

# A PROBABILISTIC VIEW ON THE ECONOMICS OF GAMBLING

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ABSTRACT. This paper begins by defining a probability space and establishing probability functions in this space over discrete random variables. We then define the expectation and variance of a random variable with the ultimate goal of proving the Weak Law of Large Numbers. We then apply these ideas to the concept of gambling to show why casinos exist and when it is beneficial to an individual to gamble as defined by individual utility.

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## 1. THE PROBABILITY SPACE

To motivate the probability space we'll begin by stating that a random experiment is one whose results may vary between trials even under identical conditions. To try to predict how often each trial will produce a certain result we need to look at the set  $\Omega$  of all possible outcomes of the random experiment. We then denote  $\mathcal{A}$  as a subset of the power set of  $\Omega$ , which is the collection of subsets of  $\Omega$ . Finally, we need a probability measure  $\mathcal{P}$  that measures the likelihood of each event in  $\mathcal{A}$  occurring as a result of a random experiment. With these ideas in mind, we can define the notion of a probability space.

**Definition 1.1.** A Probability Space is a triple  $\langle \Omega, \mathcal{A}, \mathcal{P} \rangle$  representing the probability of every possible event in a system occurring, where  $\Omega$  is the set of all possible outcomes (the sample space),  $\mathcal{A}$  is a collection of subsets of  $\Omega$  (events) which satisfies:

$$\begin{aligned} \emptyset &\in \mathcal{A}, \\ \text{for an event } a_i &\in \mathcal{A}, (\Omega \setminus a_i) \in \mathcal{A}, \\ \text{if } a_i &\in \mathcal{A} \text{ for } i=1,2,\dots \text{ then } \bigcup_i a_i \in \mathcal{A}, \end{aligned}$$

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and  $\mathcal{P}$  is the probability measure where  $\mathcal{P}: \mathcal{A} \rightarrow [0, 1]$  is a function satisfying  $\mathcal{P}(\Omega) = 1$  and for a countable set of  $n$  disjoint events  $a_1, a_2, \dots, a_n$ ,  $\mathcal{P}\left(\bigcup_{i=1}^n a_i\right) = \sum_{i=1}^n \mathcal{P}(a_i)$ .

Because it is clear each event  $a_i$  is a subset of  $\mathcal{A}$ , we can extend the notion of set theory to the events. By introducing the binary operations of union and intersection, for any two events  $a_j$  and  $a_k$ , we say that

$a_j \cup a_k$  is the event “either  $a_j$  or  $a_k$  or both,”

$a_j \cap a_k$  is the event “both  $a_j$  and  $a_k$ .”

Note that if the events  $a_j$  and  $a_k$  are mutually exclusive, then  $a_j \cap a_k = \emptyset$ , meaning that both events cannot occur simultaneously. Looking further at these conditions, if the probability that  $a_j \cap a_k$  occurs depends only on the probability that  $a_j$  occurs disjoint from the probability that  $a_k$  occurs, the two events are said to be independent.

**Definition 1.2.** Two events are independent if and only if  $\mathcal{P}(a_j \cap a_k) = \mathcal{P}(a_j)\mathcal{P}(a_k)$ .

We would now like to extend the probability measure to our sample space to motivate the probability mass function and distribution function. First, we will need to define the random variable.

## 2. RANDOM VARIABLES AND PROBABILITY FUNCTIONS

We are now going to take the sample space and assign a number to each outcome  $w \in \Omega$  where each number contains some information about the outcome. This function  $X$  is called a random variable.

**Definition 2.1.** A random variable  $X$  is a function  $X: \Omega \rightarrow \mathbb{R}$ .

**Example 2.2.** Suppose we toss a coin three times. Then  $\Omega = \{\text{HHH, HHT, HTH, HTT, THH, TTH, THT, TTT}\}$ , where H represents the coin landing on heads and T represents the coin landing on tails. Let's define our random variable  $X$  to be the function that sends each event containing an even number of heads to 1 and each event containing an odd number of heads to 0. Thus a table of our function looks like:

Table 1-1

Outcome	HHH	HHT	HTH	HTT	THH	TTH	THT	TTT
X	0	1	1	0	1	0	0	1

**Definition 2.3.** A discrete random variable is a random variable that takes on a finite or countably infinite number of values.

*Remark 2.4.* For a discrete random variable  $X$ , we can arrange the possible values that  $X$  takes on as  $x_1, x_2, \dots$  where  $x_i \geq x_j$  for  $i > j$ . Each of these values of  $X$  will have a probability associated with it,  $\mathcal{P}(X = x_i)$ , where  $1 \leq i \leq n$  for  $n$  distinct values of  $X$  and  $n \in [1, \infty)$ . From this we can define the probability mass function and distribution function of  $X$ .

**Definition 2.5.** For a discrete random variable  $X$ ,  $f(x_i)$  is the probability mass function of  $X$  if

$$f(x_i) = \mathcal{P}(X = x_i)$$

for all  $i$ .

**Definition 2.6.** The distribution function  $F(x)$  for a discrete random variable  $X$  is defined as

$$F(x) = \mathcal{P}(X \leq x) = \sum_{y \leq x} f(y),$$

where  $x \in \mathbb{R}$ .

Note: For any  $y \neq x_j$ , where  $x_j$  is a value taken on by  $X$ ,  $\mathcal{P}(y) = 0$ . Therefore, for discrete random variables,

$$\sum_{y \leq x} f(y)$$

is equivalent to writing

$$\sum_{x_j \leq x} f(x_j).$$

Note: Aside from discrete random variables, there are also continuous random variables which do not satisfy Definition 2.3.

**Definition 2.7.** A random variable  $X$  is continuous if its distribution function can be written as

$$F(x) = \mathcal{P}(X \leq x) = \int_{-\infty}^x f_X(u) du,$$

where  $f_X$  is the probability density function of  $X$ .

*Remark 2.8.* The probability density function is the continuous random variable's analog to the discrete random variable's probability mass function. While this definition is important in the field of probability, this paper will only focus on discrete random variables.

The previous ideas can be extended to the case with two or more discrete random variables. Once we define the joint distribution for the discrete random variables  $X$  and  $Y$ , generalizations can be made to the case with more than two random variables.

**Definition 2.9.** For two discrete random variables  $X$  and  $Y$ ,  $f(x_i, y_i)$  is the joint probability mass function of  $X$  and  $Y$  if

$$f(x_i, y_i) = \mathcal{P}(X = x_i, Y = y_i)$$

for all  $i$ .

**Definition 2.10.** For two discrete random variables  $X$  and  $Y$ , the joint distribution function of  $X$  and  $Y$  is defined by

$$F(x, y) = \mathcal{P}(X \leq x, Y \leq y) = \sum_{x_i \leq x} \sum_{y_i \leq y} f(x_i, y_i),$$

where  $x_i$  are the values of  $X$  less than or equal to  $x$  and  $y_i$  are the values of  $Y$  less than or equal to  $y$ .

Just as with two distinct events, the concept of independence also applies to random variables. If each pair of events taken from the random variables  $X$  and  $Y$  is independent, where one event is taken from  $X$  and one from  $Y$ , we say that  $X$  and  $Y$  are independent.

**Definition 2.11.** If for all  $x_i$  and  $y_i$  the events  $\{X = x_i\}$  and  $\{Y = y_i\}$  are independent, we say that X and Y are independent random variables. Then

$$\mathcal{P}(X = x_i, Y = y_i) = \mathcal{P}(X = x_i)\mathcal{P}(Y = y_i),$$

which means

$$f(x_i, y_i) = f_X(x_i)f_Y(y_i),$$

where  $f_X$  and  $f_Y$  are the probability mass functions of X and Y.

### 3. EXPECTATION AND VARIANCE

Now that we are able to calculate the probability that a certain value of X will result from a trial, we want to know beforehand what the expected outcome of the trial will be. For a discrete random variable X we can write the possible values of X as  $x_1, x_2, \dots, x_n$  and associate with each the probability  $\mathcal{P}(X = x_i)$  to find the expectation of X.

**Definition 3.1.** For a discrete random variable X the Expectation of X, denoted  $\mathbb{E}(X)$ , is defined by

$$\mathbb{E}(X) = x_1\mathcal{P}(X = x_1) + \dots + x_n\mathcal{P}(X = x_n) = \sum_{x_i \in \Omega} x_i\mathcal{P}(X = x_i) = \sum_{x_i \in \Omega} x_i f(x_i),$$

for  $n \in [1, \infty)$ .

Note: The expectation of X is also called the expected value of X or the mean of X and is often denoted by  $\mu$ .

**Example 3.2.** Suppose that a fair coin is about to be flipped. If it lands on heads, you win 30 dollars. If it lands on tails, you lose 20 dollars. We can define X to be the random variable that sends the result of the flip to the amount of money that you win. Then for heads,  $x_1$  is +30 and  $f(x_1)$  is 0.50. For tails,  $x_2$  is -20 and  $f(x_2)$  is 0.50. Thus the expectation is

$$\mathbb{E}(X) = (+30)(0.50) + (-20)(0.50) = +5.$$

Therefore, on each trial of this game, you are expected to earn 5 dollars in profit.

*Remark 3.3.* The concept of expectation is of extreme importance to mathematical gambling theory. We will see later that expectation is closely related to net profit when playing a large number of games in a casino.

For two random variables X and Y it can be shown that the expectation of the sum or product of the variables is equal to the sum or product of the expectations under certain conditions. The following two theorems will only be proved in the discrete case. Generalizations can easily be made in the case that X and Y are continuous variables, but these generalizations are not necessary to this paper.

**Theorem 3.4.** *If X and Y are random variables, then*

$$\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y).$$

*Proof.* Let  $f(x,y)$  be the joint probability mass function of  $X$  and  $Y$ . Then we have

$$\begin{aligned}\mathbb{E}(X + Y) &= \sum_x \sum_y (x + y)f(x, y) \\ &= \sum_x \sum_y xf(x, y) + \sum_x \sum_y yf(x, y) \\ &= \mathbb{E}(X) + \mathbb{E}(Y).\end{aligned}$$

□

**Theorem 3.5.** *If  $X$  and  $Y$  are independent random variables, then*

$$\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y).$$

*Proof.* Let  $f(x,y)$  be the joint probability mass function of  $X$  and  $Y$ .  $X$  and  $Y$  are independent, so  $f(x,y)=f_1(x)f_2(y)$ . Therefore,

$$\begin{aligned}\mathbb{E}(XY) &= \sum_x \sum_y xyf(x, y) = \sum_x \sum_y xyf_1(x)f_2(y) \\ &= \sum_x [xf_1(x) \sum_y yf_2(y)] \\ &= \sum_x [xf_1(x)\mathbb{E}(y)] \\ &= \mathbb{E}(X)\mathbb{E}(Y).\end{aligned}$$

□

Also noteworthy is the fact that due to the nature of sums, a constant  $c$  can be pulled through the expectation operator.

**Theorem 3.6.** *Let  $X$  be a random variable. If  $c$  is any constant, then*

$$\mathbb{E}(cX) = c\mathbb{E}(X).$$

*Proof.*

$$\begin{aligned}\mathbb{E}(cX) &= \sum_{x_i \in \Omega} cx_i f(x_i) \\ &= c \sum_{x_i \in \Omega} x_i f(x_i) \\ &= c\mathbb{E}(X).\end{aligned}$$

□

Now that we have begun working with the expectation of a random variable  $X$ , we'd like to have an idea of how widely the values of  $X$  are spread around  $\mathbb{E}(X)$ . Recalling that  $\mathbb{E}(X)$  is denoted by  $\mu$ , we can define the concepts of variance and standard deviation.

**Definition 3.7.** For a random variable  $X$  with mean  $\mu$  the variance of  $X$ , denoted  $\text{Var}(X)$ , is defined by

$$\text{Var}(X) = \mathbb{E}[(X - \mu)^2].$$

**Definition 3.8.** The standard deviation of  $X$  is defined as the positive square root of the variance of  $X$  and is given by

$$\sigma = \sqrt{\text{Var}(X)} = \sqrt{\mathbb{E}[(X - \mu)^2]}.$$

*Remark 3.9.* For a discrete random variable  $X$  with probability mass function  $f(x)$ , we can write the variance of  $X$  as

$$\text{Var}(X) = \mathbb{E}[(X - \mu)^2] = \sigma^2 = \sum_{x_i \in \Omega} (x_i - \mu)^2 f(x_i).$$

With the previous definitions established, we are now able to prove a few theorems necessary in proving the Law of Large Numbers, which is essential to our justification of using the expected value of a random variable in determining long-term economic success of a casino.

**Theorem 3.10.** For any constant  $c$ ,

$$\text{Var}(cX) = c^2 \text{Var}(X).$$

*Proof.* Let  $\mathbb{E}(X) = \mu$  and  $\mathbb{E}(cX) = c\mu$ . Then

$$\text{Var}(cX) = \mathbb{E}[(cX - c\mu)^2] = \mathbb{E}[c^2(X - \mu)^2] = c^2 \text{Var}(X).$$

□

**Theorem 3.11.** If  $X$  and  $Y$  are independent random variables, then

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) = \text{Var}(X - Y).$$

*Proof.* Let  $\mu_X$  be the expectation of  $X$  and  $\mu_Y$  be the expectation of  $Y$ . Then we have

$$\begin{aligned} \text{Var}(X + Y) &= \mathbb{E}[(X + Y) - (\mu_X + \mu_Y)]^2 \\ &= \mathbb{E}[(X - \mu_X) + (Y - \mu_Y)]^2 \\ &= \mathbb{E}[(X - \mu_X)^2 + 2(X - \mu_X)(Y - \mu_Y) + (Y - \mu_Y)^2] \\ &= \mathbb{E}[(X - \mu_X)^2] + 2\mathbb{E}[(X - \mu_X)(Y - \mu_Y)] + \mathbb{E}[(Y - \mu_Y)^2]. \end{aligned}$$

From here we notice that because  $X$  and  $Y$  are independent,  $(X - \mu_X)$  and  $(Y - \mu_Y)$  are also independent. Also, because  $\mathbb{E}(X)$  is defined as  $\mu_X$  and  $\mathbb{E}(Y)$  is defined as  $\mu_Y$ , then  $\mathbb{E}(X - \mu_X) = \mathbb{E}(Y - \mu_Y) = 0$ . Therefore we have

$$\begin{aligned} \text{Var}(X + Y) &= \mathbb{E}[(X - \mu_X)^2] + 2\mathbb{E}[(X - \mu_X)(Y - \mu_Y)] + \mathbb{E}[(Y - \mu_Y)^2] \\ &= \mathbb{E}[(X - \mu_X)^2] + 2[\mathbb{E}(X - \mu_X)\mathbb{E}(Y - \mu_Y)] + \mathbb{E}[(Y - \mu_Y)^2] \\ &= \mathbb{E}[(X - \mu_X)^2] + \mathbb{E}[(Y - \mu_Y)^2] \\ &= \text{Var}(X) + \text{Var}(Y). \end{aligned}$$

To complete this proof, we notice that  $\text{Var}(X - Y) = \text{Var}(X) + \text{Var}((-1)Y)$ . Then by Theorem 3.10 this equals  $\text{Var}(X) + (-1)^2 \text{Var}(Y) = \text{Var}(X) + \text{Var}(Y) = \text{Var}(X + Y)$ . □

*Remark 3.12.* Notice that the above proof can be generalized to any number of independent random variables. Therefore, we can see that the variance of a sum of independent random variables is equal to the sum of the variances.

**Theorem 3.13. (Chebyshev's Inequality)** Suppose  $X$  is a random variable with finite mean and variance. Then for any  $\varepsilon > 0$ ,

$$\mathcal{P}(|X - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}.$$

*Proof.* Let  $f(x)$  be the probability mass function of  $X$ . Then

$$\sigma^2 = \mathbb{E}[(X - \mu)^2] = \sum_{x_i \in \Omega} (x_i - \mu)^2 f(x_i).$$

But if we limit the sum over the  $x$  such that  $|x - \mu| \geq \varepsilon$ , the value of the sum can only decrease since we are only working with nonnegative terms. Therefore we have

$$\sigma^2 \geq \sum_{|x-\mu| \geq \varepsilon} (x - \mu)^2 f(x) \geq \sum_{|x-\mu| \geq \varepsilon} \varepsilon^2 f(x) = \varepsilon^2 \sum_{|x-\mu| \geq \varepsilon} f(x) = \varepsilon^2 \mathcal{P}(|x - \mu| \geq \varepsilon).$$

Dividing both sides by  $\varepsilon^2$  gives the desired result of

$$\mathcal{P}(|X - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}.$$

□

Using Chebyshev's Inequality, we get an interesting result when we plug in  $\varepsilon = k\sigma$ , namely that

$$\mathcal{P}(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

This result shows that for  $k \geq 1$ , the probability that a certain value of  $X$  lies outside of the range  $(\mu - k\sigma, \mu + k\sigma)$  is bounded above by  $\frac{1}{k^2}$  for any finite  $\mu$  and  $\sigma$ .

**Example 3.14.** Let  $k=3$ . Then using Chebyshev's Inequality,

$$\mathcal{P}(|X - \mu| \geq 3\sigma) \leq \frac{1}{3^2}.$$

This means that the probability of any value of  $X$  being within three standard deviations of the mean is at least about 89%. Consequently, there is at most about an 11% chance that the value lies  $3\sigma$  away from  $\mu$ .

*Remark 3.15.* Chebyshev's Inequality provides only a generous upper bound on the probability that a random variable lies within a certain number of standard deviations. For many distributions, the probability is much smaller. For example, in the normal distribution there is about a .13% chance that the random variable lies three standard deviations from the mean.

We are now in position to state one of the most important theorems in casino economics. Using the concept of variance and Chebyshev's Inequality, we can prove the Weak Law of Large Numbers, showing that the expected value of a casino game is directly related to the game's long-term profit.

**Theorem 3.16. Weak Law of Large Numbers** Let  $X_1, X_2, \dots, X_n$  be mutually independent random variables, each with finite mean and variance. Then for any  $\varepsilon > 0$ , if  $S_n = X_1 + X_2 + \dots + X_n$ , then

$$\lim_{n \rightarrow \infty} \mathcal{P} \left( \left| \frac{S_n}{n} - \mu \right| \geq \varepsilon \right) = 0.$$

*Proof.* By Theorem 3.4, we have

$$\mathbb{E}\left(\frac{S_n}{n}\right) = \mathbb{E}\left(\frac{X_1 + X_2 + \cdots + X_n}{n}\right) = \frac{1}{n}[\mathbb{E}(X_1) + \mathbb{E}(X_2) + \cdots + \mathbb{E}(X_n)] = \frac{1}{n}(n\mu) = \mu,$$

and by an extension of Theorem 3.11,

$$\text{Var}(S_n) = \text{Var}(X_1 + X_2 + \cdots + X_n) = \text{Var}(X_1) + \text{Var}(X_2) + \cdots + \text{Var}(X_n) = n\sigma^2,$$

so by Theorem 3.10, we have that

$$\text{Var}\left(\frac{S_n}{n}\right) = \frac{1}{n^2}\text{Var}(S_n) = \frac{\sigma^2}{n}.$$

Now we can invoke Chebyshev's Inequality. By letting  $X = \frac{S_n}{n}$ , we have

$$\mathcal{P}\left(\left|\frac{S_n}{n} - \mu\right| \geq \varepsilon\right) \leq \frac{\sigma^2}{n\varepsilon^2}.$$

Now we need only take the limit as  $n$  approaches infinity of both sides to get

$$\lim_{n \rightarrow \infty} \mathcal{P}\left(\left|\frac{S_n}{n} - \mu\right| \geq \varepsilon\right) = 0.$$

□

This theorem states that as the number of trials approaches infinity, the probability that the average of the results deviates from the expected value approaches zero. A good example is that of a large number of coin flips. While the disparity between heads and tails may be large for a small sample space, as more and more flips occur, the total amount of heads will end up very close to the total amount of tails. This idea is essential to the business model of a casino. From this theorem, we can deduce why casinos choose to open and why they stay in business.

#### 4. CASINO ECONOMICS

Business firms work in the long run. That is, they are willing to spend money or forgo profits now to ensure a higher profit in the big picture of their company. Casinos are no different. They are willing to face days where they make a negative profit as long as they end up in the positive at the end of some larger amount of time. Therefore, casino operators need to ensure that they actually will see a positive net income if they stay in business long enough. Because of this, they rely heavily on the Law of Large Numbers.

Let's look at one of the simplest games a casino can offer: roulette. A roulette wheel consists of 38 evenly spaced pockets numbered 0, 00, 1, 2,  $\dots$ , 36, where 0 and 00 are colored green, and the other 36 numbers are split into two groups of eighteen, with one group colored black and the other colored red. The wheel is spun and a ball is dropped into it while bets are placed on where the ball will land. When played according to design, this game relies completely on the laws of probability. Therefore, it makes sense to quote the expected value of the bets on the table.

**Example 4.1.** The payout for betting on red is 1 to 1. Since there are 38 pockets on the wheel, 18 of them being red, we notice that the odds of winning this bet are



$\frac{18}{38}$ . Therefore the odds of losing the bet are  $\frac{20}{38}$ . We then see that

$$\mathbb{E}(\text{Red}) = \sum a_i p_i = (1) \left(\frac{18}{38}\right) + (-1) \left(\frac{20}{38}\right) = -\frac{1}{19} \approx -.0526.$$

This says that for every one dollar bet we place on red, we are expected to lose about 5.26 cents, or the casino makes about 5.26 cents.

This calculation can be carried out for all of the bets to show that **every** bet has a negative expected value. Furthermore, this holds true for every chance-based game offered by the casino. Recall that the Weak Law of Large Numbers states for any  $\varepsilon > 0$ ,

$$\lim_{n \rightarrow \infty} \mathcal{P} \left( \left| \frac{S_n}{n} - \mu \right| \geq \varepsilon \right) = 0.$$

Therefore as the casino remains in business longer and the number of bets customers place increases, the probability that the casino does not earn their expected value approaches zero. Seeing as the expected value of the casino is positive, the casino can expect with probabilistic certainty that they will make a positive net profit. In other words, the house always wins.

## 5. GAMBLER'S UTILITY

So if the house is always expected to come out on top in the long run, why do people gamble? To answer this we need to define a gambler's utility function. Let's denote the utility function of a gambler as  $U(x_i)$ . We can work under the assumption that all rational people would prefer, under identical circumstances, to have more money than not. Therefore,  $U(x_i)$  is increasing.

For every event  $x_i$  there is an associated utility  $U(x_i)$  that will add or subtract from the gambler's total utility, depending on the outcome. Therefore each gamble has with it an associated expected utility.

**Definition 5.1.** For a discrete random variable  $X$ , let  $\mathcal{P}(X = x_i) = f(x_i)$ . Then the Expected Utility of  $X$ , denoted  $\varphi(X)$ , is defined as

$$\varphi(X) = \sum_{x_i \in \Omega} U(x_i) f(x_i).$$

*Remark 5.2.* We notice that a gambler will accept a gamble if  $\varphi(X) > 0$  and will decline a gamble if  $\varphi(X) < 0$ . In the case that  $\varphi(X) = 0$ , the gambler is indifferent as to whether or not he accepts the gamble.

Let's now go back to our roulette example. The gambler who plays roulette is willing to do so even though he has a negative expected return. This shows that for him,  $\varphi(X) \geq 0$ , which implies that he receives some amount of positive utility from the act of gambling itself. This can be likened to thinking of gambling as a form of entertainment, such as going to a baseball game. The expected return of going to a baseball game is negative since there is a cost for tickets, parking etc., but there is also positive utility gained from the experience.

Since the gambler was willing to accept the bet with a negative expected return, he would surely opt to take part in a similar gamble with an expected return of 0. Because of this, we say that he is risk loving.

**Definition 5.3.** A Risk Loving person will always accept a fair gamble. In other words, given the choice of earning the same amount of money through gamble or through certainty, the risk loving person will opt for the gamble.

*Remark 5.4.* From this definition we can see that a risk loving person has increasing marginal utility. This means that the person derives more utility from winning a certain amount of money than from losing that same amount. Therefore, his marginal utility is increasing and concave up.

Returning to the previous definition, we know that for any positive amount of money  $c$ ,  $|U(x_i) - U(x_i - c)| \leq |U(x_i + c) - U(x_i)|$ . But if the expected return of a gamble is not 0, such as in roulette, we can define the utility that a person receives from gambling.

**Definition 5.5.** The Utility of Risk, denoted by  $U_R$ , is defined as

$$U_R = \max[(b - c)]$$

for  $b, c$  satisfying  $|U(x_i) - U(x_i - b)| \leq |U(x_i + c) - U(x_i)|$ .

Using this equation we can determine for each individual how much utility they derive from the experience of gambling.

**Example 5.6.** We've quoted the expected value of a roulette bet to be about -5.26 cents per dollar. We know that someone who makes this bet has an expected utility greater than or equal to zero, so using expected value we can say

$$|U(x_i) - U(x_i - 1)| \leq |U(x_i + .9474) - U(x_i)|.$$

Thus, this person has a utility of risk greater than or equal to .0526.

Therefore while gambling against the house seems illogical mathematically, the concept of utility can be applied to show that gambling is indeed logical. Furthermore, using an individual's utility of risk and their utility function we can calculate the maximum expected loss a bet can carry that will still not lower the individual's expected utility. The combination of probability and economics can therefore justify the concept of gambling both for the casino and for the gambler.

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