, \beta, \beta + \alpha, ..., \beta + q\alpha$ has at most four elements in a string, and $p - q = n_{\beta\alpha}$.*

(2) If α, β are roots with $\beta \neq \pm\alpha$ then $(\beta, \alpha) > 0$ if and only if $\alpha - \beta$ is a root, $(\beta, \alpha) < 0$ if and only if $\alpha + \beta$ is a root, and $(\beta, \alpha) = 0$ if and only if $\alpha - \beta$ and $\alpha + \beta$ are simultaneously roots or nonroots.

(3) If α and β are distinct simple roots, then $\alpha - \beta$ and $\beta - \alpha$ are not roots.

(4) The angle between two distinct simple roots cannot be acute.

(5) The simple roots are linearly independent.

(6) The simple positive roots constitute a basis for the underlying Euclidean space E. Each positive root can be written uniquely as a non-negative integral linear combination of simple roots.

(7) Given simple roots α, any positive root β can be written in the form $\beta = \alpha_{i_1} + ... + \alpha_{i_k}$ where $\alpha_{i_1} + ... + \alpha_{i_m}$ is a positive root for each $1 \leq m \leq k$.

Proof. (1) The reflection W_α satisfies

$$\beta - p\alpha = W_\alpha(\beta + q\alpha) = (\beta - n_{\beta\alpha}) - q\alpha,$$

which shows $n_{\beta\alpha} = p - q$. The bound on $p+q$ follows from the constraint $|n_{\beta\alpha}| \leq 3$.

(2) follows immediately from (1).

(3) follows from the definition of simple roots, for if $\alpha - \beta$ is a root then $\beta = \alpha + (\beta - \alpha)$ is not simple. Similarly, $\beta - \alpha$ is not a root.

(4) follows immediately from (2) and (3).

(5) From basic linear algebra, a set of vectors lying on one side of a hyerplane with all mutual angles at least 90 degrees is linearly independent.

(6) Since the roots span E and the simple roots are linearly independent, the simple roots constitute a basis for E. Suppose for a contradiction β is a positive root with minimal $l(\beta)$ that cannot be written as a nonnegative integral linear combination of simple roots. Then since β is not itself simple, we can write $\beta = \lambda + \gamma$, where λ and γ are positive roots. But then $l(\lambda) < l(\beta)$.

(7) By (6), any positive root β can be written as a sum $\beta = \alpha_{i_1} + ... + \alpha_{i_k}$. Suppose the result holds for k, and let $\beta = \alpha_{i_1} + ... + \alpha_{i_{k+1}}$. Then $0 < (\beta, \beta) = \sum_{m=0}^{n} (\beta, \alpha_{i_m})$, so $(\beta, \alpha_{i_m}) > 0$ for some α_{i_m}. By (2) this implies $\beta - \alpha_{i_m}$ is a positive root. □

It is natural to consider to what extent a Lie algebra can be recovered from its root system. In the proceeding discussion let $\mathfrak{g}$ denote a Lie algebra with a fixed Cartan subalgebra $\mathfrak{h}$.

Lemma 9.4. *Let α and $\beta \neq \pm\alpha$ be roots, and suppose $\alpha + \beta$ is also a root. Then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.*

Proof. Since the root spaces are one-dimensional, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta]$ returns either $\mathfrak{g}_{\alpha+\beta}$ or 0. In the latter case, consider the representation $W = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_{\beta + n\alpha}$ of $\mathfrak{s}_\alpha$. Note W does not include $\mathfrak{h}$, for otherwise there exists an integer k such that $\beta = k\alpha$, which contradicts $\beta \neq \pm\alpha$. Since the root spaces are one-dimensional, so is each summand in W. By corollary (7.5) W is an irreducible representation. But if $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$ then $W = \bigoplus_{n \leq 0} \mathfrak{g}_{\beta + n\alpha}$ is a proper subrepresentation, yielding a contradiction. □

Since any positive root β can be written as a sum of primitive roots $\beta = \alpha_{i_1} + ... + \alpha_{i_k}$ such that each truncated sum $\alpha_{i_1} + ... + \alpha_{i_m}$ is a root, the preceding lemma shows the root spaces $\mathfrak{g}_\alpha$ of the simple roots generate all the root spaces and thus the entire Lie algebra by proposition (8.5).

An isomorphism of root spaces $R \subset E$ and $R' \subset E'$ is a linear isomorphism $\phi : E \to E'$ such that $(\phi(\alpha), \phi(\beta)) = (\alpha, \beta)$ for each pair of roots α and β in R. We are now prepared for the isomorphism theorem:

Proposition 9.5. *Given simple Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ with Cartan subalgebras $\mathfrak{h}$ and $\mathfrak{h}'$ and directions on the roots given by linear functionals l and l', their root systems $R \subset E$ and $R' \subset E'$ are isomorphic if and only if $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic.*

Proof. Recall that for each root α of $\mathfrak{g}$ there is a corresponding subalgebra $\mathfrak{s}_\alpha$ spanned by $x_\alpha \in \mathfrak{g}_\alpha$, $y_\alpha \in \mathfrak{g}_\alpha$ and $h_\alpha = [x_\alpha, y_\alpha]$ determined by the scalar multiple of x_α. Suppose $\phi : E \to E'$ is an isomorphism of root spaces. Then ϕ maps simple positive roots α_i bijectively to simple positive roots α'_i, inducing an isomorphism $\psi : \mathfrak{h} \to \mathfrak{h}'$ taking the corresponding h_i to h'_i. Choose x_i and x'_i in the root spaces of $\mathfrak{g}$ and $\mathfrak{g}'$ corresponding to the simple roots. We shall show there exists an isomorphism $\Psi : \mathfrak{g} \to \mathfrak{g}'$ extending ψ and mapping each x_i to x'_i.

We construct the isomorphism by considering the diagonal subalgebra $\tilde{\mathfrak{g}}$ of $\mathfrak{g} \oplus \mathfrak{g}'$ generated by each $\tilde{h}_i = h_i \oplus h'_i$, $\tilde{x}_i = x_i \oplus x'_i$, and $\tilde{y}_i = y_i \oplus y'_i$. We shall show the two projections from $\tilde{\mathfrak{g}}$ to $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphisms. The kernel of the second projection is of the form $\mathfrak{t} \oplus 0$, where $\mathfrak{t}$ is an ideal of $\mathfrak{g}$ and hence must be either 0 or $\mathfrak{g}$ by the simplicity of $\mathfrak{g}$. It is easy to see the latter case would imply $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}'$.

Suppose for a contradiction $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}'$. Take a maximal positive root $\beta \in \mathfrak{g}$ such that $\beta + \alpha$ is not a root for any simple root α. Take nonzero vectors x_β and $x'_{\beta'}$ in the corresponding maximal root spaces, and let $\tilde{x}_\beta = x_\beta \oplus x'_{\beta'}$. Let W be the subspace of $\tilde{\mathfrak{g}}$ obtained by successively applying all the $\tilde{y}_i$'s via the adjoint representation. Then $W \cap (\mathfrak{g}_\beta \oplus \mathfrak{g}'_\beta)$ is one-dimensional, spanned by the vector $\tilde{x}_\beta$. On the other hand, $\tilde{\mathfrak{g}}$ preserves W. Indeed, it is immediately clear each $\tilde{y}_i$ preserves W. For $\tilde{h}_i$, first note that $ad\tilde{h}_i \tilde{x}_\beta = 2\tilde{x}_\beta \in W$. Supposing that $ad\tilde{h}_i$ sends $\tilde{w} = ad\tilde{y}_{j_n} ... ad\tilde{y}_{j_1}(\tilde{x}_\beta)$ to W, then since ad is a derivation we have

$$\begin{aligned} ad\tilde{h}_i ad\tilde{y}_{j_{n+1}}(\tilde{w}) =& [\tilde{h}_i, [\tilde{y}_{j_{n+1}}, \tilde{w}] \\ &= [[\tilde{h}_i, \tilde{y}_{j_{n+1}}], \tilde{w}] + [\tilde{y}_{j_{n+1}}, [\tilde{h}_i, \tilde{w}]] \\ &= -2[\tilde{y}_{j_{n+1}}, \tilde{w}] + [\tilde{y}_{j_{n+1}}, [\tilde{h}_i, \tilde{w}]] \in W. \end{aligned}$$

By induction on n, $ad\tilde{h}_i$ preserves W. For $\tilde{x}_i$, the maximality of β and β' implies $ad\tilde{x}_i(\tilde{x}_\beta) = 0 \in W$. Suppose $\tilde{x}_i$ sends $\tilde{w} = ad\tilde{y}_{j_n} ... ad\tilde{y}_{j_1}(\tilde{x}_\beta)$ to W. First note that proposition (9.4) implies $[\tilde{x}_i, \tilde{y}_j] = 0$ for any $j \neq i$ since $\alpha_i - \alpha_j$ and $\alpha'_i - \alpha'_j$ are not roots. Thus $ad\tilde{x}_i$ and $ad\tilde{y}_j$ commute when $i \neq j$, so it suffices to consider the case when $j_{n+1} = i$. By the derivation property,

$$\begin{aligned} ad\tilde{x}_i ad\tilde{y}_i(\tilde{w}) &= [\tilde{x}_i, [\tilde{y}_i, \tilde{w}]] \\ &= [[\tilde{x}_i \tilde{y}_i], \tilde{w}] + [\tilde{y}_i, [\tilde{x}_i, \tilde{w}]] \\ &= [\tilde{h}_i, \tilde{w}] + [\tilde{y}_i, [\tilde{x}_i, \tilde{w}]] \in W. \end{aligned}$$

By induction, $ad\tilde{x}_i$ preserves W, proving that $\tilde{g}$ preserves W. Now, to say that $\tilde{g}$ preserves W is to say that W is an ideal of $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}'$. Since W is clearly not 0, $\mathfrak{g} \oplus 0$ or $0 \oplus \mathfrak{g}'$, it must equal $\mathfrak{g} \oplus \mathfrak{g}'$. But this implies $W \cap (\mathfrak{g}_\beta \oplus \mathfrak{g}'_\beta)$ is two-dimensional, yielding a contradiction.

The opposite implication largely consists in showing that the choice of Cartan subalgebra and direction do not change the isomorphism class of the resulting root system. See page 338 of [2] for details. □

We say that a root system R is *reducible* if it can be partitioned into two subsets S and S' such that each root in S is orthogonal to every root in S'. A root system is *irreducible* if it is not reducible.

Proposition 9.6. *The root system R of a simple Lie algebra $\mathfrak{g}$ is irreducible.*

Proof. Suppose for a contradiction $R = S \cup S'$ is a partition of R into nonempty orthogonal components. Consider the subalgebra K of $\mathfrak{g}$ generated by all the $\mathfrak{g}_\alpha$ for roots α in S. Then for β in S', we have $(\alpha+\beta,\alpha) \neq 0$ and $(\alpha+\beta,\beta) \neq 0$, so $\alpha+\beta$ is not a root. By lemma (9.4) this implies $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$. Thus $[K, \mathfrak{g}_\beta] = 0$ for each root β in S'. Since the center of $\mathfrak{g}$ is trivial, K must be a proper subalgebra. It is also clearly an ideal, contradicting the simplicity of $\mathfrak{g}$. □

10. The Classification Theorem

Since the simple roots contain essentially all the information of the root system, we can represent a root system by its Dynkin diagram.

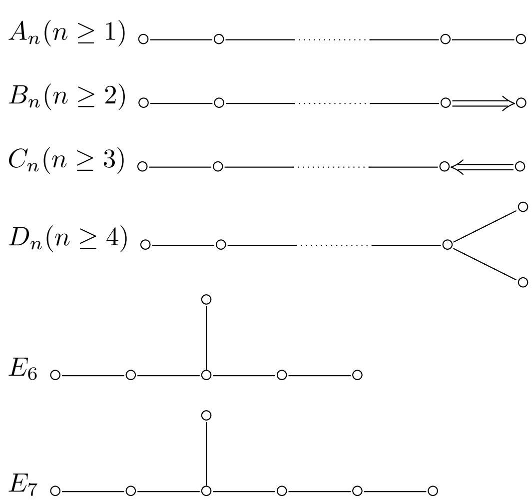
Definition 10.1. The *Dynkin diagram* of a root system is obtained by drawing a node ∘ for each simple root and joining two nodes by a number of lines depending on the angle between them. The number of lines joining the nodes associated to roots α and β is $-n_{\beta\alpha}$.

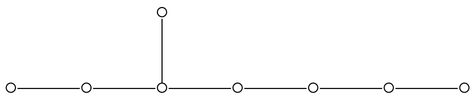
Example 10.2. By our description of the angles between the simple roots of $\mathfrak{sl}_{n+1}(\mathbb{C})$ we see that the Dynkin diagram is:

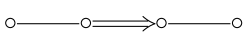
By convention we call this diagram A_n.

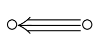
It is clear that a root system is irreducible if and only if its Dynkin diagram is connected. Moreover, each Dynkin diagram corresponds to the root system of a unique semisimple Lie algebra. We have thus reduced the project of classifying simple Lie algebras to determining connected Dynkin diagrams, which we can catalog by a series of combinatorial and geometric arguments. In this section we state and prove the classification of connected Dynkin diagrams.

Theorem 10.3. *The connected Dynkin diagrams are precisely the following:*



E_8 

F_4 

G_2 

Note that in most cases the restrictions on n are necessary to avoid duplicated isomorphism types.

Proof. To start the proof, we shall first simplify matters by considering undirected Dynkin diagrams. A Dynkin diagram without arrows to indicate relative lengths is called a *Coxeter diagram.* A Coxeter diagram with n nodes thus corresponds to a system of n linearly independent unit vectors $e_1, ..., e_n$ in some Euclidean space E such that the angle between e_i and e_j is $\frac{\pi}{2}$, $\frac{2\pi}{3}$, $\frac{3\pi}{4}$, or $\frac{5\pi}{6}$ depending on whether there are $0, 1, 2$, or 3 lines between their corresponding nodes. A diagram is said to be *admissable* if it corresponds to such a system of vectors. Conversely, any choice of arrows on an admissable diagram yields a Dynkin diagram by scaling the corresponding vectors appropriately.

Since the e_i are unit vectors, their inner-product gives the cosine of the angle between them. In other words,

$$(e_i, e_j) = 0, -\frac{1}{2}, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{3}}{2}$$

depending on the number of lines between them. It follows that

$$4(e_i, e_j)^2$$

gives the number of lines between e_i and e_j.

Now we can classify connected admissable diagrams by a series of steps. Let N denote an admissable diagram with n nodes.

(1) A subdiagram of N obtained by removing some nodes and all lines connected to them is admissable.

(2) The number of pairs of nodes connected by lines is less than or equal to $n-1$. If e_i and e_j are connected then $2(e_i, e_j) \leq -1$. Thus the inequality

$$0 < (\sum e_i, \sum e_i) = n + \sum_{i<j} (e_i, e_j)$$

implies the result. This immediately implies:

(3) N has no loops.

(4) A node has no more than three lines connected to it.

By (1), it suffices to consider the case where e_1 is connected to each of the other $n-1$ nodes. By (2) no other nodes are connected to each other. Since $e_2, ..., e_n$ are perpendicular unit vectors whose span does not contain e_1,

$$1 = (e_1, e_1)^2 > \sum_{i=2}^{n} (e_1, e_i)^2.$$

This shows the number of lines to e_1 is $\sum_{i=2}^{n} 4(e_1, e_i)^2 < 4$. This immediately implies:

(5) The only admissable diagram containing a pair of nodes with three lines between them is the Coxeter graph of G_2.

(6) Any string of nodes connected to each other by one line, with none but the ends of the string connected to any other nodes, can be collapsed to one node to yield an admissable diagram.

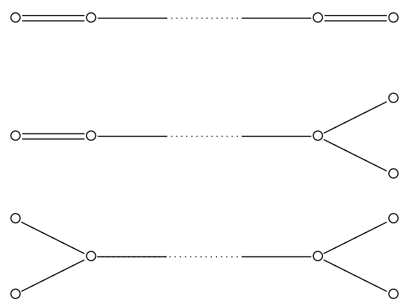
If $e_1, ..., e_r$ are unit vectors corresponding to a string, then $e' = \sum_{i=1}^{r} e_i$ is also unit vector since

$$(e', e') = r + 2((e_1, e_2), ..., (e_{r-1}, e_r)) = r - (r - 1) = 1.$$

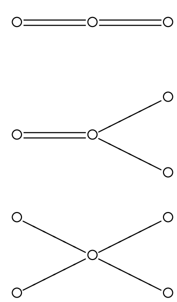
This new collection of unit vectors is linearly independent. Moreover, e' preserves the lines to the rest of the diagram since any other e_j with $j > r$ can be connected to only e_1 or e_r, which implies $(e', e_j) = (e_1, e_j)$ or (e_r, e_j).

Now we can eliminate several possibilities.

(7) N contains no subdiagram of the form:

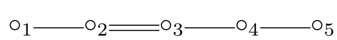


By (1), any such subdiagram must be admissable. By (6), removal of the string in the center must yield an admissable diagram.



However, in each case the resulting diagram has a node with more than three lines connected to it, contradicting (4).

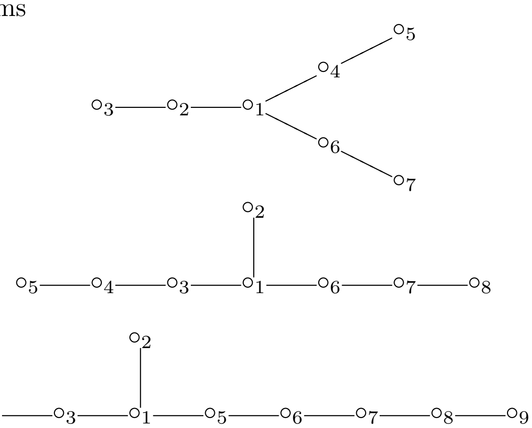
(8) The diagram



is not admissable. Index the corresponding unit vectors e_i according to their order in the diagram. Let $v = e_1 + 2e_2$ and $w = 3e_3 + 2e_4 + e_5$. Since v and w are linearly independent, the Cauchy-Schwarz inequality implies $(v, w)^2 < ||v||^2||w||^2$. However, a quick calculation shows $(v, w)^2 = 18$, $||v||^2 = 3$ and $||w||^2 = 6$, giving a contradiction. Thus (7) and (8) show that the only connected admissable graphs with a double-line are the Coxeter graphs of B_n and F_4.

(9) We are left with classifying admissable diagrams with single-lines. By (3) and (7), any such diagram can have only one branch point and no loops.

Now, the diagrams



are not admissable. Suppose otherwise for a contradiction. In the first case, consider the three vectors $u = \frac{2e_2+e_3}{\sqrt{3}}$, $v = \frac{2e_4+e_5}{\sqrt{3}}$, and $w = \frac{2e_6+e_7}{\sqrt{3}}$. The fact that these orthogonal unit vectors do not span e_1 implies

$$1 = |e_1| > (e_1, u)^2 + (e_2, v)^2 + (e_3, w)^2 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1.$$

The arguments for the other two diagrams are similar. For example, for the second diagram use the same computation on the vectors $u = e_2$, $v = \frac{2e_3+e_4}{\sqrt{6}}$, and $w = \frac{3e_6+2e_7+e_8}{\sqrt{6}}$, and similarly for the third diagram. By (1), any diagram containing one of the above three as a subdiagram is also not admissable.

Thus the only remaining connected diagrams correspond to the Coxeter diagrams of the Dynkin diagrams listed in the theorem. It is possible to exhibit vector systems for each of the remaining diagrams to show that they are indeed admissable, as we have already done for A_n. This completes the proof.

□

We have seen that the Lie algebra corresponding to A_n is $\mathfrak{sl}_{n+1}(\mathbb{C})$. For the sake of brevity we quote from page 326 of [2] the simple Lie algebras corresponding to the four main families of Dynkin diagrams:

$$\begin{array}{cc} A_n & \mathfrak{sl}_{n+1}(\mathbb{C}) \\ B_n & \mathfrak{so}_{2n+1}(\mathbb{C}) \\ C_n & \mathfrak{sp}_{2n}(\mathbb{C}) \\ D_n & \mathfrak{so}_{2n}(\mathbb{C}) \end{array}.$$

The exceptional Dynkin diagrams E_6, E_7, E_8, F_4, and G_2 also correspond to exceptional Lie algebras, although this fact is far from obvious. We shall omit the proof. See page 101 of [5] for an argument that any irreducible root system corresponds to some Lie algebra.

From what we have seen, this list of all the connected Dynkin diagrams translates into a list of all irreducible root systems. This then describes all simple complex Lie algebras, whereby we may obtain the simple real Lie algebras and simply-connected simple Lie groups. Taking quotients by discrete central subgroups, we finally find all simple Lie groups. While there is substantially more computational work to do for a complete classification, this major step in the description of simple Lie groups offers a compelling illustration of how to convert a difficult mathematical problem among equivalent forms to the point of becoming lucidly straightforward.

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