SOME L-VALUES

Example 1. Let χ be the primitive character of conductor 3, so

$$L(\chi,1) = 1 - 1/2 + 1/4 - 1/5 + 1/7 - 1/8 + \cdots .$$

If we define

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{3n+1}}{3n+1} - \frac{x^{3n+2}}{3n+2},$$

then $L(\chi,1) = f(1)$. (There are analytic issues here which we ignore; clearly the series will converge absolutely and uniformly when $|x| < 1$, and then some Abelian or Tauberian argument will show that the limiting values as $x \to 1$ will coincide with the (conditionally convergent) L-value.) We will compute $f(x)$, by first determining its derivative. Evidently,

$$f'(x) = \sum_{n=0}^{\infty} x^{3n} - x^{3n+1} = \frac{1-x}{1-x^3} = \frac{1}{1+x+x^2} = \frac{1}{(x+1/2)^2 + 3/4}.$$

Thus

$$f(x) = \frac{2}{\sqrt{3}} \arctan\big(\frac{2}{\sqrt{3}}(x+1/2)\big) - \frac{\pi}{3\sqrt{3}}.$$

(The constant of integration is determined by the requirement that $f(0) = 0$) Finally, we get that

$$L(\chi,1) = f(1) = \frac{2\pi}{3\sqrt{3}} - \frac{\pi}{3\sqrt{3}} = \frac{\pi}{3\sqrt{3}}.$$

Example 2. Let χ be the character of conductor 8 whose values are

$$\chi(1) = \chi(7) = 1, \quad \chi(3) = \chi(5) = -1,$$

so that

$$L(\chi,1) = 1 - 1/3 - 1/5 + 1/7 + 1/9 - 1/11 - 1/13 + 1/15 + \cdots .$$

Defining

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{1+8n}}{1+8n} - \frac{x^{3+8n}}{3+8n} - \frac{x^{5+8n}}{5+8n} + \frac{x^{7+8n}}{7+8n},$$

we have that $L(\chi,1) = f(1)$. (As in the preceding example, we ignore the analytic issues inherent in this formula.) We compute

$$f'(x) = \sum_{n=0}^{\infty} x^{8n} - x^{8n+2} - x^{8n+4} + x^{8n+6} = \frac{1 - x^2 - x^4 + x^6}{1 - x^8} = \frac{1-x^2}{1+x^4}.$$

A partial fractions computation shows that

$$f'(x) = \frac{1+\sqrt{2}x}{2+2\sqrt{2}x+2x^2} + \frac{1-\sqrt{2}}{2-2\sqrt{2}x+2x^2} = \frac{1+\sqrt{2}x}{1+(1+\sqrt{2}x)^2} + \frac{1-\sqrt{2}x}{1+(1-2\sqrt{2}x)^2}.$$

Antidifferentiating, we find that

$$f(x) = \frac{1}{2\sqrt{2}} \log\bigl(1 + (1+\sqrt{2}x)^2\bigr) - \frac{1}{2\sqrt{2}} \log\bigl(1 + (1-\sqrt{2}x)^2\bigr)$$
$$= \frac{1}{2\sqrt{2}} \log(2+2\sqrt{2}x+2x^2) - \frac{1}{2\sqrt{2}} \log(2-2\sqrt{2}x+2x^2) = \frac{1}{2\sqrt{2}} \log \frac{1+\sqrt{2}x+x^2}{1-\sqrt{2}x+x^2}.$$

(Again, the constant of integration, which in this case is 0, is determined by the requirement that $f(0) = 0$.) Finally, we compute that

$$L(\chi, 1) = f(1) = \frac{1}{2\sqrt{2}} \log \frac{2+\sqrt{2}}{2-\sqrt{2}} = \frac{1}{2\sqrt{2}} \log \frac{1+\sqrt{2}}{-1+\sqrt{2}} = \frac{1}{2\sqrt{2}} \log(1+\sqrt{2})^2$$
$$= \frac{1}{\sqrt{2}} \log(1+\sqrt{2}).$$

Exercise 3. As a reality check, write a program which can compute the first n terms of the series for $L(\chi, 1)$ (where $n = 100$ or 1000 or whatever you can compute in a reasonable time) in the above two examples (or maybe just ask Wolfram Alpha to do it for you!), and compare the value you get with the exact value computed above.

Exercise 4. Compute $L(\chi, 1)$ by the method of the preceding examples for each of the following cases: χ is the primitive quadratic character mod 4, so $\chi(1) = 1, \chi(3) = -1$; and χ is the primitive quadratic character mod 5, so $\chi(1) = \chi(4) = 1, \chi(2) = \chi(3) = -1$.

Exercise 5. Analyze the preceding method, and show that if $\chi(-1) = -1$, then the L-value will always involve an arctan function in its evaluation, while if $\chi(1) = 1$, then the L-value will always involve a log function in its evaluation.

Exercise 6. What is the ring of integers in $\mathbb{Q}(\sqrt{5})$? Find a generator of the group of units in the ring of integers, i.e. the "minimal" $\alpha \in \mathcal{O}^\times_{\mathbb{Q}(\sqrt{5})}$ such that $\alpha > 1$. Compare this with Exercise 4.