**SOME $L$-VALUES**

**Example 1.** Let $\chi$ be the primitive character of conductor 3, so

$$L(\chi, 1) = 1 - 1/2 + 1/4 - 1/5 + 1/7 - 1/8 + \cdots.$$ 

If we define

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{3n+1}}{3n+1} - \frac{x^{3n+2}}{3n+2},$$

then $L(\chi, 1) = f(1)$. (There are analytic issues here which we ignore; clearly the series will converge absolutely and uniformly when $|x| < 1$, and then some Abelian or Tauberian argument will show that the limiting values as $x \to 1$ will coincide with the (conditionally convergent) $L$-value.) We will compute $f(x)$, by first determining its derivative. Evidently,

$$f'(x) = \sum_{n=0}^{\infty} x^{3n} - x^{3n+1} = \frac{1}{1 - x} - \frac{1}{1 + x + x^2} = \frac{1}{(x + 1/2)^2 + 3/4}.$$ 

Thus

$$f(x) = \frac{2}{\sqrt{3}} \arctan \left( \frac{2}{\sqrt{3}} (x + 1/2) \right) - \frac{\pi}{3\sqrt{3}}.$$

(The constant of integration is determined by the requirement that $f(0) = 0$)

Finally, we get that

$$L(\chi, 1) = f(1) = \frac{2\pi}{3\sqrt{3}} - \frac{\pi}{3\sqrt{3}} = \frac{\pi}{3\sqrt{3}}.$$

**Example 2.** Let $\chi$ be the character of conductor 8 whose values are

$$\chi(1) = \chi(7) = 1, \quad \chi(3) = \chi(5) = -1,$$

so that

$$L(\chi, 1) = 1 - 1/3 - 1/5 + 1/7 + 1/9 - 1/11 - 1/13 + 1/15 + \cdots.$$ 

Defining

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{1+8n}}{1 + 8n} - \frac{x^{3+8n}}{3 + 8n} - \frac{x^{5+8n}}{5 + 8n} + \frac{x^{7+8n}}{7 + 8n},$$

we have that $L(\chi, 1) = f(1)$. (As in the preceding example, we ignore the analytic issues inherent in this formula.) We compute

$$f'(x) = \sum_{n=0}^{\infty} x^{8n} - x^{8n+2} - x^{8n+4} + x^{8n+6} = \frac{1 - x^2 - x^4 + x^6}{1 - x^8} - \frac{1 - x^2}{1 + x^4}.$$ 

A partial fractions computation shows that

$$f'(x) = \frac{1 + \sqrt{2}x}{2 + 2\sqrt{2}x + 2x^2} + \frac{1 - \sqrt{2}}{2 - 2\sqrt{2}x + 2x^2} = \frac{1 + \sqrt{2}x}{1 + (1 + \sqrt{2})x^2} + \frac{1 - \sqrt{2}x}{1 + (1 - 2\sqrt{2})x^2}.$$
Antidifferentiating, we find that
\[ f(x) = \frac{1}{2\sqrt{2}} \log(1 + (1 + \sqrt{2}x)^2) - \frac{1}{2\sqrt{2}} \log(1 + (1 - \sqrt{2}x)^2) \]
\[ = \frac{1}{2\sqrt{2}} \log(2 + 2\sqrt{2}x + 2x^2) - \frac{1}{2\sqrt{2}} \log(2 - 2\sqrt{2}x + 2x^2) = \frac{1}{2\sqrt{2}} \log \frac{1 + \sqrt{2}x + x^2}{1 - \sqrt{2}x + x^2}. \]

(Again, the constant of integration, which in this case is 0, is determined by the requirement that \( f(0) = 0 \).) Finally, we compute that
\[ L(\chi, 1) = f(1) = \frac{1}{2\sqrt{2}} \log \frac{2 + \sqrt{2}}{2 - \sqrt{2}} = \frac{1}{2\sqrt{2}} \log \frac{1 + \sqrt{2}}{-1 + \sqrt{2}} = \frac{1}{2\sqrt{2}} \log(1 + \sqrt{2})^2 \]
\[ = \frac{1}{\sqrt{2}} \log(1 + \sqrt{2}). \]

**Exercise 3.** As a reality check, write a program which can compute the first \( n \) terms of the series for \( L(\chi, 1) \) (where \( n = 100 \) or 1000 or whatever you can compute in a reasonable time) in the above two examples (or maybe just ask Wolfram Alpha to do it for you!), and compare the value you get with the exact value computed above.

**Exercise 4.** Compute \( L(\chi, 1) \) by the method of the preceding examples for each of the following cases: \( \chi \) is the primitive quadratic character mod 4, so \( \chi(1) = 1, \chi(3) = -1 \); and \( \chi \) is the primitive quadratic character mod 5, so \( \chi(1) = \chi(4) = 1, \chi(2) = \chi(3) = -1 \).

**Exercise 5.** Analyze the preceding method, and show that if \( \chi(-1) = -1 \), then the \( L \)-value will always involve an arctan function in its evaluation, while if \( \chi(1) = 1 \), then the \( L \)-value will always involve a log function in its evaluation.

**Exercise 6.** What is the ring of integers in \( \mathbb{Q}(\sqrt{5}) \)? Find a generator of the group of units in the ring of integers, i.e. the “minimal” \( \alpha \in \mathcal{O}^\times_{\mathbb{Q}(\sqrt{5})} \) such that \( \alpha > 1 \). Compare this with Exercise 4.